CONVERSE THEOREM AND THE GLOBAL LANGLANDS CORRESPONDENCE FOR FUNCTION FIELDS

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This is the notes for the last talk of the Drinfeld–Lafforgue–Lafforgue seminar. Having proven the automorphic to Galois part of the global Langlands correspondence for *general* reductive groups over function fields, we will see how converse theorems can be used to prove the Galois to automorphic part for GL_n. We will follow the treatment of L. Lafforgue [Laf02].

Fix distinct primes p and ℓ, and fix an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$. The Galois representations in this article will have ℓ-adic coefficient fields, but we will generally use $\mathbf{C}$-valued L-functions for psychological reasons. Automorphic forms will typically also have coefficients in $\mathbf{C}$, but this really does not matter. Let F be a global field of characteristic $p > 0$ defined over $\mathbf{F}_q$. For each place v, F_v is the completion of F at v with ring of integers $\mathcal{O}_v$, uniformizer ϖ_v, and residue field size q_v. Its ring of adeles is denoted by $\mathbb{A}$, with a maximal compact subgroup $\mathbb{O} = \prod_v \mathcal{O}_v$. If S is a finite set of places, then $\mathbb{A}_S = \prod_{v\in S} F_v$ and $\mathbb{A}^S = \prod'_{v\notin S} F_v$.

0. Where we stand

We are trying to prove the following theorems.

Theorem 0.1 (Cuspidal global Langlands correspondence for GL_n)**.** *Consider the sets*

- Φ_n*: global Langlands parameters, i.e. continuous semisimple representations* $\mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ *which is unramified almost everywhere, defined over a finite extension of* $\mathbf{Q}_\ell$*, and has a finite-order determinant character.*
- Π_n*: automorphic representations of* $\mathrm{GL}_n(\mathbb{A})$*, i.e. irreducible admissible subquotients of*

$$\bigcup_{K,\chi} C^\infty(\mathrm{GL}_n(\mathbf{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K, \overline{\mathbf{Q}}_\ell)_\chi$$

as K ranges over open compact subgroups of $\mathrm{GL}_n(\mathbb{A})$ *and χ ranges over finite order characters of* $F^\times\backslash\mathbb{A}^\times$*.*

Let Φ_n^{Irr} be the irreducible parameters, and let Π_n° be the cuspidal representations. There exists a canonical bijection $\Phi_n^{\mathrm{Irr}} \leftrightarrow \Pi_n^\circ$ satisfying the property that if $\sigma \leftrightarrow \pi$, then

(1) σ and π have the same set of ramified places.
(2) If both are unramified at v, then the Satake parameters of σ are the Frobenius eigenvalues of π.
(3) The central character χ_σ matches with $\det\pi$ under global class field theory.
(4) The L-functions and ϵ-factors match (to be defined).

Theorem 0.2 (Generalized Ramanujan Conjecture)**.** *If $\pi \in \Pi_n^\circ$, then π is tempered.*

Theorem 0.3 (Deligne's purity conjecture)**.** *If $\sigma \in \Phi_n^{\mathrm{Irr}}$, then it is pure of weight 0.*

Remark 0.4. Given an element $q^s \in \overline{\mathbf{Q}}_\ell$, we can construct a character $F^\times\backslash\mathbb{A}^\times \to \overline{\mathbf{Q}}_\ell^\times$ by sending x to $|x|^s$ and a Weil representation $W_F \to \overline{\mathbf{Q}}_\ell^\times$ sending Frob to q^s. By twisting, we can recover from above the slightly more general form of the conjecture giving a bijection between irreducible W_F-representations and cuspidal automorphic forms of arbitrary central character.

The main theorem we have proven is the construction of the map $\Pi_n^\circ \to \Phi_n$. More precisely.

Theorem 0.5 (V. Lafforgue [Laf18])**.** *Let χ be a character of $F^\times\backslash\mathbb{A}^\times$ of finite order. There exists a canonical decomposition as $\mathrm{GL}_n(\mathbb{A})$-modules*

$$C_c^{\mathrm{cusp}}(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}), \overline{\mathbf{Q}}_\ell)_\chi = \bigoplus_\sigma \mathfrak{h}_\sigma$$

where the direct sum ranges over all global Langlands parameters. Moreover, the correspondence sending a subrepresentation to its parameter σ satisfies the compatibility properties (1)–(3).

The goal of this talk is to go the other way. This will be done in the following three steps

(1) Given $\sigma \in \Phi_n^{\mathrm{Irr}}$, construct a candidate π using the unramified correspondence.
(2) If $\pi' \in \Pi_{n'}^{\circ}$ where $n' < n$, use Lafforgue's map to attach $\sigma' \in \Phi_{n'}$ to it. Study the behaviour of $L(s, \sigma \otimes \sigma')$ using geometry, in particular getting a functional equation.
(3) The converse theorem roughly says that if we have functional equations for $L(s, \pi \times \pi')$ for all $\pi' \in \Pi_{n'}^{\circ}$ for all $n' < n$, then π is automorphic.

We will begin by recalling the various definitions of the L-function, then state a version of the converse theorem, and finally come back to complete the proof of the global Langlands correspondence.

1. L-functions and ϵ-factors: Galois side

1.1. **Review of geometric L-functions.** Most members of the seminar should be very familiar with this, so we will be very brief. Let X be the smooth projective curve whose function field is F.

Let $\sigma \in \Phi_n$. We view it as a Galois representation on an ℓ-adic vector space. Geometrically, it also defines a constructible ℓ-adic sheaf on X which is lisse on a dense open subset $U \subseteq X$. The vector space and the sheaf will also be denoted by σ. Let σ_v be the restriction of σ to the decomposition group $\mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$. This corresponds to pulling back the sheaf σ under $\operatorname{Spec} \mathcal{O}_v \hookrightarrow X$.

Definition 1.1. Let $\sigma \in \Phi_n$. For each place v, define the local L-factor to be

$$L(Z, \sigma_v) = \det(1 - Z^{\deg v} \mathrm{Frob}_v | \sigma_v^{I_v})^{-1}$$

where I_v is the inertial group at v. The global L-function of σ is

$$L(Z, \sigma) = \prod_v L(Z, \sigma_v)$$

This is viewed as a formal power series in Z with coefficient in $\overline{\mathbf{Q}}_\ell$.

Remark 1.2. Since I_v is a normal subgroup of $G_v = \mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$, the invariants $\sigma_v^{I_v}$ is also G_v-stable. Therefore, if σ_v is G_v-irreducible and non-trivial, then $V^{I_v} = 0$, so $L(Z, \sigma_v) = 1$. On the automorphic side, σ_v corresponds to a supercuspidal representation, and the same equation holds.

Theorem 1.3 (Grothendieck)**.** *Let σ be a Galois representation as above.*

(1) The global L-function is a rational function. More precisely,

$$L(Z, \sigma) = \prod_{i=0}^{2} \det(1 - Z\mathrm{Frob} | H_c^i(U_{/\bar{\mathbf{F}}_q}, \sigma))^{(-1)^{i+1}}$$

(2) It satisfies a functional equation

$$L(Z, \sigma) = \epsilon(Z, \sigma) L((qZ)^{-1}, \sigma^\vee)$$

where $\sigma^\vee$ is the dual representation, and the global ϵ-factor is defined by

$$\epsilon(Z, \sigma) = \prod_{i=0}^{2} \det(-Z\mathrm{Frob}^{-1} | H_c^i(U_{/\bar{\mathbf{F}}_q}, \sigma))^{(-1)^{i+1}}$$

Using these expressions, Deligne's purity theorem gives us very precise information on the zeros and poles of the global L-function, for example the Riemann hypothesis. They will be compared with analogous results coming from the automorphic side in our main induction argument.

