INTRODUCTION TO LOCAL HARMONIC ANALYSIS

SHILIN LAI

These are notes for an introductory talk on representation theory over non-archimedean local fields, mainly focusing on the unramified setting. The usual references for smooth representations of p-adic groups include Cartier's article in Corvallis [BC79], Casselman's notes [Cas95], and Bernstein's Harvard notes [BR92]. A comprehensive reference for abstract harmonic analysis is [Dix77], but we can state the results we actually need more explicitly.

1. Representations

In this section, G is a second countable, Hausdorff, locally compact group. Concretely, think of $\mathbb{R}$, or $\mathrm{GL}_2(\mathbb{C})$, or $G(F)$ where G is an algebraic group and F is a locally compact field.

1.1. **Unitary representations.** Let X be a topological space with a right G-action. Further suppose that X has a G-invariant measure, which is fixed throughout. Recall that $L^2(X)$ is the space of (complex valued) measurable functions on X which are square integrable with respect to μ. This has a natural *left* G-action by right translation

$$(R_g f)(x) = f(xg)$$

The central question in harmonic analysis (for our purpose) is to decompose $L^2(X)$ into irreducible representations of G, for "interesting" choices of X.

A key feature of $L^2(X)$ is that it is *unitary*, namely it has a positive definite Hermitian inner product

$$\langle f, g\rangle_X := \int_X \overline{f(x)} g(x) dx$$

which is moreover invariant under the action of G. As a result, each of its subrepresentation is also unitary. It therefore makes sense to only concentrate on those representations.

Definition 1.1.

(1) A *unitary representation* of G is a Hilbert space $\mathscr{H}$ with a group homomorphism $\pi : G \to \mathrm{U}(\mathscr{H})$ which is continuous in the sense that $g \mapsto \pi(g)v : G \to \mathscr{H}$ is continuous for all $v \in \mathscr{H}$.
(2) A unitary representation $(\pi, \mathscr{H})$ is *irreducible* if it contains no closed invariant subspaces other than $\{0\}$ and $\mathscr{H}$.
(3) The *unitary dual* of G is the set of equivalence classes of topologically irreducible unitary representations of G. We denote it by $\Pi_{\mathrm{unit}}(G)$.[1]

By an abuse of notation, it is typical for π to also denote the Hilbert space.

There is a natural topology on $\Pi_{\mathrm{unit}}(G)$ called the *Fell topology.* It is roughly compact-open convergence of matrix coefficients. Using this, we can make the above questions more precise.

Question 1.2.

(1) **Unitary dual**: Describe $\Pi_{\mathrm{unit}}(G)$ as a topological space.
(2) **Plancherel formula**: Find a G-equivariant decomposition

$$L^2(X) = \int_{\Pi_{\mathrm{unit}}(G)} \pi^{\oplus m_\pi} d\mu(\pi)$$

where m_π is the multiplicity, and μ is a positive Borel measure called the *Plancherel measure*.[2]

We will not define the direct integral construction. Instead, we will do some examples which illustrate its relation with the usual Plancherel formula.

[1] Most sources would use $\hat{G}$, but that will mean the dual reductive group for us later.
[2] As we will see in the examples, "the" Plancherel measure is not unique.

Example 1.3.

(1) Let $G = S^1$. For each $n \in \mathbb{Z}$, let $e_n(z) = z^n$. This is a unitary character of G. Let $\mathbb{C}(e_n)$ be the one-dimensional space where G acts by e_n, then $\Pi_{\text{unit}}(G) = \{\mathbb{C}(n) | n \in \mathbb{Z}\}$, with the discrete topology.

Moreover, $e_n \in L^2(G)$ is an eigenfunction for the action of G. This defines a closed embedding $\mathbb{C}(e_n) \hookrightarrow L^2(G)$. The associated projection map is

$$f \mapsto \hat{f}(n)e_n \in \mathbb{C}(n), \quad \hat{f}(n) := \int_{S^1} f(z)\overline{e_n(z)}dz$$

The basic result in Fourier series is that we have a decomposition of $L^2(G)$ into irreducibles

$$L^2(S^1) \simeq \int_{\mathbb{Z}} \mathbb{C}(e_n)d\mu(n), \quad f \mapsto (\hat{f}(n)e_n)_{n\in\mathbb{Z}}$$

where the Plancherel measure $\mu(n)$ is just the counting measure. The direct integral here is just suitably (namely L^2) completed direct sum.

