Periods and L-functions

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Let Ξ be a smooth projective curve defined over $\mathbb{F}_q$, then we can form its function field $F = \mathbb{F}_q(\Xi)$, integral adeles $\mathbb{O}$, and adeles $\mathbb{A}$. We fix an isomorphism of coefficient fields $\mathbb{C} \cong \bar{\mathbb{Q}}_\ell$ for convenience.

Let G be a reductive group over $\mathbb{F}_q$. The classical (unramified) global Langlands conjecture is roughly a picture of the form

$$\left\{\begin{matrix}\text{Unramified automorphic}\\ \text{representations of } G(\mathbb{A})\end{matrix}\right\} \longleftrightarrow \left\{\check{G}\text{-local systems on } \Xi\right\}$$

In particular, there are L-functions defined on both sides, and they are supposed to agree. The definition of L-functions on either side depends on additional data, and one part of the relative Langlands framework is to clarify this. The picture is now roughly

$$\begin{array}{ccc}\left\{\begin{matrix}\text{Unramified automorphic}\\ \text{representations of } G(\mathbb{A})\end{matrix}\right\} & \longleftrightarrow & \left\{\check{G}\text{-local systems on } \Xi\right\} \\ \Big\downarrow X & & \Big\downarrow \check{X} \\ \text{Period} & \overset{\approx}{\longleftrightarrow} & \text{Values of } L\text{-function}\end{array}$$

This talk will explain the constructions of period functions and L-functions. We will also illustrate the connection in the Iwasawa–Tate case, which was explained in Kaletha's talk from two days ago. The following talk will categorify both sides into sheaves.

1. Automorphic side

The A-side TQFT $\mathcal{A}_G$ evaluated on the 3-dimensional object Ξ gives the vector space of functions on $\mathrm{Bun}_G(\mathbb{F}_q)$. A boundary condition gives an element of this space. Let X be any G-variety, then the boundary theory $\mathcal{A}_{G,X}$ produces the following element.

Definition 1.1. The *theta series* for X is the function $\Theta_X : \mathrm{Bun}_G(\mathbb{F}_q) \to \mathbb{C}$ defined by

$$\Theta_X(g) := \sum_{x \in X(F)} \mathbf{1}_{X(\mathbb{O})}(xg)$$

for all $g \in G(\mathbb{A})$. Here, $\mathbf{1}_{X(\mathbb{O})}$ is the indicator function of the open compact set $X(\mathbb{O}) \subseteq X(\mathbb{A})$.

Remark 1.2. In general, the presence of a non-trivial $\mathbb{G}_{\mathrm{gr}}$-action introduces an additional twist by a square root of the canonical bundle, cf. [BZSV (10.6)].

Example 1.3. In the Iwasawa–Tate case, $G = \mathbb{G}_m$ acts on $X = \mathbb{A}^1$ by scaling, so

$$\Theta_X(a) = \sum_{x \in F} \mathbf{1}_{\mathbb{O}}(ax) = \#\{x \in F \mid x \in a^{-1}\mathbb{O}\}.$$

This is exactly the number of global sections of the line bundle attached to a.

This example is a part of the following geometric reinterpretation of the theta series, which follows by the same type of computation.

Lemma 1.4. *Let $\mathcal{G}$ be the G-bundle attached to $g \in G(\mathbb{A})$, then $\Theta_X(g)$ counts the number of X-sections of the X-bundle associated to $\mathcal{G}$.*

Given a function $\varphi : \mathrm{Bun}_G(\mathbb{F}_q) \to \mathbb{C}$, we can pair it against the theta series

$$\langle \Theta_X, \varphi \rangle = \int_{G(F)\backslash G(\mathbb{A})} \Theta_X(g)\varphi(g)\, dg = \sum_{x \in \mathrm{Bun}_G(\mathbb{F}_q)} \frac{1}{\#\mathrm{Aut}_x} \Theta_X(x)\varphi(x)$$

We call this the *period* of φ. In other words, $\langle \Theta_X, - \rangle$ is the period functional on automoprhic forms.

Example 1.5. If $X = H\backslash G$ is homogeneous, then in our everywhere unramified setting, an easy computation shows that

$$\langle \Theta_X, \varphi \rangle = \int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\, dh$$

is the usual period.

2. Spectral side

Let Ξ be a $\check{G}$-local system. Its L-function depends on the additional datum of an algebraic representation

$$r : \check{G} \to \mathrm{GL}_n.$$

This determines a rank n local system $r \circ \Xi$ on Ξ.

Definition 2.1. Let $\mathcal{L}$ be a local system on Ξ. Its L-function is defined to be

$$L(s, \mathcal{L}) = \det\left(1 - q^{-s}\mathrm{Frob} | \mathrm{R}\Gamma(\Xi_{/\bar{\mathbb{F}}_q}, \mathcal{L})\right)^{-1}.$$

The determinant is graded, namely the odd degree terms carry an exponent of -1.