1.2. **Local ϵ-factors.** It will be necessary in the comparison with the automorphic side to factor $\epsilon(Z, \sigma)$ into a product of local ϵ-factors. They are functions $\epsilon(Z, \sigma_v, \psi_v)$, which depend on a finite dimensional ℓ-adic representation σ_v of $\mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$ and a local additive character $\psi_v : F_v \to \overline{\mathbf{Q}}_\ell^\times$. They need to satisfy the following properties:

(1) $\epsilon(Z, -, \psi_v)$ is multiplicative in short exact sequences.
(2) If E_v/F_v is a finite extension, and σ_v is a virtual representation of $\mathrm{Gal}(F_v^{\mathrm{sep}}/E_v)$ of dimension 0, then $\epsilon(Z, \sigma_v, \psi_v \circ \mathrm{Tr}_{E_v/F_v}) = \epsilon(Z, \mathrm{Ind}_{E_v}^{F_v} \sigma_v, \psi_v)$.
(3) If σ_v is a character, then the ϵ-factor agrees with the automorphic ϵ-factor defined by Tate.

For the last item and more properties, see Section 3.4 of [Tat79]. Tate required a measure on F_v as an additional piece of data, but we follow Langlands and fix it to be the self-dual measure with respect to ψ_v. Given existence, these axioms uniquely determine ϵ using Brauer's theorem. The product formula for Artin representations (no geometric monodromy) follow easily.

Proving existence comes down to verifying certain identities among Gauss sums coming from decompositions of induction of characters. They were shown in a long incomplete paper of Langlands. In the function field case with ℓ-adic coefficients, Laumon [Lau87] gave a purely local construction of the ϵ-factors and proved the product formula. The work is based on the ℓ-adic Fourier transform and his principle of stationary phase.

Theorem 1.4 (Deligne, Laumon). *There exists a unique function $\epsilon(Z, \sigma_v, \psi_v)$ of the form cZ^f which depends only on the local information and satisfies conditions (1)–(3) above. Moreover, we have the product formula*

$$\epsilon(Z, \sigma) = \prod_v \epsilon(Z, \sigma_v, \psi_v)$$

Remark 1.5. The exponent f is equal to $a(\sigma_v) + n \cdot \mathrm{cond}(\psi_v)$, where $a(\sigma_v)$ is the Artin conductor of σ_v. It is already a non-trivial fact that $a(\sigma_v)$ is an integer. The coefficient c contains more subtle information.

Before Laumon's work, Deligne [Del73] gave a short proof of existence using global methods, but he could only prove the product formula if the representation has finite monodromy. The idea of his proof is to start with global ϵ-factor and somehow only keep information from one local place. A key lemma is the following stability property.

Lemma 1.6 (Deligne). *Let σ_v be an n-dimensional ℓ-adic representations of* $\mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$. *Suppose χ_v is sufficiently ramified at v, then*

$$\epsilon(Z, \chi_v\sigma_v, \psi_v) = \epsilon(Z, \chi_v, \psi_v)^{n-1}\epsilon(Z, \chi_v \det(\sigma_v), \psi_v), \quad L(Z, \chi_v\sigma_v) = 1$$

Proof. The assertion about the L-factor comes from Remark 1.2. The assertion about the ϵ-factor follows from the proof of Lemma 4.16 of [Del73] together with his definition of an "(α, ψ)-system of local constants". □

This process of deducing local information from global information by twisting away places with unknown behaviour will be used several times, in conjunction with the corresponding property for the automorphic ϵ-factor (Theorem 2.12).

2. L-functions and ϵ-factors: automorphic side

For this and the next section, we transfer coefficients to $\mathbf{C}$ both for psychological reasons and so that the treatment holds for number fields as well. It does not matter that our identification is not continuous, since all L-functions will be formal Laurent series in a formal variable Z. The only difference in the case of $\mathbf{C}$-coefficients is that we can write $Z = q^{-s}$ and treat s as the variable.

2.1. Introduction: Hecke's L-function.

For this historical motivation, we let f be a classical cusp form of level 1 and weight k. Hecke considered the integral

$$\Phi(f, s) = \int_0^\infty f(iy)y^s \frac{dy}{y}$$

It is a version of the Fourier transform known as the Mellin transform. Because f is rapidly decreasing at the cusps 0 and $i\infty$, this integral converges absolutely for all $s \in \mathbf{C}$. Suppose $f(z) = \sum_{n\geq 1} a_n q^n$ is its Fourier expansion, then

$$\Phi(f, s) = \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y} = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s} = (2\pi)^{-s}\Gamma(s)L(f, s)$$

Finally, we can perform a change of variable $y \mapsto y^{-1}$ and apply modularity $z \mapsto -\frac{1}{z}$ to get

$$\Phi(f, s) = \int_0^\infty f\Big(\frac{i}{y}\Big)y^{-s}\frac{dy}{y} = \int_0^\infty f(iy)(iy)^k y^{-s}\frac{dy}{y} = i^k\Phi(f, k - s)$$

This gives the functional equation for $L(f, s)$.

Under the dictionary between classical modular form and automorphic forms, for some normalization of the central character, Hecke's integral can be rewritten as

$$\Phi(\varphi_f, s) = \int_{\mathbf{Q}^\times \backslash \mathbb{A}_{\mathbf{Q}}^\times} \varphi_f\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) |a|^{s-\frac{1}{2}}\, da$$

The analogue of the Fourier expansion is the Whittaker expansion

$$\varphi_f(g) = \sum_{\gamma \in \mathbf{Q}^\times} W_f\left(\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right)$$

where W_f is a special function on $\mathrm{GL}_2(\mathbb{A})$. Plugging this into the integral gives

$$\Phi(\varphi_f, s) = \int_{\mathbf{Q}^\times \backslash \mathbb{A}_{\mathbf{Q}}^\times} \sum_{\gamma \in \mathbf{Q}^\times} W_f\left(\begin{smallmatrix} \gamma a & 0 \\ 0 & 1 \end{smallmatrix}\right) |a|^{s-\frac{1}{2}}\, da = \int_{\mathbb{A}_{\mathbf{Q}}^\times} W_f\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) |a|^{s-\frac{1}{2}}\, da \quad (\text{if } \mathrm{Re}(s) \gg 1)$$

This gives the factorization of the L-function into local factors, in exactly the same way the archimedean Fourier expansion separated out the archimedean L-factor $(2\pi)^{-s}\Gamma(s)$.

Finally, we note that if we had worked over a function field, then φ_f is compactly supported by Harder's theorem, so the integral becomes a finite sum. Therefore, $L(s, \pi)$ is a polynomial in q^{-s}.

2.2. **General principle.** For the purpose of converse theorem, we need to define the L-function of a pair of automorphic representations following works of Jacquet, Piatetski-Shapiro, and Shalika. On the Galois side, this is the operation of taking tensor products. The method will be outlined here, and the details can be found in Cogdell's lectures [Cog04]. This subsection will be used to fix notations and give a broad overview of what will happen.

Fix an additive character $\psi = \bigotimes_v' \psi_v : F\backslash\mathbb{A} \to \mathbf{C}^\times$. Let $\pi \in \Pi_n$ with factorization $\pi = \bigotimes_v' \pi_v$ where π_v is an irreducible admissible representation of $\mathrm{GL}_n(F_v)$. The global representation π is realized in the space of cusp forms, and given a vector $\varphi \in \pi$, we can define $\tilde{\varphi}(g) = \varphi(g^{-t}) \in \tilde{\pi}$. Finally, fix $n' < n$ and let $\pi' \in \Pi_{n'}$, with a similar set of notations. Note that at this point, we are not requiring them to be cuspidal. In fact, for applications to the converse theorem, we need to consider the case when one of them is not.