More concretely, the direct integral decomposition means there is a Fourier inversion formula

$$f(z) = \sum_{n\in\mathbb{Z}} \hat{f}(n)z^n$$

Applying it to $\bar{f} * f$ gives the classical Plancherel formula

$$\|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(\Pi_{\text{unit}}(G))} = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2$$

Note that all the formulae above depends on various choices: the Haar measure on S^1 was normalized to have total volume 1, and the eigenfunctions e_n were chosen so that $e_n(1) = 1$. This is a general feature: the Plancherel measure depends on these choices, but the resulting L^2-space does not change.

(2) Let $G = \mathbb{R}$. For each $x \in \mathbb{R}$, let $e_x : G \to S^1,\ y \mapsto e^{2\pi ixy}$, then $\Pi_{\text{unit}}(G) = \{e_x | x \in \mathbb{R}\} \simeq \mathbb{R}$ with the usual topology.

The new feature here is that $e_x \notin L^2(G)$. In fact, G acting on $L^2(G)$ has no eigenvectors. Instead, e_x approximate eigenvectors. This is a feature of the continuous spectrum, and the direct integral construction is used to decompose $L^2(G)$ as a "direct sum" of a continuum number of approximate eigenspaces. In this case, it is just the Fourier inversion formula

$$L^2(G) \simeq \int_{\mathbb{R}} \mathbb{C}(e_x)\,dx,\ f(x) = \int_{\mathbb{R}} \hat{f}(y)e_x(y)dy, \quad \text{where } \hat{f}(x) = \int_{\mathbb{R}} f(y)\overline{e_x(y)}dy$$

The Plancherel measure is a suitably normalized Haar measure on $\mathbb{R}$, and we have the classical Plancherel formula $\|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(\Pi_{\text{unit}}(G))}$.

If we allow x to be a complex number, then e_x defined by the same formula is still a character $G \to \mathbb{C}^\times$, but it is no longer unitary, so it does not appear in the Plancherel formula. It might still make sense to consider them in a classification, and we will discuss this later.

(3) Let $G = \mathbb{R}^\times$. This is isomorphic to $\mathbb{R} \times \{\pm 1\}$, so the unitary is two copies of $\mathbb{R}$. It is usual to index them differently here. Let $s \in i\mathbb{R}$, then define

$$e_s^\pm(x) = |x|^s \cdot \begin{cases} 1 & + \\ x/|x| & - \end{cases}$$

so $\Pi_{\text{unit}}(G) \simeq i\mathbb{R} \sqcup i\mathbb{R}$, with the usual topology.

Let $X = \mathbb{A}^1 = \mathbb{R}$, then the projection of $L^2(X)$ to approximate eigenspaces is classically called the Mellin transform

$$(\mathcal{M}^\pm f)(s) = \int_X f(x)\overline{e_s^\pm(x)}d^\times x = \int_0^\infty f^\pm(x)x^{-s}\frac{dx}{x}$$

where $f = f^+ + f^-$ is the decomposition into even and odd parts. Correspondingly, the disintegration of $L^2(X)$ is just the Mellin inversion formula

$$L^2(X) \simeq \int_{\Pi_{\text{unit}}(G)} \mathbb{C}(e_s^\pm)\,ds,\ f(x) = \sum_{\epsilon\in\{\pm\}} \int_{i\mathbb{R}} (\mathcal{M}^\epsilon f)(s)e_s(x)ds$$

The Plancherel measure is again just the Haar measure on $i\mathbb{R}$, suitably normalized.

We have carefully avoided convergence issues. Before now, everything worked in the dense subspace $C_c^\infty(G) \subseteq L^2(G)$, but now the integral defining $\mathcal{M}f$ might not converge even if $f \in C_c^\infty(X)$. The issue is the point $0 \in X$, and this was prominently featured in Tate's thesis. He had to work with non-unitary e_s, where $\mathrm{Re}(s) \gg 0$. This is another reason to think about non-unitary representations.

(4) We now move on to some non-abelian groups. Let $G = \mathrm{SU}(2)$, a compact Lie group. Its unitary dual is $\Pi_{\mathrm{unit}}(G) = \{\pi_n | n \in \mathbb{Z}_{\geq 1}\}$, where π_n is the irreducible representation of dimension n.

