OVERVIEW OF AUTOMORPHIC FORMS ON FUNCTION FIELDS

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This is the notes for the first talk of the Drinfeld–Lafforgue–Lafforgue seminar. The aim is to rapidly recall the definitions and foundations of the theory of automorphic forms and representations. Along the way, we will draw attention to certain differences between the function field and number field case. The classical starting reference is the Corvallis volumes [BC79a, BC79b], but also see the new GTM textbook [GH19].

Fix distinct primes p and ℓ. Let $E = \overline{\mathbf{Q}}_\ell$. Unless stated otherwise, all representations and functions in this article have coefficient field E. In the function field case, the automorphic side does not care about the topology on E, unlike in the number field case where it is essential that $E = \mathbf{C}$.

1. Smooth and admissible representations

1.1. **Basic definitions.** We will follow the exposition of Cartier in his article in [BC79a]. Paul Garrett also has a set of notes on his website, but his Haar measure is the right Haar measure, which causes some differences in the formulae.

In this section, G is a second countable td-group, where the td property means that the identity of G has a neighbourhood basis of open compact subgroups. This is satisfied if G is the R-points of an algebraic group, when R is a non-archimdean local field or the adèles over a function field. In particular, G is locally compact, Hausdorff, and totally disconnected, which also characterizes td-groups. Let μ be a fixed left-invariant Haar measure on G.

Definition 1.1. A representation (π, V) of G is *smooth* if the stabilizer of every vector is open.

We will often abuse notation and let π also denote V. The category of smooth representations of G will be denoted by $\mathrm{Sm}(G)$. It is also the module category of the Hecke algebra, which we introduce now.

Definition 1.2. Let K be an open compact subgroup of G, then the Hecke algebra $\mathcal{H}(G, K)$ is the space of all compactly supported K-bi-invariant functions with multiplication given by

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)dx$$

This is a unital associative algebra with unit $e_K = \mu(K)^{-1}\mathbf{1}_K$. If $G = \bigsqcup_\alpha Kg_\alpha K$, then a basis for $\mathcal{H}(G, K)$ is $\{\mu(K)^{-1}\mathbf{1}_{Kg_\alpha K}\}_\alpha$. Multiplication is given by computing the index of certain intersections of double cosets.

If $K' \subseteq K$, then $\mathcal{H}(G, K) \subseteq \mathcal{H}(G, K')$ as associative algebras, but it does not preserve the unit. The direct limit is denoted by $\mathcal{H}(G) = \bigcup_K \mathcal{H}(G, K)$. This is also the convolution algebra of compactly supported smooth (i.e. locally constant) functions. It is not unital.

Lemma 1.3. *(1) The category of smooth representations is equivalent to the category of non-degenerate $\mathcal{H}(G)$-modules (non-degenerate means the map $\mathcal{H}(G) \times V \to V$ is surjective).*

Moreover, if K is an open compact subgroup, then the subcategory of representations with K-fixed vectors and $\mathcal{H}(G, K)$-modules are also equivalent.

(2) Schur's lemma holds: if V is smooth irreducible, then any intertwining operator is a multiple of the identity. In particular, V has a central character.

Proof. (1) Let (π, V) be a smooth representation and $f \in \mathcal{H}(G)$, then define

$$\pi(f)(v) = \int_G f(g)\pi(g)(v)dg$$

It is easy to verify that this is an algebra homomorphism $\mathcal{H}(G) \to \mathrm{End}_E(V)$. Moreover, let K be an open compact subgroup in the stabilizer of v, then $\pi(e_K)(v) = v$, so the representation is non-degenerate.

Conversely, let V be a non-degenerate $\mathcal{H}(G)$-module, and let $g \in G$. Let $(K_n)_{n\geq 1}$ be a decreasing sequence of open compact subgroups forming a basis around $1 \in G$. Define

$$\pi(g)(v) = \lim_{n\to\infty} \mu(K_n)^{-1}\mathbf{1}_{gK_n} \cdot v$$

By non-degeneracy, there exists $f_1, \cdots, f_r \in \mathcal{H}(G)$ and $v_1, \cdots, v_r \in V$ such that $v = \sum_{i=1}^r f_i \cdot v_i$. If n is large, then all f_i are constant on all translates of K_n, so the limit stabilizes. It is then standard to check that this defines a smooth representation, and the two constructions are mutual inverses. The second statement is easier to check since $\mathcal{H}(G, K)$ is unital.

(2) The algebra of intertwining operators on V is a unital division algebra over E. Its dimension is countable since G is second countable. Since E is algebraically closed and uncountable, it has to be E. □

Let Z be a closed subgroup of the centre of G, and let $\chi : Z \to E^\times$ be a character. We say a representation π has Z-character χ if $\pi(z) = \chi(z)$ for all $z \in Z$. Similarly, we introduce the χ-Hecke algebra $\mathcal{H}_\chi(G)$ of functions f such that

1. f is smooth.
2. $f(zg) = \chi(z)^{-1}f(g)$ for all $z \in Z$.
3. f is compactly supported modulo Z.

The convolution is now taken by integrating over G/Z. Note that f is not compactly supported on G if Z is not compact. Similarly as in the lemma, $\mathcal{H}_\chi(G)$-modules are equivalent to smooth representations with Z-character χ. We will usually use this with Z being the split central torus.

Definition 1.4. Let (π, V) be a smooth representation, then its *contragredient* $(\tilde{\pi}, \widetilde{V})$ consists of the smooth vectors of the dual representation defined by

$$\langle \pi(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle$$

Just like for infinite dimensional vector spaces, it is not necessarily true that $\tilde{\tilde{\pi}}$ is isomorphic to π. We now introduce a more refined class of representations with good finiteness properties.

Definition 1.5. A representation (π, V) of G is *admissible* if it is smooth and for every open compact subgroup $K \subseteq G$, the invariants V^K is finite dimensional.