The theory of Eulerian integral representations starts with an adelic integral

$$\Phi(s, \varphi, \varphi'), \quad s \in \mathbf{C},\ \varphi \otimes \varphi' \in \pi \boxtimes \pi'$$

It is usually called a zeta integral. It is supposed to be chosen so that when φ and φ' are pure tensors, it factors into a product of local zeta integrals. We then use local representation theory to study them, in particular defining local L-factors from them. The global zeta integral is then related to the L-function we are interested in, and we can transfer analytic properties of the integral to get analytic properties of the L-function. There will be a global functional equation for Φ, which in our case will just be a change of variable. Combined with the local functional equations, we will get the functional equation for L-functions.

Remark 2.1. There is a parallel theory if $n' = n$, where in the global integral, we need the additional data of a Bruhat–Schwartz function which is used to form an Eisenstein series. This is closer in spirit to Tate's thesis and Godement–Jacquet's integral representation of the standard L-function for GL_n. All results in this section carry through. In fact, once we have sketched the theory for $n' < n$, we will drop that hypothesis.

In the next three sections, we will (1) define Whittaker models, which will give us the factorization of the global zeta integral (2) define the global zeta integral and state its properties, and (3) state the properties of the local zeta integral. The local part is where non-trivial inputs from representation theory shows up.

2.3. **Whittaker model.** Let N_n be the group of $n \times n$ upper triangular matrices with diagonal entries 1. The additive character ψ gives rise to a character $\psi : N_n(F)\backslash N_n(\mathbb{A}) \to \mathbf{C}^\times$

$$\psi(g) = \psi(g_{12} + g_{23} + \cdots + g_{n-1,n})$$

If φ is an automorphic form on $\mathrm{GL}_n(\mathbb{A})$, then its Whittaker function is the ψ-Fourier coefficient of φ, i.e.

$$W_\varphi(g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} \varphi(ng)\psi(n)^{-1} dn$$

If $n = 2$, then $N_2 \simeq \mathbf{G}_a$ is abelian, so we have a classical Fourier expansion, with the character space identified with $F^\times$. In general, we can inductively establish the following.

Proposition 2.2. *Let P be the subgroup $\{\left(\begin{smallmatrix}\gamma & v\\ 0 & 1\end{smallmatrix}\right)\}$, where $\gamma \in \mathrm{GL}_{n-1}$, then*

$$\varphi(g) = \sum_{p \in N_n(F)\backslash P(F)} W_\varphi(pg) = \sum_{\gamma \in N_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\varphi\left(\left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right)g\right)$$

By definition, the Whittaker function satisfies the condition

$$W_\varphi(ng) = \psi(n)W_\varphi(g), \quad n \in N_n(\mathbb{A}),\ g \in \mathrm{GL}_n(\mathbb{A})$$

Let π be an irreducible automorphic representation, then it follows from the Whittaker expansion that there is an injection

$$\pi \hookrightarrow \mathcal{W}(\psi) := \{W \in C^\infty(\mathrm{GL}_n(\mathbb{A})) \,|\, W(ng) = \psi(n)W(g)\} = \mathrm{Ind}_{N_n(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \psi$$

We denote its image by $\mathcal{W}(\psi, \pi)$ and call it the (global) Whittaker model of π.

Note that $\mathcal{W}(\psi)$ factors as a restricted tensor product of local inductions, so we have an injection

$$\pi_v \hookrightarrow \mathcal{W}_v(\psi_v) := \mathrm{Ind}_{N_n(F_v)}^{\mathrm{GL}_n(F_v)} \psi_v$$

for each place v. The first serious result we encounter is the fact that $\mathcal{W}_v(\psi_v)$ is multiplicity-free.

Theorem 2.3 (Local uniqueness)**.** *For any irreducible admissible representation π_v of $\mathrm{GL}_n(F_v)$,*

(1) $\dim \mathrm{Hom}_{\mathrm{GL}_n(F_v)}(\pi_v, \mathcal{W}_v(\psi_v)) \leq 1$. *We say π_v is* generic *if the dimension is non-zero. This notion is independent of the chosen ψ.*

(2) If π_v is generic, then so is $\widetilde{\pi}_v$, and moreover if W is a ψ_v-Whittaker function in π_v, then $\widetilde{W}(g) := W(wg^{-t})$ is a ψ_v^{-1}-Whittaker function for $\widetilde{\pi}_v$, where w is the matrix with 1s on the anti-diagonal and 0s elsewhere.

Proof. The proof in the non-archimedean case is originally due to Gelfand and Kazhdan [GK75] using the now standard method of showing that an anti-involution fixes certain invariant distributions. The involution in this case is $g \mapsto wg^tw$. Though we do not need it, the archimedean case was done in [Sha74]. □

Corollary 2.4 (Global uniqueness)**.** *If π is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$, then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})}(\pi, \mathcal{W}(\psi)) \leq 1$$

We say π is generic if the dimension is 1.

Corollary 2.5 (Multiplicity one)**.** *(1) Cusp forms on $\mathrm{GL}_n(\mathbb{A})$ are generic.*

(2) Representations appear with multiplicity one in the cuspidal spectrum.

Remark 2.6. We will need Piatetski-Shapiro's strong multiplicity one result, which says if π_1 and π_2 are cuspidal automorphic representations which agree locally almost everywhere, then they are equal in the cuspidal spectrum. This shows that the Galois to automorphic correspondence is uniquely specified by the unramified correspondence. In Theorem 2.14, we will prove this using L-functions, following Jacuqet–Shalika.

Not all admissible representations are generic. For example, finite dimensional representations are never generic. In the case of GL_2, they are the only non-examples, though for $n > 2$, there are many more. The following theorem [Kud94, Theorem 2.3.1] gives a satisfactory answer in general.

Theorem 2.7 (Rodier [Rod73], Zelevinsky [Zel80])**.** *If $\sigma_1, \cdots, \sigma_r$ are supercuspidal representations of $\mathrm{GL}_{n_1}(F_v)$, $\cdots, \mathrm{GL}_{n_r}(F_v)$ respectively, then for $n = n_1 + \cdots + n_r$ and $P \subseteq \mathrm{GL}_n$ the parabolic subgroup associated to this partition, the induced representation*

$$\mathrm{Ind}_{P(F_v)}^{\mathrm{GL}_n(F_v)}(\sigma_1 \boxtimes \cdots \boxtimes \sigma_r)$$

has a unique generic constituent, which can be specified using the Berstein–Zelevinsky classification.

2.4. **Global picture I.** The global integral in our case has the form

$$\Phi(s,\varphi,\varphi')=\int_{\mathrm{GL}_{n'}(F)\backslash\mathrm{GL}_{n'}(\mathbb{A})}\mathbb{P}_{n',\psi}\varphi\left(\begin{smallmatrix}h&0\\0&1\end{smallmatrix}\right)\varphi'(h)\,|\det h|^{s-\frac{1}{2}}\,dh$$

where $\mathbb{P}_{n',\psi}$ is a certain projection of φ onto $\mathrm{GL}_{n'}$. If $h\in\mathrm{GL}_{n'}(\mathbb{A})$, then

$$\mathbb{P}_{n',\psi}\varphi\left(\begin{smallmatrix}h&0\\0&1\end{smallmatrix}\right):=|\det h|^{-\frac{n-n'+1}{2}}\int_{Y(F)\backslash Y(\mathbb{A})}\varphi\left(y\left(\begin{smallmatrix}h&0\\0&1\end{smallmatrix}\right)\right)\psi(y)^{-1}dy=\sum_{\gamma\in N_{n'}(F)\backslash\mathrm{GL}_{n'}(F)}W_\varphi\left(\begin{smallmatrix}\gamma h&0\\0&1\end{smallmatrix}\right)$$

In the integral, Y is the unipotent radical of the parabolic subgroup of GL_n associated with the partition $(n'+1,1,\cdots,1)$. The detail is not important, but we note that

- The domain of integration is compact.
- If φ is cuspidal, then we only need to assume φ' is of moderate growth to get absolute convergence.
- If $n=n'+1$, then the projection does nothing. In particular, the formula simplifies to the one in the introduction if $n=2$.