The Peter–Weyl theorem can be written in the following form

$$L^2(G) \simeq \int_{\Pi_{\mathrm{unit}}(G)} \pi^* \otimes \pi \, d\mu(\pi), \quad f \mapsto \pi(f) := \int_G f(g)\pi(g)dg \in \mathrm{End}(\pi)$$

One new feature is that the π-isotypic component is no longer multiplicity-free. Instead of a natural embedding $\pi \hookrightarrow L^2(G)$, matrix coefficients gives a natural embedding $\pi^* \otimes \pi \to L^2(G)$, namely

$$v^* \otimes v \mapsto (g \mapsto \langle v^*, \pi(g)v\rangle) \in L^2(G)$$

This is the equivalent of normalizing the eigenfunctions. With this normalization, the Plancherel measure is no longer the counting measure. In fact, if $f \in L^2(G)$, then

$$f(g) = \sum_{n=1}^{\infty} n \cdot \mathrm{Tr}(\pi_n(f)\pi_n(g))$$

In other words, μ assigns a weight of $\dim \pi$ to each $\pi \in \Pi_{\mathrm{unit}}(G)$. This is holds more generally for any compact groups. The proof is analogous to the character orthogonality relations for finite groups.

The multiplicity feature can be explained away in general. From the point of view of spherical varieties, the correct action should be $G \times G$ acting on $X = \triangle G \backslash G \times G$. Equivalently, this is $G \times G$ acting on G by left and right multiplication. The direct integral is now multiplicity free, and the Plancherel measure is supported on the the diagonal of $\Pi_{\mathrm{unit}}(G \times G) = \Pi_{\mathrm{unit}}(G)^2$.

(5) The group $G = \mathrm{SU}(2)$ also acts on $X = S^2$ via the double cover $G \twoheadrightarrow \mathrm{SO}(3)$. Due to central character constraint, only the odd dimensional representations show up in $L^2(X)$. More precisely, we have the following multiplicity-free decomposition.

$$L^2(X) \simeq \int_{n \text{ odd}} \pi_n d\mu(\pi_n)$$

To obtain this decomposition, the usual technique is to use the Lie group action $\mathfrak{g}$ on the smooth functions $C^\infty(X)$. The Casimir operator $\Omega \in \mathfrak{g}$ becomes the Laplace operator, and it acts by a different scalar on each π_n, so we get a lot of information from the spectral decomposition of the Laplacian on S^2. This transfer from $L^2(X)$ to $C^\infty(X)$ will be replicated in the p-adic setting using smooth representations.

(6) Finally, we look at a complicated example. Let $G = \mathrm{PGL}_2(F)$, where $F = \mathbb{F}_q((t))$. The unitary dual for general $\mathrm{GL}_n(F)$ was first computed by Tadić [Tad86], but the $n = 2$ case should be known before. The picture is already quite complicated.

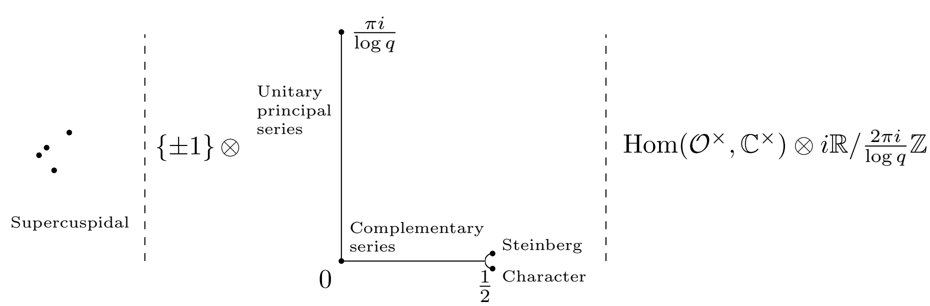

Figure 1. Unitary dual of $\mathrm{PGL}_2(F)$

We briefly explain this picture. Fix a uniformizer, so $F^\times \simeq \mathcal{O}_F^\times \times \mathbb{Z}$. Given a character χ in the discrete set $\mathrm{Hom}(\mathcal{O}^\times, S^1)$, we can form the twist $\chi|\cdot|^s$, which is a character on $F^\times$. The parameter s

runs over the cylinder $\mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z}$, and the character is unitary exactly when $s \in i\mathbb{R}$. There is a general process of parabolic induction. Using it, we can form a (not necessarily unitary) representation $\operatorname{Ind}(\chi\,|\cdot|^s, \chi^{-1}\,|\cdot|^{-s})$. If $\chi \neq \chi^{-1}$, then this is unitary exactly when $s \in i\mathbb{R}$, creating the right most part, which is a countable disjoint union of circles.