If π is admissible, then its isotypic components with respect to the action of an open compact subgroup K are finite dimensional, so in particular, we have $\tilde{\tilde{\pi}} \simeq \pi$. Moreover, if $f \in \mathcal{H}(G)$, then $\pi(f)$ is a finite rank operator on V, so we can define the character of π as a distribution

$$\theta_\pi(f) = \operatorname{Tr}\pi(f), \quad f \in \mathcal{H}(G)$$

The classical linear independence of character still holds here.

Lemma 1.6. *If $\pi_1, \cdots, \pi_n$ are irreducible admissible representations which are pairwise non-isomorphic, then their characters $\theta_{\pi_1}, \cdots, \theta_{\pi_n}$ are linearly independent distributions.*

Proof. Choose a sufficiently small open compact subgroup K such that all π_i^K are non-empty, then $\pi_1^K, \cdots \pi_n^K$ are finite dimensional representations of the unital associative algebra $\mathcal{H}(G, K)$. By irreducibility of π_i, the K-fixed vectors are irreducible and pairwise non-isomorphic. It follows from the classical theorem that θ_{π_i} are linearly independent, even when restricted to $\mathcal{H}(G, K)$. □

In the local case, we have the following deep theorem of Harish-Chandra in harmonic analysis, a proof of which can be found in the long expository article [Kot05].

Theorem 1.7 (Harish-Chandra)**.** *Let G be the rational points of a reductive group over a non-archimdean local field. Let π be an irreducible admissible representation on G, then the trace distribution θ_π lies in $L^1_{\mathrm{loc}}(G)$, and it is locally constant on the regular semisimple elements.*

Finally, we classify smooth representations based on growth properties of their matrix coefficients.

Definition 1.8. Let χ be a character of G. Let Z be a closed subgroup of the center of G. Suppose a smooth representation π has Z-character χ, then it is *supercuspidal* (resp. *square integrable* resp. *tempered*) with respect to Z if for all $v \in V$, $\tilde{v} \in \widetilde{V}$, the relative matrix coefficient

$$\pi_{v,\tilde{v}} \in C^\infty(G/Z) : g \mapsto \langle \tilde{v}, \chi(g)^{-1}\pi(g)v\rangle$$

is compactly supported (resp. lies in $L^2(G/Z)$ resp. lies in $L^{2+\epsilon}(G/Z)$ for all $\epsilon > 0$).

Remark 1.9.

(1) If χ is not unitary, this is usually called essentially square integrable (resp. essentially tempered) in the literature. The definition of supercuspidal representations typically also require admissibility, but it is easy to see that this is automatic if π is finitely generated (cf. Theorem 2.11).
(2) The article [CHH88] gives an elementary proof that unitary tempered representations are exactly those which are weakly contained in the regular representation. In this case, the trace character is a tempered distribution, which explains the name.

1.2. **Change of group.**

Definition 1.10. Let $\varphi : G \to G'$ be a continuous homomorphism of second countable td-groups, both equipped with a choice of left-invariant Haar measure. The pullback (or restriction) functor is defined by

$$\varphi^* : \mathrm{Sm}(G') \to \mathrm{Sm}(G), \quad (\pi', V') \mapsto (\pi' \circ \varphi, V)$$

The pushforward functor is defined by

$$\varphi_* : \mathrm{Sm}(G) \to \mathrm{Sm}(G'), \quad (\pi, V) \mapsto \mathcal{H}(G') \otimes_{\mathcal{H}(G)} V$$

where the right $\mathcal{H}(G)$-module structure on $\mathcal{H}(G')$ is the convolution

$$(f' *_\varphi f)(g') = \int_G f'(g'\varphi(g)^{-1}) f(g) \Delta_{G'}(\varphi(g)^{-1}) dg$$

Remark 1.11. We briefly review the modular character Δ. Given a locally compact topological group G, it is defined by the relation $\mu(Ag) = \Delta_G(g)\mu(A)$ for all measurable $A \subseteq G$. In particular, we have

$$\int_G f(x)dx = \Delta_G(g) \int_G f(xg)dx$$

A right-invariant Haar measure is then $\Delta_G^{-1} d\mu$. It is a character $G \to E^\times$ whose image lies in a $\mathbf{Q}^\times$-line, so if $K \subseteq G$ is compact, then $\Delta_G|_K$ is trivial.

The above formalism is typically used in the following three cases:
– If $\iota : H \hookrightarrow G$, then ι^* is the restriction Res_H^G.
– If $\iota : H \hookrightarrow G$ is an embedding, then ι_* is a twisted compact induction $\pi \mapsto \text{c-ind}_H^G(\pi \otimes \Delta_G/\Delta_H)$.
– If $p : G \twoheadrightarrow G/H$ is a topological quotient, then p_* is the trivial H-cotype functor

$$V \mapsto V_H := V/\langle \pi(h)v - v | h \in H, v \in V\rangle$$

It is helpful to give another characterization of induction.

Definition 1.12. Let H be a subgroup of G. Let π be a smooth H-representation. Define

$$\mathrm{Ind}_H^G \pi = \{f : G \to \pi \mid f(hg) = hf(g) \text{ for all } h \in H,\ g \in G\}^{\mathrm{sm}}$$

where the action is $(\gamma \cdot f)(g) = f(g\gamma)$, and the superscript denotes the subspace of smooth vectors (those are the uniformly locally constant functions). The compact induction is the subspace

$$\text{c-ind}_H^G \pi = \{f \in \mathrm{Ind}_H^G \pi \mid f \text{ is compactly supported modulo } H\}$$

so if $H\backslash G$ is compact, then $\text{c-ind}_H^G = \mathrm{Ind}_H^G$.

The intertwining operator between $\iota_*\pi$ and $\text{c-ind}_H^G \pi$ is given by

$$I(f \otimes v)(g) = \int_H f(g^{-1}h)(\pi(h)v) \frac{\Delta_G(h)}{\Delta_H(h)} dh$$

where $f \in \mathcal{H}(G)$ and $v \in \pi$.

Lemma 1.13 (Frobenius reciprocity). *Let $\varphi : H \to G$ be a continuous homomorphism. Let $\sigma \in \mathrm{Sm}(H)$ and $\pi \in \mathrm{Sm}(G)$, then*

(1) If φ is open, then $\mathrm{Hom}_H(\sigma, \varphi^*\pi) \simeq \mathrm{Hom}_G(\varphi_*\sigma, \pi)$.
(2) If $H \subseteq G$, then $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma)$.
(3) The functor c-ind_H^G is exact.