We will summarize some properties of the integral, with a view towards properties of L-functions. The ones we list are mainly intricate manipulations, and the details can be found in the two papers [JS81b, JS81a] where the bulk of the theory was first developed.

Proposition 2.8. *Suppose π is cuspidal and π' is generic, then*

- *$\Phi(s,\varphi,\varphi')$ converges absolutely for all $s\in\mathbf{C}$ and defines polynomial in q^{-s} (in the number field case, replace polynomial by bounded in vertical strips).*
- *After a change of variable $h\mapsto h^{-t}$, we find a closely related integral $\widetilde{\Phi}$ such that*

$$\Phi(s,\varphi,\varphi')=\widetilde{\Phi}(1-s,\tilde{\varphi},\tilde{\varphi}')$$

- *Suppose φ has ψ-Whittaker function $W=\prod_v W_v$, and φ' has ψ^{-1}-Whittaker function $W'=\prod_v W'_v$, then there are factorizations*

$$\Phi(s,\varphi,\varphi')=\prod_v\Phi(s,W_v,W'_v),\quad \widetilde{\Phi}(s,\tilde{\varphi},\tilde{\varphi}')=\prod_v\widetilde{\Phi}(s,\iota\widetilde{W}_v,\widetilde{W}'_v)$$

where ι is right translation by $\left(\begin{smallmatrix}I_{n'}&\\&w_{n-n'}\end{smallmatrix}\right)$, and the terms on the right hand side are explicit integrals over local groups. The infinite products and the local integrals converge absolutely for $\mathrm{Re}(s)\gg 0$.

We now give the formulae for the local zeta integrals in the above theorem, only to illustrate that the factorization is entirely formal. For $1\le j\le n-n'-1$, define

$$\Phi_j(s,W_v,W'_v)=\int_{N_{n'}(F_v)\backslash\mathrm{GL}_{n'}(F_v)}\int_{M_{j,n'}(F_v)}W_v\begin{pmatrix}h&&\\x&I_j&\\&&I_{n-n'-j}\end{pmatrix}W'_v(h)\,|\det h|^{s-\frac{n-m}{2}}\,dxdh$$

then $\Phi=\Phi_0$ and $\widetilde{\Phi}=\Phi_{n-n'-1}$. The calculation is a very unpleasant version of the introductory case, which corresponds to $\mathrm{GL}_2\times\mathrm{GL}_1$.

The task at hand is now reduced to studying the local integrals. The main tool there is the explicit description of smooth representations of $\mathrm{GL}_n(F_v)$ in various models.

2.5. **Local picture.** To the local zeta integrals, one attaches in a systematic way local L-factors, γ-factors, and ϵ-factors. For applications to the converse theorem, we allow reducible π_v satisfying the property that

$$\dim\mathrm{Hom}_{\mathrm{GL}_n(F_v)}(\pi_v,\mathcal{W}(\psi_v))=1$$

which we call a representation of Whittaker type. Another statement of Theorem 2.7 is that a parabolic induction $\mathrm{Ind}_P^G\sigma$ is of Whittaker type if and only if σ is as well. The main reason is that we want to study how various things behave with respect to the Berstein–Zelevinsky classification, which requires as an intermediary studying reducible induced representations. From now on, fix two representations π_v and π'_v of Whittaker type on $\mathrm{GL}_n(F_v)$ and $\mathrm{GL}_{n'}(F_v)$ respectively.

We now summarize these constructions, which can be found in the two papers [JS81b, JS81a] by Jacquet and Shalika. There are two key ingredients in the proof of the following theorem: (1) a uniform control of functions in the Whittaker model to establish convergence and bounded denominator, and (2) a multiplicity one result giving the local functional equation.

Theorem 2.9.

- *For* $\mathrm{Re}(s) \gg 0$*, the local integrals* Φ_j *converge absolutely to a rational function in* q_v^{-s}*. If both* π_v *and* π'_v *are unitary, then* $\mathrm{Re}(s) \geq 1$ *is enough.*
- *The collections of functions*

$$\mathcal{I}_j(\pi_v \times \pi'_v) = \{\Phi_j(s, W_v, W'_v) \mid W_v \in \mathcal{W}(\psi_v, \pi_v), W'_v \in \mathcal{W}(\psi_v^{-1}, \pi'_v)\} \subseteq \mathbf{C}(q_v^{-s})$$

  is a $\mathbf{C}[q_v^{\pm s}]$*-ideal independent of* j*.*
- *There exists a unique generator of this ideal of the form* $P(q_v^{-s})^{-1}$*, where* $P(Z) \in \mathbf{C}[Z]$*,* $P(0) = 1$*. This is the* local L-factor $L(s, \pi_v \times \pi'_v)$*.*
- *The two maps* $(W_v, W'_v) \mapsto \Phi_0(s, W_v, W'_v)$ *and* $(W_v, W'_v) \mapsto \Phi_{n-n'-1}(1-s, \iota\widetilde{W}_v, \widetilde{W}'_v)$ *both lie in*

$$\mathrm{Hom}_{\mathrm{GL}_{n'}(F_v)\times Y(F_v)}\Big(\mathcal{W}(\psi_v, \pi_v) \times \mathcal{W}(\psi_v^{-1}, \pi'_v), \mathbf{C}(q_v^{-s})\Big(\big(-s + \frac{n-n'}{2}\big) \boxtimes \psi_v\Big)\Big)$$

  where Y *is the unipotent subgroup introduced in the global section. This space is* 1*-dimensional.*
- *There exists a meromorphic function* γ *such that the* local functional equation *holds*

$$\widetilde{\Phi}(1-s, \iota\widetilde{W}_v, \widetilde{W}'_v) = \omega_{\pi'_v}(-1)^{n-1}\gamma(s, \pi_v \times \pi'_v, \psi)\Phi(s, W_v, W'_v)$$

- *The* local ϵ-factor *is defined by*

$$\epsilon(s, \pi_v \times \pi'_v, \psi_v) = \gamma(s, \pi_v \times \pi'_v, \psi_v)\frac{L(s, \pi_v \times \pi'_v)}{L(1-s, \tilde{\pi}_v \times \tilde{\pi}'_v)}$$

  It follows from the functional equation that the ϵ*-factor can be written as the form*

$$\epsilon(s, \pi_v \times \pi'_v, \psi_v) = \epsilon_v q_v^{-f(s-\frac{1}{2})}$$

  where $|\epsilon_v| = 1$*, and* f *is an integer.*
- *If* $n' = 1$ *and* π'_v *is the trivial character, then the above definitions for* $\pi_v \times \pi'_v$ *agrees with the Godement–Jacquet theory for* π_v*.*

From now on, we suppose the parallel theory for $n' = n$ **has been developed and drop the hypothesis that** $n' < n$. Suppose π_v is any irreducible admissible representation of $\mathrm{GL}_n(F_v)$, then by the Langlands classification, it is the irreducible quotient of a representation ξ_v induced from tempered representations, and ξ_v is uniquely determined. Similarly, suppose π'_v is a quotient of ξ'_v. Both ξ_v and ξ'_v are of Whittaker type (by Theorem 2.7). We will *define* the local factors of $\pi_v \times \pi'_v$ to be the corresponding factors of $\xi_v \times \xi'_v$, defined using the previous theorem. If π_v is already generic, then $\pi_v = \xi_v$, so the notation does not conflict.

Under the local Langlands correspondence, the local factors should match. More precisely, if $\pi_v \leftrightarrow r_v$ and $\pi'_v \leftrightarrow r'_v$, where r_v, r'_v are ℓ-adic representations of $\mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$, then we expect

$$L(s, \pi_v \times \pi'_v) = L(s, r_v \otimes r'_v), \quad \epsilon(s, \pi_v \times \pi'_v, \psi_v) = \epsilon(s, r_v \otimes r'_v, \psi_v)$$

Proving this compatibility requires verifying the above definition is compatible with induction, which is the main theorem of [JPSS83]. It is more subtle than just saying $L(s, \sigma \times (\pi_1 \boxplus \pi_2)) = L(s, \sigma \times \pi_1)L(s, \sigma \times \pi_2)$ since on the Galois side, we need to take the monodromy coinvariants to get a genuine Galois representation. We will state it in a vague way and refer to the recipe in Section 3.2 of [Kud94] for details.