There are exactly two characters with $\chi = \chi^{-1}$. For those, $s \leftrightarrow -s$ does not change the representation, so the circle becomes a line segment. On the other hand, if $s \in (0, \frac{1}{2})$, the a priori non-unitary inner product can be modified into a unitary one. At $s = \frac{1}{2}$, this representation becomes reducible, and $\Pi_{\text{unit}}(G)$ is not Hausdorff there.

Finally, there is a countable discrete set of unitary representations which do not arise this way. They are known as the supercuspidal representations. They are at the heart of the local Langlands correspondence. All of them are infinite dimensional.

The Plancherel measure is only supported on the unitary principle series, supercuspidals, and the two Steinberg points: the complementary series does not contribute to $L^2(G)$. This is reflected in the asymptotic property of their matrix coefficients. Unlike in the previous cases, the Plancherel measures on the vertical lines are *not the Haar measure*. For example, on the component corresponding to the trivial character, it is given by

$$\frac{L(1-2it,\mathbf{1})L(1+2it,\mathbf{1})}{L(-2it,\mathbf{1})L(2it,\mathbf{1})} \cdot \frac{q+1}{2} \cdot \frac{\log q}{\pi} dt$$

where $L(s,\mathbf{1}) := (1-q^{-s})^{-1}$. This is a consequence of Macdonald's spherical Plancherel formula, which we will see later. The explicit Plancherel measure for GL_n can be found in [AP05].

This picture gets even more complicated for groups beyond GL_2. The standard reference is [Wal03], but there are still ongoing works to make it explicit beyond GL_n. Luckily for us, we will be looking at the unramified representations. This is a more algebraic theory, so we will want to introduce an algebraic replacement of Hilbert space representations, which also allows for non-unitary representations.

1.2. **Smooth and admissible representations.** In this section, we restrict further and suppose G is a second countable td-group, where the td property means that the identity of G has a neighbourhood basis of open compact subgroups. In particular, G is locally compact, Hausdorff, and totally disconnected, which also characterizes td-groups. We continue to fix a Haar measure on G.

Concretely, think of $G = \mathbf{G}(F)$, where $\mathbf{G}$ is an algebraic group and F is a local field. Even more concretely, think of $G = \mathrm{GL}_n(\mathbb{F}_q((t)))$.

Definition 1.4. Let (π, V) be an algebraic representation of G.

(1) A vector $v \in V$ is *smooth* if its stabilizer is open in G.
(2) The representation π is smooth if every vector is smooth.
(3) If V is an arbitrary representation, then the subset of smooth vectors is a smooth representation of G, which we denote by V^∞.
(4) The *contragredient* of (π, V) is the representation $(V^*)^\infty$, usually denoted by $(\tilde{\pi}, \widetilde{V})$.

We will often abuse notation and let π also denote V. The category of smooth representations of G will be denoted by $\mathrm{Sm}(G)$. The equivalence classes of irreducible objects will be denoted by $\Pi_{\text{sm}}(G)$.

The algebraic counterpart of $L^1(G)$ is the Hecke algebra $\mathcal{H}(G) := C_c^\infty(G)$ under convolution. A new feature is that G has many open compact subgroups, so it makes sense to introduce certain refinements.

Definition 1.5. Let K be an open compact subgroup of G, then the Hecke algebra $\mathcal{H}(G, K)$ is the space $C_c^\infty(K\backslash G/K, \mathbb{C})$ with multiplication given by

$$(f_1 * f_2)(x) = \int_G f_1(xg) f_2(g^{-1}) dg$$

This is a unital associative algebra with unit $e_K = \mathrm{vol}(K)^{-1}\mathbf{1}_K$.

If $K' \subseteq K$, then $\mathcal{H}(G,K) \subseteq \mathcal{H}(G,K')$ as associative algebras, but it does not preserve the unit. The direct limit over all K is just $\mathcal{H}(G)$, which is no longer unital.