Since Ξ is a smooth curve, the cohomology is supported in degrees $[0, 2]$, and really the only interesting degree is 1. Let $V = \mathrm{H}^1(\Xi_{/\bar{\mathbb{F}}_q}, \mathcal{L})$, then the corresponding term in the above product is

$$\det(1 - q^{-s}\mathrm{Frob}|V) = \mathrm{Tr}(q^{-s}\mathrm{Frob}| \wedge^\bullet V).$$

Again, the trace is a graded trace.

Classically, L-functions are usually defined as an Euler product over the places of a global field. In the function field case, this connection is given by the Grothendieck–Lefschetz trace formula.

Theorem 2.2. *There is a factorization*

$$L(s, \mathcal{L}) = \prod_{v \in |\Xi|} L(s, \mathcal{L}_v),$$

where each term is defined by

$$L(s, \mathcal{L}_v) = \det(1 - (q^{-s}\mathrm{Frob})^{\deg v} | \mathcal{L}_v)^{-1}.$$

Example 2.3. Let $\mathbf{1}$ be the constant local system of rank 1, then for each v,

$$L(s, \mathbf{1}_v) = \frac{1}{1 - q^{-s \deg v}}.$$

The global L-function is equal to

$$L(s, \mathbf{1}) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})} = \prod_{v \in |\Xi|} L(s, \mathbf{1}_v),$$

where $P(T)$ is some polynomial of degree $2g$, $g = \text{genus}(\Xi)$. Note that this is a meromorphic function of $s \in \mathbb{C}$, and it has simple poles at $s = 0$ and $s = 1$. In this case, Poincaré duality implies the functional equation $L(s, \mathbf{1}) = L(1 - s, \mathbf{1})$.

3. Numerical relations

In the Tate–Iwasawa setting of $G = \mathbb{G}_m$ acting on $X = \mathbb{A}^1$, we will compute the period

$$\langle \Theta_X, q^{-s \deg} \rangle,$$

where $\deg : \text{Pic}_\Xi(\mathbb{F}_q) \to \mathbb{Z}$ is the degree function. This was exactly the computation in Tate's thesis that Kaletha explained previously, but we will say it in a slightly different way.

By definition,

$$\begin{aligned}
\langle \Theta_X, q^{-s \deg} \rangle &= \sum_{\mathcal{L} \in \text{Pic}_\Xi(\mathbb{F}_q)} q^{-s \deg(\mathcal{L})} \Theta_X(\mathcal{L}) \\
&= \sum_{\mathcal{L} \in \text{Pic}_\Xi(\mathbb{F}_q)} q^{-s \deg(\mathcal{L})} \# \mathrm{H}^0(\mathcal{L}) \\
&\text{“}=\text{”} \sum_{D \in \text{Div}^+ \Xi} q^{-s \deg D}.
\end{aligned}$$

For the final equality, the zero section contributes a divergent term, which also appeared in Tate's thesis. It can be handled by a suitable regulariation procedure.

An effective divisor is a formal linear combination of closed points of Ξ with non-negative integer coefficients, so the final sum above factors

$$\begin{aligned}
\sum_{D \in \text{Div}^+ \Xi} q^{-s \deg D} &= \prod_{v \in |\Xi|} \sum_{n \geq 0} q^{-s \cdot n \deg v} \\
&= \prod_{v \in |\Xi|} \frac{1}{1 - q^{-s \deg v}}.
\end{aligned}$$

The final expression is the Euler product of an L-function, so we get an equality

$$\langle \Theta_X, q^{-s \deg} \rangle = L(s, \mathbf{1}).$$

In the remaining talks, we will see a categorified version of this numerical equality.

Remark 3.1. We now make two important remarks about the above example.

(1) A better formulation of the equality is that

$$\langle \Theta_X, q^{-s\deg} \rangle = L(0, |\cdot|^s)$$

for any fixed $s \in \mathbb{C}$. Here, the point of evaluation on the right hand side is determined by the $\mathbb{G}_{\mathrm{gr}}$-action on the dual variety $\check{M} = T^*\mathbb{A}^1$, and it is independent of the automorphic representation.

We get an equality of L-functions since we are allowed to twist the automorphic form by central character. This is not possible in the Gross–Prasad case, and the period integral only sees one particular L-value (namely the central one) instead of the entire L-function.

(2) The Riemann–Roch theorem gives the identity

$$\Theta_X(x) = q^{-(g-1)}|x|^{-1}\Theta_X(\mathfrak{d}x^{-1}),$$

where $\mathfrak{d}$ is the canonical divisor. Replacing x by $\mathfrak{d}^{\frac{1}{2}}x$ removes the extra $\mathfrak{d}$ from the formula, and this is exactly the correction alluded to in Remark 1.2 if we used the scaling $\mathbb{G}_{\mathrm{gr}}$-action.

The Riemann–Roch theorem can be derived from the Poisson summation formula. The Fourier transform can be interpreted as switching between the two different ways of polarizing the Hamiltonian G-space $M = T^*X$, and the above identity states that the theta series attached to both polarizations agree. In Kaletha's talk, we saw that this additional symmetry (of M) leads to the functional equation.