Theorem 2.10. *In the local Langlands correspondence, suppose the local* L*-factors and* ϵ*-factors for pairs match for supercuspidal representations, then they match for all representations.*

Corollary 2.11.

- *If* π_v *is parabolically induced from* $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ *and* π'_v *is parabolically induced from* $\pi'_1 \boxtimes \cdots \boxtimes \pi'_s$*, where each individual factor is generic, then*

$$\gamma(s, \pi \times \pi', \psi_v) = \prod_{i,j} \gamma(s, \pi_i \times \pi'_j, \psi_v)$$

- *If* π_v *and* π'_v *are unramified with Satake parameters* $z_1, \cdots, z_n$ *and* $z'_1, \cdots, z'_{n'}$ *respectively, then*

$$L(s, \pi_v \times \pi'_v) = \prod_{i,i'} \frac{1}{1 - z_i z'_{i'} q_v^{-s}}, \quad \epsilon(s, \pi_v \times \pi'_v, \psi_v) = \epsilon(s, \omega_\pi^{n'}\omega_{\pi'}^{n}, \psi_v)$$

  Moreover, if ψ_v *is also unramified, then the local* ϵ*-factor is* 1*.*

– *If* π_v *and* π'_v *are supercuspidal, then*

$$L(s,\pi_v\times\pi'_v)=\begin{cases}L(s,\chi_v) & \text{if } \tilde{\pi}_v\chi_v=\pi'_v\\ 1 & \text{otherwise}\end{cases}$$

The ϵ-factor is hard to compute in general, but there is a stabilization phenomenon.

Theorem 2.12 (Jacquet–Shalika [JS85])**.** *Let* π_1,π_2 *be two irreducible admissible representations of* $\mathrm{GL}_n(F_v)$ *with the same central character, and let* π'_v *be an irreducible admissible representation of* $\mathrm{GL}_{n'}(F_v)$*. If* η_v *is a highly ramified character of* $F_v^\times$*, depending on* π_1,π_2,π'_v*, then*

$$\gamma(s,\pi_1\eta_v\times\pi'_v,\psi_v)=\gamma(s,\pi_2\eta_v\times\pi'_v,\psi_v),\ L(s,\pi_1\eta_v\times\pi'_v)=L(s,\pi_1\eta_v\times\pi'_v)=1$$

The stable form can be computed in terms of abelian ϵ*-factors.*

This is the analogue of Lemma 1.6. In global applications, this theorem can be used to hide away complications at a finite set of places, usually the ramified places. We will see this principle applied when we prove the global Langlands correspondence, where we do not want to use the local Langlands correspondence for ramified places.

2.6. **Global picture II.** The local calculations can now be assembled to produce global theorems.

Theorem 2.13. *Let* π*,* π' *be (unitary) cuspidal automorphic representations of* $\mathrm{GL}_n(\mathbb{A})$ *and* $\mathrm{GL}_{n'}(\mathbb{A})$ *respectively* (n,n' *arbitrary). Define their* L*-function by*

$$L(s,\pi\times\pi'):=\prod_v L(s,\pi_v\times\pi'_v)$$

Then we have the following properties

– *The infinite product converges absolutely if* $\mathrm{Re}(s)>1$*. It follows that* $L(s,\pi\times\pi')$ *has no zero or pole in the region* $\mathrm{Re}(s)>1$*.*
– $L(s,\pi\times\pi')$ *is a rational function in* q^{-s} *(only in the function field case), and hence admits a meromorphic continuation in* s*.*
– *We have a functional equation*

$$L(s,\pi\times\pi')=\epsilon(s,\pi\times\pi')L(1-s,\tilde{\pi}\times\tilde{\pi}')$$

where $\epsilon(s,\pi\times\pi')=\prod_v\epsilon(s,\pi_v\times\pi'_v,\psi_v)$*. It is independent of* ψ*.*
– *If* $n'\neq n$*, then* $L(s,\pi\times\pi')$ *is entire. If* $n'=n$*, then* $L(s,\pi\times\tilde{\pi}')$ *has a pole at* $s=1$ *if and only if* $\pi'=\pi$*, in which case the pole is simple and the residue is the adjoint* L*-value of* π*.*

Given what we have stated, the hard part of this theorem is the absolute convergence in the region $\mathrm{Re}(s)>1$ (as opposed to an unspecified half-plane), which is done in [JS81a]. Beyond that, everything follows from the following relation between the global zeta integral and the global L-function

$$\Phi(s,\varphi,\varphi')=L(s,\pi\times\pi')\prod_{v\in S}\frac{\Phi(s,W_v,W'_v)}{L(s,\pi_v\times\pi'_v)}$$

where S is a finite set of places such that away from S, all data is unramified. The finite product is a polynomial in $q^{\pm s}$, so the analytic properties of $L(s,\pi\times\pi')$ is entirely controlled by the global zeta integral. The pole in the last item comes from the Eisenstein series in the $n'=n$ case, which we did not cover. This can also be explained by the Langlands–Shahidi method.

As usual, if S is a finite set of places, we will let $L^S(s,-):=\prod_{v\notin S}L(s,-_v)$. The difference between $L^S(s,-)$ and $L(s,-)$ is a finite product of local factors. Moreover, if S contains all bad places, then $L^S(s,-)$ has the following functional equation

$$L^S(s,\pi\times\pi')=\prod_{v\in S}\gamma(s,\pi_v\times\pi'_v,\psi_v)\times L^S(1-s,\tilde{\pi}\times\tilde{\pi}')$$

which is sometimes more convenient in applications.

Theorem 2.14 (Strong multiplicity one)**.** *Suppose* π_1,π_2 *are two irreducible subrepresentations of the space of cusp forms such that* $\pi_{1,v}\simeq\pi_{2,v}$ *for almost all* v*, then* $\pi_1=\pi_2$*.*

Proof. If S is sufficiently large, then $L^S(s,\pi_1\times\tilde{\pi}_2)=L^S(s,\pi_1\times\tilde{\pi}_1)$ by hypothesis. By Theorem 2.9, the local factors converge absolutely in the region $\mathrm{Re}(s)\geq 1$, so they do not contribute poles in that region. By the main theorem, the $L(s,\pi_1\times\tilde{\pi}_1)$ has a pole at $s=1$, so $L(s,\pi_1\times\tilde{\pi}_2)$ also has a pole at $s=1$, which implies $\pi_1\simeq\pi_2$. The claim now follows from the multiplicity one theorem we have already proven. □

3. Converse theorems

The converse theorem in the form we are stating was first proven by Piatetski-Shapiro in the function field case due to analytic difficulties at the archimedean places. The number field case was later completed by Cogdell–Piatetski-Shapiro [CPS94], who also further weakened the hypothesis. The proof is quite involved. To motivate it, we start with Hecke's converse theorem, which motivated the modern treatment. After that, we will highlight some aspects of the proof. The details can be found in the references above.

3.1. Introduction: Hecke's converse theorem.

Theorem 3.1 (Hecke)**.** *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers and $k\geq 2$ be an integer. Define*

$$\Lambda(s)=(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

Suppose we have the following properties

1. *There exists a $\gamma>0$ such that $a_n=O(n^\gamma)$.*
2. *The function $\Lambda(s)$ has an analytic continuation with the functional equation*
$$\Lambda(s)=i^k\Lambda(k-s)$$
3. *The function $\Lambda(s)$ is bounded in vertical strips of finite width.*

then $\sum_{n\geq 1}a_nq^n$ is a modular form of level 1 and weight k.