Lemma 1.6. *(1) The category of smooth representations is equivalent to the category of non-degenerate $\mathcal{H}(G)$-modules (non-degenerate means the map $\mathcal{H}(G) \times V \to V$ is surjective).*

Moreover, if K is an open compact subgroup, then the subcategory of representations generated by K-fixed vectors and $\mathcal{H}(G,K)$-modules are also equivalent.

(2) *Schur's lemma holds: if V is smooth irreducible, then any intertwining operator is a multiple of the identity. In particular, V has a central character.*

We need to impose more finiteness conditions to make the algebraic theory nicer. In particular, we want things like $\tilde{\tilde{\pi}} \simeq \pi$ and Frobenius reciprocity, which are not true algebraically if the underlying vector spaces are infinite dimensional.

Definition 1.7. A representation (π, V) of G is *admissible* if it is smooth and for every open compact subgroup $K \subseteq G$, the invariants V^K is finite dimensional.

One can think of this as finite multiplicity of K-types for irreducible unitary representations of real Lie groups. In fact, for the groups we are interested in, there are similar automatic admissibility statements. The following two results are very difficult.

Theorem 1.8. *Let G be the F-points of a reductive group, where F is a local field.*

(1) *(Jacquet) If π is an irreducible smooth representation of G, then π is admissible.*
(2) *(Bernstein) If π is an irreducible unitary representation of G, then π^∞ is dense in π, and the representation is admissible.*
(3) *The operation of taking smooth vectors is an injection $\Pi_{\mathrm{unit}}(G) \to \Pi_{\mathrm{sm}}(G)$. The image is exactly those representations with a G-invariant positive definite linear form (which is necessarily unique up to scalar multiples).*

As a result of this theorem, there are two steps in classifying $\Pi_{\mathrm{unit}}(G)$, with different flavours.

(1) Classify $\Pi_{\mathrm{sm}}(G)$: this is the content of the local Langlands conjecture.
(2) Determine which ones are in $\Pi_{\mathrm{unit}}(G)$: this has more to do with analysis than arithmetic.

The development of the Plancherel formula for $L^2(G)$ is closely related to item (2), though as we have seen, the support of the Plancherel measure is sometimes smaller than $\Pi_{\mathrm{unit}}(G)$.

2. Unramified representations

In this section, F is a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k of order $q = p^n$. The norm is normalized by $|\varpi|_F = q^{-1}$. Let $\mathcal{G}$ be a split reductive group scheme over $\operatorname{Spec} \mathcal{O}$ with generic fibre $\mathbf{G}$. Let $G = \mathcal{G}(F)$. It has a subgroup $K = \mathcal{G}(\mathcal{O})$, which is a maximal open compact subgroup. There is a canonical normalization of the Haar measure by taking $\mathrm{vol}(K) = 1$.

Again, concretely, we may take $G = \mathrm{GL}_n(\mathbb{F}_q((\varpi)))$ and $K = \mathrm{GL}_n(\mathbb{F}_q[[\varpi]])$.

2.1. **Satake isomorphism.**

Definition 2.1. A smooth representation π of G is *spherical* or *unramified* if $\pi^K \neq \{0\}$.

Remark 2.2. This definition may depend on the choice of the smooth model $\mathcal{G}_{/\mathcal{O}}$. In the case $\mathcal{G} = \mathrm{GL}_n$, it is the case that all such choices lead to conjugate K, so the above definition is independent of choices.

Example 2.3. The trivial representation is unramified and unitary. From the point of view of L^2-harmonic analysis, it is very far from the general case.

Let $\mathcal{H}^\circ(G) = \mathcal{H}(G,K)$. This is called the *unramified Hecke algebra*, and it acts on the K-fixed vectors of a smooth representation. There is a bijection between smooth irreducible representations and simple $\mathcal{H}^\circ(G)$-modules. The Satake isomorphism gives a very simple description of the structure of $\mathcal{H}^\circ(G)$.

We first introduce more notations. Let T be a split maximal torus in G with group of cocharacters $X_*(T)$. Let B be a Borel subgroup containing T, with unipotent factor N. Let $W_T = N_G(T)/T$ be the Weyl group for T. Let T° be the integral elements of T. Finally, let $\delta(t) = |\det(\mathrm{Ad}_t|N)|_F$. This is twice the half sum of positive roots.