Proof. Define an auxiliary functor $\varphi^! : \mathrm{Sm}(G) \to \mathrm{Sm}(H)$ as follows: $\varphi^!\pi$ consists of the H-smooth vectors of the distribution space $\mathcal{H}(G)^* \otimes_{\mathcal{H}(G)} \pi$. We have a map $\mathrm{Hom}_H(\sigma, \varphi^!\pi) \simeq \mathrm{Hom}_G(\varphi_*\sigma, \pi)$,

$$(\alpha : \sigma \to \varphi^!\pi) \mapsto (\alpha' : \varphi_*\sigma \to \pi,\ f' \otimes v \mapsto \pi(f')\alpha(v))$$

It is easy to see that this is bijective, so φ_* has a right adjoint. This and the explicit description of c-ind_H^G proves (3). Moreover, if φ is open, then $\varphi^! = \varphi^*$, giving (1). Part (2) is the standard Frobenius reciprocity given by the bijections

$$(\alpha : \pi \to \mathrm{Ind}_H^G \sigma) \mapsto (\alpha' : \pi|_H \to \sigma,\ x \mapsto \alpha(x)(1))$$
$$(\beta : \pi|_H \to \sigma) \mapsto (\beta' : \pi \to \mathrm{Ind}_H^G \sigma,\ x \mapsto (g \mapsto \beta(gx)))$$

□

Not much can be said about admissibility in this generality, but in the cases we are interested in, there will be deep results showing that admissibility is preserved by these functors.

2. Local theory

In this section, F is a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k of order $q = p^n$. The norm is normalized by $|\varpi|_F = q^{-1}$. Let G be the F-points of a connected reductive group $\mathbf{G}$ over F. The modular character Δ_G is trivial since $\mathbf{G}$ is reductive, so the Haar measure on G is bi-invariant. We will use the convention that a bold faced letter denote an algebraic group over F, and the corresponding usual letter denotes its F-points. We will implicitly use the theory of buildings via the resulting matrix decomposition theorems. Proofs of them can be found in Tits's article in [BC79a].

2.1. Satake isomorphism.

Definition 2.1. The group $\mathbf{G}$ is *unramified* if it is quasi-split and splits over an unramified extension of F.

If $\mathbf{G}$ is unramified, then it has a hyperspecial maximal compact subgroup K. Concretely, this can be realized as the $\mathcal{O}_F$-points of a smooth model of $\mathbf{G}$ over $\mathcal{O}_F$. In particular, if $G = \mathrm{GL}_n(F)$, then we can take $K = \mathrm{GL}_n(\mathcal{O}_F)$. If H is a subgroup of G, we normalize the Haar measure on H by setting $\mu_H(H \cap K) = 1$. The Satake isomorphism describes the structure of the unramified Hecke algebra $\mathcal{H}(G, K)$.

We need some more notations. Let $\mathbf{B}$ be a Borel in $\mathbf{G}$ defined over F. Let $\mathbf{S}$ be the maximal F-split torus of $\mathbf{B}$, and let $\mathbf{T}$ be its centralizer, then $\mathbf{T}$ is a non-split maximal torus. Let T° denote the integral points of T, more formally the set of $t \in T$ such that $v_F(\lambda(t)) = 0$ for all $\lambda \in X^*(T)$.

Theorem 2.2 (Satake isomorphism)**.** *There is an isomorphism of algebras*

$$\mathcal{S} : \mathcal{H}(G, K) \xrightarrow{\sim} \mathcal{H}(T, T^\circ)^{W_S} \simeq E[X_*(S)]^{W_S}, \quad \mathcal{S}f(t) = \delta(t)^{\frac{1}{2}} \int_N f(tn)dn$$

In particular, $\mathcal{H}(G, K)$ is commutative.

We will not prove this. It can be found in Cartier's article. The next example covers some of the ideas in the proof, in particular the use of the Cartan decomposition.

Example 2.3. We will compute the above isomorphism for $\mathbf{G} = \mathrm{GL}_n$ by hand. Take $K = \mathrm{GL}_n(\mathcal{O}_F)$, B the upper triangular matrices, T the diagonal matrices, then T° is the maximal compact subgroup of T where the diagonal entries are units.

The group G has a Cartan decomposition

$$G = \bigsqcup_{a_1 \geq \cdots \geq a_n} K \,\mathrm{diag}(\varpi^{a_1}, \cdots, \varpi^{a_n}) K$$

which comes from the theory of elementary divisors. For each sequence $\underline{a} = (a_1 \geq \cdots \geq a_n)$, let $\varphi_{\underline{a}}$ be the indicator function of the corresponding piece. They form a basis for $\mathcal{H}(G, K)$. Moreover, let $\underline{m}_i$ be the sequence with i 1s and $n - i$ 0s, and let $\varphi_i = \varphi_{\underline{m}_i}$, then $\varphi_1, \cdots, \varphi_n, \varphi_n^{-1}$ generates $\mathcal{H}(G, K)$ as an algebra (note that φ_0 is the identity).