Proof. Let $f(z)=\sum_{n\geq 1}a_nq^n$, which already satisfies $f(z+1)=f(z)$. It remains to verify $f(-\frac{1}{z})=z^kf(z)$. Let $g(z)=z^kf(-\frac{1}{z})$. The same manipulation as before shows that

$$\int_0^\infty f(iy)y^s\frac{dy}{y}=\Lambda(s),\qquad \int_0^\infty g(iy)y^s\frac{dy}{y}=i^k\Lambda(k-s)=\Lambda(s)$$

The inverse Mellin transform formally gives

$$f(iy)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Lambda(s)y^{-s}ds=g(iy)$$

The subtlety here is that the first integral requires $\alpha\gg 0$, and the second integral requires $\alpha\ll 0$, since we a priori only know that f is rapidly decreasing at the cusp $i\infty$. Therefore, we need to shift the contour of integration from the line $\mathrm{Re}(s)=\alpha\gg 0$ to the line $\mathrm{Re}(s)=\alpha'\ll 0$. This is where the third hypothesis on boundedness in vertical strip comes into play. The Phragmen–Lindelöf principle is also required to get decay in the critical strip from decay in the region of absolute convergence. □

3.2. Statement and outline of proof.

Another way of phrasing Hecke's proof is to say that by construction, f is invariant under $\begin{pmatrix}1&*\\&1\end{pmatrix}$, and g is invariant under $\begin{pmatrix}1&\\ *&1\end{pmatrix}$, so once we have $f=g$, we have something invariant under the full modular group. To show they are equal, we paired them with the characters y^s and showed the results are equal using the functional equation. In an ideal world, this is enough because the characters span a dense subspace of certain function spaces.

To get higher level modular forms, Weil's converse theorem needed twisting by characters as well. In general, the unspecified function space will be the space of automorphic forms on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})$, Using the theory of Eisenstein series, the representations are essentially inductions of cuspidal representations on $\mathrm{GL}_{n'}(\mathbb{A})$ for $n'<n$, so we should need to understand L-functions of pairs.

Another difficulty with higher level modular forms is that the functional equation relates f with its Atkin–Lehner involution, so in Weil's converse theorem, we are actually given two sequences of Fourier coefficients. In the adelic treatment, they are packaged together in the local Whittaker model.

Theorem 3.2 (Piatestki-Shapiro)**.** *Let $\pi=\bigotimes_v'\pi_v$ be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ such that each π_v is generic. Let S be a finite set of places. Suppose the following two conditions hold*

(1) *Its central character* χ *is automorphic (i.e. invariant by* $F^\times$*).*
(2) *For each* $n' < n$ *and* $\pi' \in \Pi^\circ_{n'}$ *unramified at the places in* S*, the* L*-functions* $L(s, \pi \times \pi')$ *and* $L(s, \tilde{\pi} \times \tilde{\pi}')$ *are polynomials in* q^{-s} *satisfying a functional equation*

$$L(s, \pi \times \pi') = \epsilon(s, \pi \times \pi') L(1 - s, \tilde{\pi} \times \tilde{\pi}')$$

Then up to changing its local component at the places in S*,* π *is automorphic.*

Remark 3.3.

(1) In the number field case, the hypothesis that the L-functions are polynomials are replaced by bounded in vertical strips of finite widths.
(2) There are many further improvements such as decreasing the number of π' required or allowing poles. Cogdell has a good survey of those results [Cog14].

Outline of proof. We first assume $S = \emptyset$. By assumption, we can view π using its Whittaker model $\mathcal{W}(\psi, \pi)$. Using the Fourier expansion, define an intertwining operator

$$U : \mathcal{W}(\psi, \pi) \hookrightarrow C_c^\infty(\mathrm{GL}_n(\mathbb{A}), \mathbf{C})_\chi, \quad U_W(g) = \sum_{p \in N_n(F)\backslash P(F)} W(pg)$$

By construction, this is $P(F)$-invariant. We can construct another intertwining operator V

$$V_W(g) = \sum_{p \in \alpha^{-1} N(F) \alpha \backslash Q(F)} W(\alpha q g)$$

where $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ I_{n-1} & 0 \end{smallmatrix}\right)$ and Q is the subgroup $\{\left(\begin{smallmatrix} \gamma & 0 \\ v & 1 \end{smallmatrix}\right)\}$. The functions V_W are $Q(F)$-invariant. The groups $P(F)$ and $Q(F)$ generate $\mathrm{GL}_n(F)$, so it is enough to show that $U_W = V_W$. Equivalently, since $\mathrm{GL}_n(\mathbb{A})$ acts by right multiplication, we just need to prove that $U_W(1) = V_W(1)$ for all W.

First, we need to think about convergence. There is an identity relating V_W to $U_{\widetilde{W}}$, so we only need to look at U_W. This comes down to estimating the growth of the local and global Whittaker functions along the diagonal torus. The local estimates were also used earlier in showing the convergence of the local zeta integral. Viewing U_W and V_W as functions on $\mathrm{GL}_{n-1}(\mathbb{A})$ via the embedding $h \mapsto \left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the result is that both U_W and V_W on $\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A})$ are rapidly decreasing *modulo centre*. Moreover, $U_W\left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is rapidly decreasing as $|\det h| \to \infty$, so if $\varphi' \in C^\infty(\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}), \mathbf{C})_{\chi^{-1}}$ is of moderate growth, then the global zeta integral

$$\Phi(s, U_W, \varphi') = \int_{\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A})} U_W\left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right) \varphi'(h) \, |\det h|^{s - \frac{1}{2}} \, dh$$

converges for $\mathrm{Re}(s) \gg 0$. The corresponding integral for V_W converges for $\mathrm{Re}(s) \ll 0$, essentially since V_W is looking at the opposite cusp for $\mathrm{GL}_n(\mathbb{A})$ (in Hecke's converse theorem, f is rapidly decreasing at ∞, so g is rapidly decreasing at 0, but we don't know a priori that f is rapidly decreasing at 0).

The test functions we will use are Eisenstein series, which we briefly review. Let $n - 1 = n_1 + \cdots + n_r$ be a partition of $n - 1$. This corresponds to a standard Levi subgroup $M \simeq \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. Let φ be a cusp form on M, and let $s = (s_1, \cdots, s_r) \in \mathbf{C}^r$ correspond to a character on $M(\mathbb{A})$. To these data, we attach an Eisenstein series $E(s, \varphi)$. It satisfies the following properties

– If $\mathrm{Re}(s_i) - \mathrm{Re}(s_{i+1}) \gg 0$ for all i, then the series defining $E(s, \sigma)$ is absolutely convergent. The result is a function of moderate growth on $\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A})$.
– Let σ be the cuspidal automorphic representation of $M(\mathbb{A})$ generated by φ, then for almost all s in the region of absolute convergence, $E(s, -)$ is an intertwining operator

$$\mathrm{Ind}_{M(\mathbb{A})}^{\mathrm{GL}_{n-1}(\mathbb{A})} \sigma(s) \hookrightarrow C^\infty(\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}), \mathbf{C})$$

and the induced representation is irreducible.

It is a consequence of the spectral decomposition of $L^2(\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}))_\chi$ that if a rapidly decreasing function f pairs trivially with each $E(s, \varphi)$ for almost all s, then $f = 0$. Therefore, from now on, we will take φ' to be an Eisenstein series such that the automoprhic representation π' generated by φ' is an irreducible induced representation.