In the explicit example of $\mathrm{GL}_n(F)$

– T is the diagonal torus, $T^\circ \simeq (\mathcal{O}^\times)^n$.
– $X_*(T) \simeq \mathbb{Z}^n$, identifying $\underline{a} \in \mathbb{Z}^n$ with the map $z \mapsto \mathrm{diag}(z^{a_1}, \cdots, z^{a_n})$.
– B is the upper triangular matrices.

– N is the unipotent upper triangular matrices.
– $W \simeq S_n$ can be represented by the permutation matrices.
– $\delta(\operatorname{diag}(t_1, \cdots, t_n)) = \prod_i |t_i|_F^{n-2i+1}$.

Theorem 2.4 (Satake isomorphism)**.** *There is an isomorphism of algebras*

$$\mathcal{S} : \mathcal{H}^\circ(G) \xrightarrow{\sim} \mathcal{H}(T, T^\circ)^{W_T}, \quad \mathcal{S}f(t) = \delta(t)^{\frac{1}{2}} \int_N f(tn)dn$$

In particular, $\mathcal{H}^\circ(G)$ is commutative.

Remark 2.5. The integral is a version of the Jacquet functor construction. It is also an analogue of the constant term of global automorphic forms.

Idea of proof. It is formal to check that $\mathcal{S}$ is an algebra homomorphism. To show the image is W_T-invariant, one can rewrite $\mathcal{S}$ as an orbital integral using the Weyl integral formula, or alternatively use the theory of intertwining operators introduced later.

Let $\lambda \in X_*(T)^+$, the set cocharacters which are positive with respect to the chosen Borel subgroup B. By an abuse of notation, let $\lambda = \lambda(\varpi) \in T$. Form the indicator function

$$c_\lambda = \mathbf{1}[K\lambda K] \in \mathcal{H}^\circ(G)$$

By the Cartan decomposition, $\{c_\lambda \,|\, \lambda \in X_*(T)^+\}$ is a basis for $\mathcal{H}^\circ(G)$.

On the target space, $\lambda \mapsto \lambda(\varpi)$ gives an isomorphism of abelian groups $X_*(T) \simeq T/T^\circ$. Therefore, restriction to the positive cone identifies $\mathcal{H}(T, T^\circ)^{W_T}$ with $X_*(T)^+$.

Let $\lambda, \mu \in X_*(T)^+$, then the key computation is

$$\mathcal{S}(c_\lambda)(\mu) = \begin{cases} \delta(\lambda)^{-\frac{1}{2}} & \mu = \lambda \\ 0 & \mu \not\leq \lambda \end{cases}$$

where the partial order is defined by $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of positive coroots. From this, the matrix of $\mathcal{S}$ is upper triangular with non-zero diagonal entries with respect to the chosen bases, so $\mathcal{S}$ is bijective. This computation is very clearly explained in [Gro98]. □

Example 2.6. We will compute the above isomorphism for $\mathbf{G} = \mathrm{SL}_2$ by hand. It has a single positive coroot, which we denote by α, and the Cartan decomposition is

$$G = \bigsqcup_{n \geq 0} K t_n K, \quad t_n = \alpha(\varpi)^n = \begin{pmatrix} \varpi^n & \\ & \varpi^{-n} \end{pmatrix}$$

Concretely, a matrix g is in Kt_nK if and only if the minimal valuation of the entries of g is $-n$. Let $c_n = \mathbf{1}[Kt_nK]$, then for all $m \geq 0$,

$$\begin{aligned} \mathcal{S}(c_n)(t_m) &= \delta(t_m)^{\frac{1}{2}} \int_F c_n\Big(t_m \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\Big) dx \\ &= q^{-m} \int_F c_n\Big(\begin{pmatrix} \varpi^m & \varpi^m x \\ & \varpi^{-m} \end{pmatrix}\Big) dx \end{aligned}$$

If $m > n$, then the integral is 0. If $m = n$, then the integral is supported on the set $v(x) \geq -2n$. If $m < n$, then the integral is supported on the set $v(x) = -n - m$. Therefore,

$$\mathcal{S}(c_n)(t_m) = \begin{cases} 0 & m > n \\ q^n & m = n \\ q^{n-1}(q-1) & m < n \end{cases}$$