The right hand side of the Satake isomorphism is isomorphic to $E[T_1^{\pm 1}, \cdots, T_n^{\pm 1}]^{S_n} = E[E_1, \cdots, E_{n-1}, E_n^{\pm 1}]$, where $E_1, \cdots, E_n$ are the elementary symmetric polynomials. The Satake isomorphism is

$$\mathcal{S}(\varphi_i) = q^{\frac{i(n-i)}{2}} E_i$$

In fact, let $t = \operatorname{diag}(\varpi^{a_1}, \cdots, \varpi^{a_n})$, then

$$\begin{aligned}\mathcal{S}(\varphi_i)(t) &= \delta(t)^{\frac{1}{2}} \int_N \varphi_i(tn)dn \\ &= q^{-\frac{1}{2}\sum_{j=1}^n (n-2j+1)a_j} \mu_N \left(N \cap t^{-1} K \operatorname{diag}(\underbrace{\varpi, \cdots, \varpi}_{i}, \underbrace{1, \cdots, 1}_{n-i}) K \right)\end{aligned}$$

By considering the determinants of the principal minors, we see that if $(a_1, \cdots, a_n)$ is not a permutation of $\underline{m}_i$, then $\mathcal{S}(\varphi_i)(t) = 0$. Since K contains the permutation matrices, $\mathcal{S}(\varphi_i) \in \mathcal{H}(T, T^\circ)^W$. Therefore, we just need to compute the value when $(a_1, \cdots, a_n) = \underline{m}_i$, in which case the multiplier is $q^{-\frac{1}{2}i(n-i)}$. The usual reduction algorithm shows that the set under consideration is

$$\begin{pmatrix} N_i(\mathcal{O}_F) & \varpi^{-1}\mathrm{Mat}_{i,n-i}(\mathcal{O}_F) \\ 0 & N_{n-i}(\mathcal{O}_F) \end{pmatrix}$$

which has volume $q^{i(n-i)}$, so finally, we get that $\mathcal{S}(\varphi_i)(t) = q^{\frac{i(n-i)}{2}}$. Following through the identification of $\mathcal{H}(T, T^\circ)$ with $E[X_*(T)]$ gives the claim.

2.2. **Unramified representations and L-group.** We continue the notations from the previous subsection. In particular, we still assume that G is unramified. A more detailed discussion, in particular a proof of the unramified correspondence, can be found in Borel's article in [BC79b].

Definition 2.4. A smooth representation π of G is *unramified* if $\pi^K \neq \{0\}$.

Therefore, there is an equivalence of category between unramified representations of G and modules over $\mathcal{H}(G, K)$, which is now a commutative algebra with well-understood structure by the Satake isomorphism. In addition to being the simplest class of representations to study, they are also the generic component of a global automorphic representation.

It follows from this discussion that irreducible unramified representations are classified by characters of $\mathcal{H}(G, K) \simeq \mathbf{Z}[X_*(S)]^{W_S}$. They are in turn in bijection with the semisimple conjugacy classes of a certain algebraic group called the L-group. We first give the general definition.

Definition 2.5. Let $\mathbf{G}$ be an arbitrary (not necessarily unramified) connected reductive group over F. The L-group of G is a semi-direct product

$${}^L G = \widehat{G}(E) \rtimes W_F$$

where $\widehat{G}$ is the connected reductive group over E with the dual root datum, and the Weil group W_F acts by the Galois action on the root datum of G.

Example 2.6.

1. If $G = \mathrm{GL}_n$, then ${}^L G = \mathrm{GL}_n(E) \times W_F$.
2. If $G = U(V)$ is a unitary group with respect to a quadratic extension F'/F, then ${}^L G = \mathrm{GL}_n(E) \rtimes W_F$. The action of W_F factors through $\mathrm{Gal}(F'/F)$, with the non-trivial element acting as $g \mapsto \Phi^{-1} g^{-t} \Phi$, where $\Phi_{ij} = (-1)^{i+1}\delta_{i,n+1-j}$.

Theorem 2.7. *Let $\mathbf{G}$ be unramified. There is a bijection between smooth unramified representations of G and semisimple conjugacy classes of ${}^L G$.*

This is the unramified local Langlands correspondence, which is not particular hard given the Satake isomorphism. In the case of GL_n, this is the statement that unramified representations of $\mathrm{GL}_n(F)$ correspond to semisimple conjugacy classes of $\mathrm{GL}_n(E)$, which are in turn classified by its n eigenvalues. These are the *Satake parameters* of a representation.

2.3. **Parabolic induction and Jacquet functor.** Let $\mathbf{P} \subseteq \mathbf{G}$ be a parabolic subgroup defined over F, with Levi decomposition $\mathbf{P} = \mathbf{MN}$. Let R_P^+ be the roots of $\mathbf{G}$ which appear in N. Let $\delta(m) = |\det(\mathrm{Ad}_m|N)|_F$, which is the valuation of the sum of positive roots. We have $\Delta_P(mn) = \delta(m)^{-1}$. Let $\mathbf{A_P}$ be the maximal split torus in the centre of $\mathbf{M}$. For $\epsilon > 0$, let

$$A_P^-(\epsilon) = \{a \in A \mid |\alpha(a)|_F \leq \epsilon \text{ for all } \alpha \in R_P^+\}$$

We write $A_P^- = A_P^-(1)$. As $\epsilon \to 0$, $A_P^-(\epsilon)$ "tends to a boundary" of A_P/Z. Without subscripts, they will refer to a minimal parabolic.

We will now introduce the two functors in the title. The results on the Jacqeut functor first appeared in [Jac71], but a more accessible exposition may be [Cas95]. Consider the diagram

$$\begin{tikzcd} & P \arrow[dl, hook, "\iota"'] \arrow[dr, two heads, "p"] & \\ G & & M \end{tikzcd}$$

This induces two functors.

Definition 2.8. The functor $I_{M,G}(\sigma) = p^*\iota_*(\sigma \otimes \delta^{\frac{1}{2}}) : \mathrm{Sm}(M) \to \mathrm{Sm}(G)$ is the (normalized) *parabolic induction.* The functor $J_{G,M}(\pi) = p_*\iota^*(\pi) \otimes \delta^{\frac{1}{2}} : \mathrm{Sm}(G) \to \mathrm{Sm}(M)$ is the (normalized) *Jacquet functor.*

Explicitly, $I_{M,G}(\sigma)$ consists of all uniformly locally constant functions $f : G \to \sigma$ such that

$$f(mng) = \delta(m)^{\frac{1}{2}}\sigma(m)f(g) \quad \text{for all } m \in M,\ n \in N,\ g \in G$$

with G acting from the right. The Jacquet functor consists of first restricting to P, then taking N-coinvariants followed by a character twist. Since $\mathbf{G}/\mathbf{P}$ is projective, G/P is compact, so we have an adjunction

$$\mathrm{Hom}_M(J_{G,M}\pi, \sigma) \simeq \mathrm{Hom}_G(\pi, I_{M,G}\sigma)$$

Moreover, the results of the previous section shows that both $I_{M,G}$ and $J_{G,M}$ are exact.