The unfolding procedure before still works and gives factorizations

$$\Phi(s, U_W, \varphi') = \prod_v \Phi(s, W_v, W'_v), \quad \Phi(s, V_W, \varphi') = \prod_v \widetilde{\Phi}(1-s, \iota\widetilde{W}_v, \widetilde{W}'_v)$$

where $W' = \prod_v W'_v$ are the Whittaker functions of φ' defined using the same integral expression, and the local zeta integrals are the same ones from before. At a place where everything is unramified, the local zeta integrals are local L-factors, so we have an equation of the form

$$\Phi(s, U_W, \varphi') = L(s, \pi \times \pi') \cdot \prod_{v \in S} \frac{\Phi(s, W_v, W'_v)}{L(s, \pi_v \times \pi'_v)}$$

Similarly, we also have

$$\Phi(s, V_W, \varphi') = L(s, \tilde{\pi} \times \tilde{\pi}') \cdot \prod_{v \in S} \frac{\widetilde{\Phi}(1-s, \iota\widetilde{W}_v, \widetilde{W}'_v)}{L(s, \tilde{\pi}_v \times \tilde{\pi}'_v)}$$

We are now ready to use our hypothesis. By the choice of φ', π' is an induced representation. Compatibility under induction shows that $L(s, \pi \times \pi')$ factors into a product $\prod_{i=1}^r L(s, \pi \times \tau_r)$, where τ_r is a cusp form on some $\mathrm{GL}_{n'}(\mathbb{A})$ for $n' < n$. For each of them, we have a global functional equation by assumption. Combined with the local functional equation, we get

$$\Phi(s, U_W, \varphi') = \Phi(s, V_W, \varphi')$$

By our discussion on Eisenstein series, this should imply $U_W(1) = V_W(1)$. This deduction is not strictly correct since they have different regions of convergence, so the above equality only holds in the sense of analytic continuation. We need to first perform an inverse Mellin transformation to reduce the situation to automorphic forms on $\mathrm{SL}_{n-1}(\mathbb{A})$. Exactly as in Hecke's converse theorem, this requires certain analytic properties. In the function field case, both sides are formal Laurent series in q^{-s} or q^s, so our hypothesis that the global L-functions are polynomials replaces the inverse Mellin transform.

Finally, in the case $S \neq \emptyset$, we restrict the choice of $W \in \mathcal{W}(\psi, \pi)$ to those which are unramified at S. If π' is unramified at all places in S, then we can follow through the proof as above. Otherwise, π' is ramified at a place in S, but the test vector is unramified, so the local zeta integral is 0, so the necessary equality follows trivially. This gives $U_W = V_W$ on a certain congruence subgroup of $\mathrm{GL}_n(\mathbb{A})$, and therefore automorphy on that group. Weak approximation shows that the automorphic forms can be extended, but this comes at the cost of possibly replacing the components of π at places in S. □

4. The global Langlands correspondence

We are now ready to prove the main theorems 0.1 (GLC), 0.2 (GRC), and 0.3 (Purity) from Theorem 0.5 (Automorphic to Galois). This is an induction on n, with the base case $n = 1$ being class field theory. The induction statement we use is the following.

> There exists a bijection between $\bigsqcup_{n' \leq n} \Phi^{\mathrm{Irr}}_{n'}$ and $\bigsqcup_{n' \leq n} \Pi^{\circ}_{n'}$ satisfying (1)–(3) of Theorem 0.1 plus condition (4'): for any π, π' corresponding to σ, σ', we have
>
> (P_n) $$L(s, \pi_v \times \pi'_v) = L(s, \sigma_v \otimes \sigma'_v), \quad \epsilon(s, \pi_v \times \pi'_v, \psi_v) = \epsilon(s, \sigma_v \otimes \sigma'_v, \psi_v)$$
>
> Moreover, we assume the generalized Ramanujan conjecture (GRC) holds up to rank n, i.e. every $\pi \in \bigsqcup_{n' \leq n} \Pi^{\circ}_{n'}$ is tempered.

Notably, we are *not* deducing the GRC analytically from symmetric power functoriality. It will still come from Deligne's purity theorem. In the proof, the GRC is required to be able to identify local L-factors at bad places, which is in turn needed in the converse theorem to go to the next rank.

The purity conjecture (Theorem 0.3) will be deduced from the GRC and the global Langlands correspondence. However, a key input to the induction is that it holds for all Langlands parameter appearing in the automorphic to Galois correspondence. This is the next lemma, which is the only place in this talk we need to look at the actual constructions of V. Lafforgue beyond Theorem 0.5.

Lemma 4.1. *The global Langlands parameter appearing in the decomposition are irreducible of weight 0.*

Proof. Suppose a cuspidal automorphic representation π has a reducible parameter σ. By induction hypothesis, to each of its irreducible component τ we can attach a cuspidal automorphic representation π_τ on $\mathrm{GL}_{n_\tau}(\mathbb{A})$, so the global L-function factors

$$L(s, \pi \times \tilde{\pi}) = \prod_\tau L(s, \pi_\tau \times \tilde{\pi})$$

But the left hand side has a pole at $s = 1$ and the right hand side is entire. This is a contradiction. We remark that this also follows from the harder result proven in [JS81a] showing the existence of cuspidal support.

The purity part is now Lemma 16.2 of [Laf18]. Essentially by construction, there is an intertwining operator from certain cohomology groups to $\sigma \boxtimes \sigma^\vee$, so it is pure. The weight is then necessarily 0 since $\det \sigma$ has finite order by hypothesis. □

Now suppose $\sigma \in \Phi_n^{\mathrm{Irr}}$ is unramified away from the finite set of places S. For each $v \notin S$, the unramified local Langlands correspondence attaches an unramified representation π_v of $\mathrm{GL}_n(F_v)$ to $\sigma|_{\mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)}$. For the finitely many places $v \in S$, choose π_v to be an arbitrary generic irreducible admissible representation with matching central character. Let $\pi = \bigotimes_v' \pi_v$. Let χ be an adelic character which is highly ramified (depending on σ and π) at all places in S. This will be used to twist away informations of π at S.

Let $n' < n$ and π' be a cuspidal automorphic representation of $\mathrm{GL}_{n'}(\mathbb{A})$ unramified at the places in S. Let $\sigma' \in \Phi_{n'}^{\mathrm{Irr}}$ be its global Langlands parameter. If $v \notin S$, then π_v is unramified, so it is a constituent of an induction of characters. By compatibility with induction and the induction hypothesis,

$$L(s, \chi_v \sigma_v \otimes \sigma_v') = L(s, \chi_v \pi_v \otimes \pi_v'), \quad \epsilon(s, \chi_v \sigma_v \otimes \sigma_v', \psi_v) = \epsilon(s, \chi_v \pi_v \times \pi_v', \psi_v)$$

If $v \in S$, then π_v' and σ_v' are unramified by hypothesis. Using compatibility with induction on either side, we need to study the local L- and ϵ-factors of unramified character twists of $\chi\pi$. It follows from the stability properties (Theorem 2.12 and Theorem 1.6) that we still have the above equalities. Therefore,

$$L(s, \chi\sigma \otimes \sigma') = L(s, \chi\pi \times \pi'), \quad \epsilon(s, \chi\pi \times \pi') = \prod_v \epsilon(s, \chi_v \sigma_v \otimes \sigma_v', \psi_v) = \epsilon(s, \chi\sigma \otimes \sigma')$$

where we have also applied the Laumon product formula. The cohomological interpretation of L-functions gives a functional equation on the Galois side.

$$L(s, \chi\sigma \otimes \sigma') = \epsilon(s, \chi\sigma \otimes \sigma') L(1 - s, \chi^{-1}\sigma^\vee \otimes (\sigma')^\vee)$$

Therefore, the expected functional equation holds for $\pi \times \pi'$. Moreover, σ and σ' are irreducible of different ranks, so the L-functions on both sides are polynomials in q^{-s}. This verifies all hypothesis of the converse theorem, so up to changing the components at the places in S, π is automorphic. By construction, properties (1)–(3) of Theorem 0.1 are satisfied.