One can check similarly that $\mathcal{S}(c_n)(t_m^{-1}) = \mathcal{S}(c_n)(t_m)$, proving that $\mathcal{S}(c_n) \in \mathcal{H}(T, T^\circ)^{W_T} \simeq \mathbb{C}[\alpha + \alpha^{-1}]$. More precisely, the above computation gives

$$\mathcal{S}(c_n) = q^n(\alpha^n + \alpha^{-n}) + q^{n-1}(q-1) \sum_{1 \leq m < n} (\alpha^m + \alpha^{-m}) + q^{n-1}(q-1)$$

A nicer way to write this is to compute that

$$\sum_{m\leq n} \mathcal{S}(c_m) = q^n \sum_{|k|\leq n} \alpha^k$$

Moreover, note that $\sum_{|k|\leq n} \alpha^k$ is the character of the representation of PGL_2 of dimension $2n+1$.

2.2. **Unramified representations and L-group.** By our discussion before, there is an isomorphism of algebras $\mathcal{H}(T,T^\circ) \simeq \mathbb{C}[X_*(T)]$. This is the space of compactly supported functions on the free abelian group $X_*(T)$. Given $f \in \mathbb{C}[X_*(T)]$, its Fourier transform is

$$\mathcal{F}f(\alpha) = \sum_{\lambda\in X_*(T)} f(\lambda)\overline{\langle \alpha, \lambda\rangle}, \quad \alpha \in X^*(T)\otimes_{\mathbb{Z}} \mathbb{C}^\times$$

We can interpret $X^*(T)\otimes_{\mathbb{Z}}\mathbb{C}^\times$ as the $\mathbb{C}$-points of the algebraic torus $\hat{T}$ such that $X_*(\hat{T}) = X^*(T)$. By an abuse of notation, we will write $\hat{T} = \hat{T}(\mathbb{C})$. The subgroup of unitary characters $X^*(T)\otimes_{\mathbb{Z}} S^1$ will be denoted by $\hat{T}^1$. [3] In summary, $\mathcal{F}$ identifies $\mathbb{C}[X_*(T)]$ with $\mathcal{O}(\hat{T})$. Taking into account the Weyl-action, we get

Corollary 2.7. *There is a sequence of isomorphisms*

$$\mathcal{H}^\circ(G) \xrightarrow{\mathcal{S}} \mathcal{H}(T,T^\circ)^{W_T} \xrightarrow{\mathcal{F}} \mathcal{O}(\hat{T}/W_T)$$

Since $\mathcal{H}^\circ(G)$ is abelian, the irreducible unramified representations are identified with the $\mathbb{C}$-points of $\operatorname{Spec}\mathcal{H}^\circ(G)$. By the above version of the Satake isomorphism, this is just the $\mathbb{C}$-points of the quotient $\hat{T}/W_T$. This can be interpreted in terms of semisimple conjugacy classes in a dual group.

Definition 2.8. Let $\mathbf{G}$ be a split reductive group over F. Its *dual group* $\hat{\mathbf{G}}$ is the split reductive group whose root datum is the dual of the root datum for $\mathbf{G}$.

Example 2.9. Here are some pairs of dual groups

$\mathbf{G}$	GL_n	SL_n	PGL_n	Sp_{2n}	O_{2n}
$\hat{\mathbf{G}}$	GL_n	PGL_n	SL_n	O_{2n+1}	O_{2n}

Corollary 2.10. *There is a bijection between irreducible unramified representations of* $\mathbf{G}(F)$ *and semisimple conjugacy classes of* $\hat{\mathbf{G}}(\mathbb{C})$.

Example 2.11. We compute the image of the trivial representation. This corresponds to the algebra homomorphism

$$\mathcal{H}^\circ(G) \to \mathbb{C}, \quad f \mapsto \int_G f(g)dg$$

The Iwasawa decomposition gives an integral formula

$$\int_G f(g)dg = \int_T\int_N\int_K f(tnk)dkdndt = \int_T \delta^{-\frac{1}{2}}(t)(\mathcal{S}f)(t)dt$$

Therefore, it is attached to the character $\delta^{-\frac{1}{2}}$. By making a substitution $z = q^{-s}$, which changes $X^*(T)\otimes_{\mathbb{Z}}\mathbb{C}^\times$ to $X^*(T)\otimes_{\mathbb{Z}}\mathbb{C}$, this is exactly $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$.