The twists in the above definitions are used to preserve pre-unitarity: a representation is pre-unitary if it has a G-invariant bilinear form. More precisely, we have

Proposition 2.9. *The contragredient of $I_{M,G}(\sigma)$ is $I_{M,G}(\tilde{\sigma})$. The pairing can be given by*

$$\langle f, \tilde{f}\rangle = \int_K \langle f(k), \tilde{f}(k)\rangle dk$$

if K is a special maximal compact subgroup (e.g. a hyperspecial one in the unramified case).

Proof. The function $g \mapsto \langle f(g), \tilde{f}(g)\rangle$ is in $I_{M,G}(\delta^{\frac{1}{2}})$, so we need to show the integral defines a G-invariant form $I_{M,G}(\delta^{\frac{1}{2}}) \to E$. The non-degeneracy is then easy to verify. We have a surjection $\mathcal{P} : C_c^\infty(G) \to I_{M,G}(\delta^{\frac{1}{2}})$ sending $f \in C_c^\infty(G)$ to

$$(\mathcal{P}f)(g) = \int_P \delta(p)^{-1} f(pg) d_r p = \int_P f(pg)dp$$

But the Iwasawa decomposition gives

$$\int_K (\mathcal{P}f)(k)dk = \int_K \int_P f(pk)dpdk = \int_G f(g)dg$$

so $\int_K dk$ is a G-invariant linear form on $I_{M,G}(\delta^{\frac{1}{2}})$. $\square$

We now state a fundamental finiteness property of the two functors.

Theorem 2.10 (Jacquet)**.** *Both $I_{M,G}$ and $J_{G,M}$ preserves admissibility. The functor $J_{G,M}$ also sends finitely generated representations to finitely generated representations.*

Proof. For this proof only, we will use the un-normalized versions of the two functors.

Let σ be an admissible representation of M, then its extension to P is also admissible. Let K be an open compact subgroup of G, then since $P\backslash G$ is compact, there exists a finite set $C \subseteq G$ such that $G = PCK$,. Let $f \in I_{M,G}(\sigma)^K$, then f is determined by its restriction to C, and moreover, $f(x)$ is fixed by $xKx^{-1} \cap P$ if $x \in C$. Therefore $f(C)$ lands in the $P \cap \bigcap_{x\in C} xKx^{-1}$-fixed vectors of σ. This is finite dimensional since σ is admissible, so $I_{M,G}(\sigma)$ is also admissible.

We now move onto $J_{G,M}$. First, suppose $X \subseteq \pi$ is a finite set generating a smooth representation π. Choose an open compact subgroup of G such that $X \subseteq \pi^K$, then we can find C as before such that $G = PCK$. Now the co-invariants V_N are P-generated by $\{\pi(c)x \,|\, c \in C, x \in X\}$.

Finally, let (π, V) be an admissible representation of G. Let K be an open compact subgroup of G, and let $M_K = K \cap M$. Let $U \subseteq (V_N)^{M_K}$ be finite dimensional, then there exists a finite dimensional subsapce

$\widetilde{U} \subseteq V$ which surjects onto U. The idea of the proof is to the adjust vectors in $\widetilde{U}$ to lie in $V^{K'}$ for some open compact subgroup K' depending only on K.

By averaging over M_K, we can make sure $\widetilde{U} \subseteq V^{M_K}$. Now, each vector in $\widetilde{U}$ is fixed by some open compact subgroup of N^-, the opposite unipotent subgroup to N, so we can choose $N^-_{K,U} \subseteq N^-$ small enough so that $\widetilde{U} \subseteq V^{M_K N^-_{K,U}}$. It is an easy lemma that if N_1^-, N_2^- are open compact subgroups of N^-, then there exists a in A_P such that $aN_1^- a^{-1} \subseteq N_2^-$. Applying this to $N^-_{K,U}$ and $N^-_K = N^- \cap K$ gives an element a, depending on U, such that $a^{-1}N^-_K a \subseteq N^-_{K,U}$. For $n \in N^-_K$ and $u \in \widetilde{U}$, we have

$$\pi(n)\pi(a)u = \pi(an')u = \pi(a)u$$

where $n' = a^{-1}na \in N^-_{K,U}$. It follows that $\pi(a)\widetilde{U} \subseteq V^{M_K N^-_K}$. Averaging over compact subgroups of N does not change the image in V_N, so the image of $\pi(a)\widetilde{U}$ is contained in the image of $V^{N_K M_K N^-_K}$, where $N_K = N \cap K$. The operator $\pi(a)$ is invertible, so $\dim U \leq \dim V^{N_K M_K N^-_K}$. We say K is good if

- If $a \in A^-_M$, then $aN_K a^{-1} \subseteq N_K$.
- If P is a standard parabolic subgroup, then $K = N^-_K M_K N_K$, and this is a topological isomorphism.

If $\mathcal{G}$ is a smooth model of G over $\mathcal{O}_F$, then $K_n = \ker(\mathcal{G}(\mathcal{O}_F) \to \mathcal{G}(\mathcal{O}_F/\varpi^n))$ are all good, and in general, good neighbourhood basis of the identity exists. If K is good, then $N_K M_K N^-_K = K$, so by the admissibility of V, $\dim U$ is uniformly bounded, so $\dim (V_N)^{M_K} < \infty$. We have shown more: V^K surjects onto $(V_N)^{M_K}$. Since the collection of all M_K is a basis of M around 1, we have the admissibility of V_N. □

The Jacquet functor was first used to characterize supercuspidal representations as those which do not arise from parabolic inductions.

Theorem 2.11 (Jacquet)**.** *The following are equivalent for an irreducible smooth representation π of G.*

(1) π is supercuspidal (matrix coefficients are compactly supported modulo centre).
(2) $J_{G,M}(\pi) = 0$ for all parabolics $P \subseteq G$.
(3) $\mathrm{Hom}_G(\pi, I_{G,M}(\sigma)) = 0$ for all parabolics $P \subseteq G$ and smooth representations σ.