To conclude the induction step, we need to show (a) π is cuspidal, (b) π is tempered, and (c) matching L- and ϵ-factors for pairs at bad places. The proof of the first property is like the converse of Lemma 4.1. Indeed, suppose π is not cuspidal, then it is a constituent of a parabolic induction of a cusp form $\pi_1 \boxtimes \cdots \boxtimes \pi_r$. At almost all places, the Satake parameters of π agrees with the union of the Satake parameters of $\pi_1, \cdots, \pi_r$. Suppose π_i matches with σ_i, then the Satake parameters of π agree with the Frobenius eigenvalues of $\bigotimes_{i=1}^r \sigma_i$ almost everywhere. But the irreducible representation σ also satisfies this property. This is a contradiction by the Chebotarev density theorem.

We will prove properties (b) and (c) together. Both are related to the poles of the local L-factors: for property (b), this follows from Tadić's work on the unitary dual of $\mathrm{GL}_n(F_v)$; for property (c), since the local L-factors on either side are of the form $P(q^{-s})^{-1}$, where P is a polynomial, the matching of local L-factors is equivalent to showing that they have the same poles.

Lemma 4.2. *If $\pi \leftrightarrow \sigma$ and $\pi' \leftrightarrow \sigma'$ in the sense of matching Frobenius eigenvalues with Satake parameters at almost all places, then for all place v,*

$$\gamma(s, \pi_v \times \pi_v', \psi_v) = \gamma(s, \sigma_v \otimes \sigma_v', \psi_v)$$

where the γ-factor on the Galois side is defined using its ϵ-factor.

Proof. Let χ be an arbitrary character of $F^\times\backslash\mathbb{A}^\times$. For S sufficiently large, the global functional equations on both sides can be written as

$$L^S(s,\chi\pi\times\pi') = L^S(s,\chi^{-1}\tilde{\pi}\times\tilde{\pi}')\prod_{v\in S}\gamma(s,\chi_v\pi_v\times\pi'_v,\psi_v)$$
$$L^S(s,\chi\sigma\otimes\sigma') = L^S(s,\chi^{-1}\sigma^\vee\otimes(\sigma')^\vee)\prod_{v\in S}\gamma(s,\chi_v\sigma_v\otimes\sigma'_v,\psi_v)$$

By hypothesis, the incomplete L-functions match, so the product of the γ-factors match. Now, fix a place $v^*\in S$. Choose χ such that χ_{v^*} is trivial and χ_v is very ramified if $v\in S\backslash\{v^*\}$. By stability (Lemma 1.6 and Theorem 2.12), the lemma follows. □

Symbolically, $\gamma = \frac{\epsilon\tilde{L}}{L}$. Here, ϵ is an entire function, and L and $\tilde{L}$ have no zeros. Suppose we can show that the poles $\tilde{L}$ and L do not cancel on either side. Given this, we know that L and $\tilde{L}$ match, so the local ϵ-factors also match by the lemma. This proves (c).

We get non-cancellation by showing that L has no pole in the region $\mathrm{Re}(s) > \frac{1}{2}$. Since $\tilde{L}$ is evaluated at $1-s$, it has no pole in the region $\mathrm{Re}(s) < \frac{1}{2}$, and we are done. On the Galois side, by Lemma 4.1, $\sigma\otimes\sigma'$ and $\sigma^\vee\otimes(\sigma')^\vee$ are both pure of weight 0, and the claim follows from the next lemma from Weil II.

Lemma 4.3 (Deligne). *If σ is a lisse ℓ-adic sheaf on a non-empty open subscheme $U\subseteq X$, pure of weight 0, then for all places v of X, the local L-factor $L(s,\sigma_v)$ has no pole in the region $\mathrm{Re}(s) > 0$.*

Proof. In the factorization of L-functions,

$$L(s,\sigma) = L^S(s,\sigma)\prod_{v\in S}L(s,\sigma_v)$$

the Euler product defining the incomplete L-function converges absolutely for $\mathrm{Re}(s) > 1$ since σ is pure of weight 0. It follows that it has no zeros in that region. By Theorem 1.3, the poles of the global L-functions are controlled by $H^0_c(\sigma)$ and $H^2_c(\sigma)$, and it's easy to see that there are none in the region $\mathrm{Re}(s) > 1$. Therefore, $L(s,\sigma_v)$ has no pole in the region $\mathrm{Re}(s) > 1$.

We now use the tensor power trick. The previous paragraph shows that the zeros of $\det(1-q^{-s}\mathrm{Frob}|(\sigma^{\otimes k})^{I_v})$ have modulus at most q. If α is a root of $\det(1-q^{-s}\mathrm{Frob}|\sigma^{I_v})$, then α^k is one of those zeros, so $|\alpha|^k\le q$. As $k\to\infty$, we see that $|\alpha|\le 1$, which translates to $L(s,\sigma_v)$ having no pole if $\mathrm{Re}(s) > 0$. □

The automorphic side is more subtle. By hypothesis, π_v and π'_v are unitary, so the Jacquet–Shalika estimate from Theorem 2.9 gives a pole-free region of $\mathrm{Re}(s)\ge 1$. This is not enough to get non-cancellation, so we need additional information, which in this case is that they should be tempered. The Berstein–Zelevinsky parameters of unitary (better, unitarizable) representations of $\mathrm{GL}_n(F_v)$ have been classified by Tadić [Tad86]. This and the recipe referred to in Theorem 2.10 gives us the following theorem.

Theorem 4.4. *Let π_v and π'_v be unitary generic representations of $\mathrm{GL}_n(F_v)$ and $\mathrm{GL}_{n'}(F_v)$ respectively.*

(1) If one of them is tempered, then $L(s,\pi_v\times\pi'_v)$ has no pole in the region $\mathrm{Re}(s)\ge\frac{1}{2}$.
(2) If both are tempered, then this can be improved to $\mathrm{Re}(s) > 0$.
(3) Suppose for each supercuspidal representation ρ_v occurring in the Berstein–Zelevinsky decomposition of π_v, $L(s,\pi_v\times\tilde{\rho}_v)$ has no pole in the region $\mathrm{Re}(s) > 0$, then π_v is tempered.

Example 4.5. We illustrate the classification when $n=2$, in which case the complete list is

(1) Supercuspidal representations.
(2) Twisted Steinberg representations: irreducible quotient of $\mathrm{Ind}(\chi(-\frac{1}{2})\otimes\chi(\frac{1}{2}))$, χ unitary.
(3) Unitary principal series: $\mathrm{Ind}(\chi_1\otimes\chi_2)$, χ_1,χ_2 unitary, $\chi_1\ne\chi_2(\pm 1)$.
(4) Complementary series: $\mathrm{Ind}(\chi(\alpha)\otimes\chi(-\alpha))$, χ unitary, $\alpha\in(0,\frac{1}{2})$.
(5) Characters: irreducible quotient of $\mathrm{Ind}(\chi(\frac{1}{2})\otimes\chi(-\frac{1}{2}))$, χ unitary.

Items (1)–(2) form the discrete series. Adding (3) gives the tempered representations. Adding (4) gives the generic unitary representations. The fact that complementary series exists shows that the weaker estimate in (1) of the theorem is sharp.

To complete the proof, we proceed as follows:

(1) If one of π and π' is tempered at v, then Theorem 4.4 is enough for non-cancellation. By the argument above, (c) is proven for this pair, and we get the stronger estimate using Lemma 4.3.
(2) Let $\pi \in \Pi_n^\circ$ be arbitrary. For each ρ_v a supercuspidal component of π_v, choose a global cuspidal π' whose local component is $\tilde{\rho}_v$ (a simple application of the trace formula, see Lemma VI.12 of [Laf02]). The stronger estimate coming from the previous step for each of those pairs shows that π_v is tempered. This completes the proof of the GRC for all $\pi \in \bigsqcup_{n' \leq n} \Pi_{n'}^\circ$.
(3) Since the GRC is now known up to rank n, the first step can be done if both π and π' have rank at most n, which completes the inductive step. □

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