They all imply the admissibility of π.

Moreover, for any irreducible smooth representation π, there exists a parabolic subgroup $P = MN \subseteq G$ and a supercuspidal representation σ of M such that $\pi \hookrightarrow I_{G,M}(\sigma)$. In particular, π is admissible.

Proof. We first do the easy parts of the theorem. The equivalence of (2) and (3) follows from Frobenius reciprocity. Suppose (1) holds and π^K contains an infinite set of linearly independent vectors $(v_n)_{n\geq 1}$. Let $v^* \in (V^K)^*$ be 1 on them and 0 on a complementary subspace. It extends to an element of $\widetilde{V}$ using the projector $\pi(e_K)$. The matrix coefficient $\pi_{v^*,v}$ for any $v \neq 0$ is not compactly supported modulo centre. Finally, the last part is an induction on the rank of G. It uses the lemma that if M is a non-degenerate finitely generated module over an associative algebra, then M has an irreducible quotient, which is proven by constructing a maximal submodule of M using Zorn's lemma.

It remains to prove the equivalence of (1) and (2). First suppose (1) holds. First fix a good open compact subgroup K, in the sense of the proof of admissibility. The spaces π^K and $\tilde{\pi}^K$ are finite dimensional, so $\pi_{v,\tilde{v}}$ is uniformly compactly supported for $v \in \pi^K$, $\tilde{v} \in \tilde{\pi}^K$. In particular, there exists an element $a \in A^-_M$ such that $\langle \tilde{v}, \pi(a)v\rangle = 0$ for all such $v, \tilde{v}$. Since $\tilde{\pi}^K = (\pi^K)^*$, we have $\pi(e_K)\pi(a)v = 0$ for all $v \in \pi^K$. But $\pi(a)v \in V^{M_K N^-_K}$ since $a^{-1}N^-_K a \subseteq N^-_K$, so $\pi(e_K)\pi(a)v$ is just another N_K-averaging of $\pi(a)v$. Therefore, $\pi(e_K)\pi(a)v$ has the same image as $\pi(a)v$ in V_N. By our proof of admissibility, V^K surjects onto $(V_N)^{M_K}$. Given $\bar{u} \in (V_N)^{M_K}$, let v be a lift of $\pi(a^{-1})\bar{u}$ to V^K. We get that $\bar{u} = \pi(a)\bar{v} = \overline{\pi(e_K)\pi(a)v} = 0$, so $(V_N)^{M_K} = 0$. As K varies, this gives $V_N = 0$.

Finally, to prove (1) from (2), we will need a lemma.

Lemma 2.12. *There exists a unique bilinear pairing between V_N and $\widetilde{V}_{N-}$ such for each $v \in V$, $\tilde{v} \in \widetilde{V}$, there exists $\epsilon > 0$ such that for all $a \in A^-_M(\epsilon)$,*

$$\langle \tilde{v}, \pi(a)v\rangle = \langle \bar{\tilde{v}}, \pi(a)\bar{v}\rangle_M$$

It is moreover M-invariant and non-degenerate, so $\widetilde{V_N} \simeq \widetilde{V}_{N^-}$.

We defer the technical proof to after this. For now, fix $v \in V$ and $\tilde{v} \in \widetilde{V}$. Let α be a positive root of $\mathbf{G}$. Let $\mathbf{P}_\alpha$ be the maximal parabolic subgroup attached to α, then by hypothesis, $J_{G,M_\alpha}(\pi) = 0$, so by the lemma, there exists an $\epsilon > 0$ such that if $a \in A^-_{P_\alpha}(\epsilon)$, then $\langle \tilde{v}, \pi(a)v\rangle = 0$. We have shown that if $a \in A^-$, $|\alpha(a)|_F \leq \epsilon$, then $\pi_{v,\tilde{v}}(a) = 0$. Let K be a special subgroup, then we have the Cartan decomposition $G = KA^-K$, so the matrix coefficient is supported on $KA_\epsilon K$, where A_ϵ is the compact modulo centre subset of A^- consisting of all a such that $\epsilon < |\alpha(a)|_F \leq 1$ for all $\alpha \in R^+$. □

Proof of Lemma 2.12. Fix fix a good maximal compact subgroup K. The idea is to consider a subspace of V^K such that the surjection $V^K \twoheadrightarrow (V_N)^{M_K}$ is an isomorphism, and then using it to define the canonical pairing. The good subspace will be defined using $\pi(e_{KaK})$ for $a \in A^-$ sufficiently close to infinity.

The kernel of $V^K \to (V_N)^{M_K}$ is finite dimensional. Each vector of a basis has the form $\sum(\pi(n_i)v_i - v_i)$ for a finite sum, so there exists an open compact subgroup $N_0 \subseteq N$ containing all n_i which appears this way. Then $\int_{N_0} \pi(n)vdn = 0$ for all v in the kernel. There exists $\epsilon > 0$ such that $aN_0a^{-1} \subseteq N_K$ for all $a \in A^-(\epsilon)$. Fix one such a. The argument in the main proof shows that $\pi(e_K)\pi(a)V^K = \pi(e_{KaK})V$ surjects onto $(V_N)^{M_K}$. Suppose $v = \pi(e_K)\pi(a)v_0$ where $v_0 \in V^K$, then $v = \pi(e_{N_K})\pi(a)v_0$ by the invariance property of $\pi(a)v_0$. If v is sent to 0, then by our choice of N_0,

$$\int_{N_0}\int_{N_K} \pi(n_0)\pi(n)\pi(a)v_0 dndn_0 = 0$$

We can expand the outer integration to range over $a^{-1}N_Ka \supseteq N_0$ without changing the zero. Conjugation by a^{-1} expands N_K, so we can combine the two integrals into

$$\int_{a^{-1}N_Ka} \pi(n)\pi(a)v_0 dn = 0$$

Pulling $\pi(a)$ to the front conjugates n, which changes dn by a constant multiple depending on a. It follows that $\int_{a^{-2}N_Ka^2} \pi(n)v_0dn = 0$, so v_0 is in the kernel of $V^K \to (V_N)^{M_K}$. But then $\pi(e_{N_0})v_0 = 0$ by the choice of N_0, so we have

$$v = \pi(e_K)\pi(a)v_0 = \pi(a)\pi(e_{a^{-1}N_Ka})v_0 = 0$$

This proves that the projection $\pi(e_{KaK})V \to (V_N)^{M_K}$ is an isomorphism for all $a \in A^-(\epsilon)$.

Observe a stability property: if $a \in A^-$ and $a_0 \in A^-(\epsilon)$, then $\pi(e_{Kaa_0K})V \subseteq \pi(e_{KaK})V$. But $aa_0 \in A^-(\epsilon)$, so they have the same dimension by the above argument, so the two spaces are in fact equal. We denote this by V^K_{ord}, so it is equal to $\pi(e_{KaK})V$ for all $a \in A^-(\epsilon)$, and there is an isomorphism $V^K_{\mathrm{ord}} \xrightarrow{\sim} (V_N)^{M_K}$. Using it and the corresponding isomorphism for $\widetilde{V}$, we can transfer the canonical pairing to a pairing

$$(V_N)^{M_K} \otimes (\widetilde{V}_{N^-})^{M_K} \to E$$

We now need to check the identity in the lemma. Fix $v \in V$ and $\tilde{v} \in \widetilde{V}$. Let K be a good open compact subgroup fixing both of them. We are now in the setting before, so suppose $a \in A^-(\epsilon)$, then

$$\langle \tilde{v}, \pi(a)v\rangle = \langle \tilde{v}, \pi(e_K)\pi(a)v\rangle = \langle \pi(e_K)\pi(a^{-1})\tilde{v}, v\rangle$$

Because we are using N^- for $\tilde{v}$, $a^{-1} \in A^-_{P^-}$, so $\pi(e_K)\pi(a^{-1})\tilde{v} = \pi(e_{N^-_K})\pi(a^{-1})\tilde{v} = \pi(a^{-1})\pi(e_{aN^-_Ka^{-1}})\tilde{v}$. By decreasing ϵ if necessary, we can make $e(aN^-_Ka^{-1})\tilde{v} = 0$ for all $\tilde{v} \in \widetilde{V}^K$ which are sent to 0 in $\widetilde{V}_{N^-}$. The required identity follows.

Finally, we need to check uniqueness. Given this, M-invariance is easy, and non-degeneracy follows from the corresponding fact for the pairing on $V \otimes \widetilde{V}$. But in checking the identity, we showed that $\langle \tilde{v}, \pi(a)v\rangle = \langle \tilde{v}, v_{\mathrm{ord}}\rangle$ if $a \in A^-(\epsilon)$ for ϵ sufficiently small. Here, $v_{\mathrm{ord}} = \pi(e_{KaK})v \in V^K_{\mathrm{ord}}$. It follows that any M-bilinear form satisfying the property must be the one we defined. □

2.4. **Bernstein–Zelevinsky classification and local Langlands correspondence.** The local Langlands correspondence is the following conjecture.

Conjecture 2.13. *There is a finite-to-one correspondence from admissible irreducible representations of $G(F)$ to relevant Frobenius-semisimple continuous homomorphisms $W_F \to {}^LG$.*

Remark 2.14. It is essential here that the dual group is an ℓ-adic group. In the usual formulation with $\mathbf{C}$-coefficient, we need to replace W_F by the Weil–Deligne group $W_F \times \mathrm{SL}_2(\mathbf{C})$. The two categories of representations are equivalent by the Grothendieck ℓ-adic monodromy theorem.

We will not explain the terms here. The unramified representations correspond to the unramified Galois representations, namely those which factor through $W_{F^{\mathrm{ur}}/F} = \mathrm{Frob}^{\mathbf{Z}}$. In particular, they are reducible. The irreducible ones with trivial monodromy are expected to correspond to supercuspidal representations. If this is shown, then we need to understand how to build general admissible representations from supercuspidal ones, in particular understanding how parabolic inductions decompose. In the non-archimedean case, a good survey for $G = \mathrm{GL}_n$ is the paper [Kud94] in the Motives proceedings.

The first step is the p-adic analogue of the Langlands classification for real groups [Lan89]. The proof uses similar ideas, namely constructing the required parabolic by studying the growth of matrix coefficients on the torus and then identify the right representation in the composition series of the corresponding Jacquet module. It can be found in [BZ76] for $G = \mathrm{GL}_n$ and [BW00] in general.

Theorem 2.15.

(1) If σ is unitary and tempered and $\lambda \in X_*(M) \otimes_{\mathbf{Z}} \mathbf{R}$ is in the positive chamber, then $I_{M,G}(\sigma \otimes \lambda)$ has a unique irreducible quotient $Q(\sigma, \lambda)$.

(2) Every smooth irreducible representation is equal to $Q(\sigma, \lambda)$ for some choice of data. The choice of data is unique if we fix a Levi decomposition of each standard parabolic.

In the formulation of the correspondence using $\mathbf{C}$-coefficients, the tempered representations correspond to parameters $W_F \times \mathrm{SL}_2(\mathbf{C})$ which are bounded on W_F. Given a general Langalnds parameter, the image of W_F lies in the Levi of a parabolic subgroup, and it can be made bounded after twisting. This is not hard to show, and it allows us to reduce the local Langlands correspondence to the tempered case.

The classification of tempered and square-integrable representations using induction from supercuspidal representations are harder in general. For $G = \mathrm{GL}_n$, the Berstein–Zelevinsky classification completely describes it using "segments". On the Galois side, the square-integrable representations correspond to indecomposable parameters, and segments are responsible for the monodromy action. A detailed description of segments can be found in [Kud94].

3. Global theory

In this section, we change to notation to let F be a global field of characteristic $p > 0$ defined over $\mathbf{F}_q$. For each place v, F_v is the completion of F at v with ring of integers $\mathcal{O}_v$, uniformizer ϖ_v, and residue field size q_v. Its ring of adèles is denoted by $\mathbb{A}$. If S is a finite set of places, then $\mathbb{A}_S = \prod_{v \in S} F_v$ and $\mathbb{A}^S = \prod'_{v \notin S} F_v$.

3.1. **Automorphic forms.** Let G be a connected reductive group over F. The adèlic points $G(\mathbb{A})$ is of the type described in the first section. It contains a discrete subgroup $G(F)$. Let Z denote the maximal F-split central torus of G. Fix a character $\chi : Z(F)\backslash Z(\mathbb{A}) \to E^\times$.

Remark 3.1. As is usual in this theory, the split central tori cause problems with finiteness statements, essentially due to the degree map $\deg : F^\times \backslash \mathbb{A}_F^\times \to \mathbf{Z}$. One usually either fixes a central character (as we do here) or consider an appropriate subgroup $G(\mathbb{A})^0$, similarly to studying Pic^0 instead of Pic.

Definition 3.2. A function $f \in C^\infty(G(F)\backslash G(\mathbb{A})/K)$ is cuspidal if

$$\int_{N(F)\backslash N(\mathbb{A})} f(nx)dn = 0$$

where N is the unipotent radical of any proper F-parabolic subgroup of G.

Definition 3.3. Let $K \subseteq G(\mathbb{A})$ be an open compact subgroup. An *automorphic form* on G with level K and character χ is a function f in

$$C^\infty(G(F)\backslash G(\mathbb{A})/K)_\chi = \{f : G(F)\backslash G(\mathbb{A})/K \to E \,|\, f(zg) = \chi(z)f(g) \text{ for all } z \in Z(\mathbb{A}),\ g \in G(\mathbb{A})\}$$

such that the representation of $G(\mathbb{A})$ generated by f is admissible. It is a *cusp form* if it is also cuspidal. We will let $\mathcal{A}(G, K, \chi)$ denote the space of automorphic forms of level K and let $\mathcal{A}^\circ(G, K, \chi)$ denote the space of cusp forms of level K. Let $\mathcal{A}(G, \chi) = \bigcup_K \mathcal{A}(G, K, \chi)$, and similarly define $\mathcal{A}^\circ(G, \chi)$.

In the number field case, one usually requires $Z(\mathcal{U}(\mathfrak{g}_\infty))$-finiteness, but the foundational works of Harish-Chandra [HC68] proves a finiteness theorem for them, which implies admissibility. In the function field case, the corresponding result is due to Harder.

Theorem 3.4 (Harder, [Har74, Corollary 1.2.3])**.** *Cuspidal functions in $C^\infty(G(F)\backslash G(\mathbb{A})/K)_\chi$ are uniformly compactly supported. In particular, the space $\mathcal{A}^\circ(G,K,\chi)$ is finite dimensional, and cuspidal functions are automatically cusp forms.*

3.2. **Automorphic representations and L-functions.**

Definition 3.5. A smooth irreducible representation of $G(\mathbb{A})$ is *automorphic* if it appears as a subquotient of the representation of $G(\mathbb{A})$ on the space $\mathcal{A}(G,\chi)$ of all automorphic forms. It is *cuspidal* if it is a subspace of the space of all cusp forms.

Remark 3.6. The cuspidal part decomposes completely, so we don't need to take subquotient to define a cuspidal automorphic representation.

By Harder's theorem from the previous section, automorphic representations are admissible. The general theory of decomposing admissible representations explained in Flath's article in [BC79a] gives us the factorization theorem.

Proposition 3.7. *Let π be an automorphic representation, then for each place v, there exists an admissible irreducible representation π_v of $G(F_v)$ such that*

(1) For almost all v, π_v is unramified with spherical vector π_v°.
(2) $\pi = \bigotimes_v' \pi_v$, where the restricted tensor product is taken with respect to the spherical vectors π_v°.

Now suppose $G = \mathrm{GL}_n$. Given an automorphic representation π and a place v such that π_v is unramified, we can consider the Satake parameters $\alpha_1, \cdots, \alpha_n$ of π_v. The local L-factor of π at v is defined by

$$L(\pi_v, s) = \frac{1}{\prod_{i=1}^n (1-\alpha_i q_v^{-s})}$$

There is a systematic method using integral representations or the Berstein–Zelevinsky classification for defining local L-factors at the bad places. The global L-function of π is then

$$L(\pi, s) = \prod_v L(\pi_v, s)$$

This is a rational function in q^{-s}, which one can show using the standard integral representation.

Example 3.8. Let $G = \mathrm{GL}_2$, then the global zeta integral is

$$Z(\varphi, s) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi\big(\left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right)\big)\, |a|^{s-\frac{1}{2}}\, da$$

Let π be a cuspidal automorphic representation. Suppose $\varphi \in \pi$, then it is a classical computation that $Z(\varphi, s)$ factors into an Euler product, where each local term is a multiple of $L(\pi_v, s)$. Moreover, by choosing a good test vector at each unramified places, we can show that $Z(\varphi, s)$ is a rational multiple of $L(\pi, s)$. On the other hand, using Harder's finiteness theorem, the integral is in fact a finite sum, so it is rational.

We are now ready to state Langland's conjecture for GL_n, and also the main result of this seminar.

Theorem 3.9. *There is a bijection between almost everywhere unramified continuous irreducible n-dimensional ℓ-adic representations of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ with finite order determinants and cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ with finite order central characters. This bijection is uniquely specified by matching L-functions and ϵ-factors.*

More explicitly, if σ corresponds to the automorphic representation π, then for almost all places v, σ and π are unramified at v, and $\sigma(\mathrm{Frob}_v)$ has eigenvalues equal to the Satake parameters of π_v.

Remark 3.10.

(1) We can remove the conditions on determinants and central characters by using the Weil group W_F, which replaces with $\mathrm{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q) \simeq \hat{\mathbf{Z}}$ quotient by $\mathbf{Z}$.
(2) The book of Moeglin and Waldspurger [MW95] seems to also cover the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$ in the function field setting. In particular, analogous to the local case, there is a method of building up general automorphic forms from cusp forms using *Eisenstein series*. They describe completely the continuous part of the spectrum. The discrete, non-cuspidal part is more mysterious, analogous to the situation of the tempered representations which are not super-cuspidal.

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