Let $\Pi^\circ_{\mathrm{sm}}(G) \subseteq \Pi_{\mathrm{sm}}(G)$ denote the set of unramified representations, then we have a very concrete description $\Pi^\circ_{\mathrm{sm}}(G) = \hat{T}/W_T$. A natural question is to identify which of them are unitarizable, since this should be related to the L^2-harmonic analysis questions we are interested in. Let $\hat{T}^1 := \mathrm{Hom}(X_*(T), S^1)$ be the subset of unitary characters, then it is not hard to show that points in $\hat{T}^1/W_T$ are unitary representations.

However, this is not the full set. In particular, the trivial representation corresponds to the character $\delta^{-\frac{1}{2}}$, which is very far from unitary. There is also the complementary series, as we saw in the $\mathrm{PGL}_2(F)$ example. A lot of the difficulties for identifying $\Pi_{\mathrm{unit}}(G)$ is already present in the unramified setting.

Remark 2.12. If $\mathbf{G}$ is not split, then the above results need to be modified to take into account the Galois action on T. This is one origin of the Langlands dual group ${}^L\mathbf{G}$, which is a semidirect product of $\hat{\mathbf{G}}$ with a Galois group.

[3] In some references, $\hat{T}$ denotes the group of unitary characters of T.

2.3. **Unramified Plancherel formula.** Suppose X is a G-variety as before. We were looking for a direct integral decomposition

$$L^2(X) = \int_{\Pi_{\mathrm{unit}}(G)} \pi^{\oplus m_\pi} d\mu(\pi)$$

Since we understand $\Pi^\circ_{\mathrm{sm}}(G)$ very well, it makes sense to consider the K-invariant vectors of the above decomposition.

$$L^2(X)^K = L^2(X/K) = \int_{\hat{T}/W_T} (\pi_\chi^K)^{\oplus m_\chi} d\mu(\chi)$$

where we have used the Satake isomorphism to rewrite the smooth dual in a concrete way. Moreover, since $\mathcal{H}^\circ(G)$ is abelian, the spaces π_χ^K are 1-dimensional. The flavour is therefore close to the abelian Fourier transform examples we did before.

To simplify notations, we now assume all the multiplicities are 1. This holds in many cases, but in general, they should be labelled by a dual torus attached to X. Further fix a base point $x_0 \in X$. For each $\pi \in \Pi^\circ_{\mathrm{unit}}(G)$ that appears in the decomposition, let Ω_π be the eigenfunction normalized so that $\Omega_\pi(x_0) = 1$. This gives a normalization of the spherical Fourier transform

$$f \mapsto \hat{f}(\pi) := \langle f, \Omega_\pi \rangle_{L^2(X)}$$

Explicitly, the Plancherel measure needs to satisfy the Fourier inversion formula

$$f(x) = \int_{\hat{T}/W_T} \hat{f}(\pi)\Omega_\pi(x) d\mu(\pi)$$

This is equivalent to an L^2-version, which is more often known as the Plancherel formula

$$\langle f, g\rangle_{L^2(X)} = \int_{\hat{T}/W_T} \hat{f}(\pi)\overline{\hat{g}(\pi)} d\mu(\pi)$$

The determination of the Plancherel measure is closely related to calculation expressions for Ω_π.

In the case $X = G$ (viewed as a $G \times G$-space), such a formula was computed in general by Macdonald [Mac71] and Casselman [Cas80] using a different method. The results are analogous to Harish-Chandra's formulae in the semisimple Lie group case. This was generalized by Sakellaridis [Sak13] to many spherical cases, and the book [BZSV23] contains a conjectural categorification of these formulae.

2.4. **Group case: statement of results.** The group case is $G \times G$ acting on $X = \triangle G\backslash G \times G$ on the right. The base point is chosen to be $x_0 = (1, 1)$. For the unramified theory, this unwinds to the left action of the unramified Hecke algebra $\mathcal{H}^\circ(G)$ on the space $L^2(K\backslash G/K)$, given by the formula

$$(f \cdot \varphi)(x) = \int_G \varphi(xg) f(g) dg$$

Unfortunately, this differs from the convolution $\varphi * f$ by an inverse. Instead, $f \cdot \varphi = f^\vee * \varphi$, where $f^\vee$ is the pullback of f along $g \mapsto g^{-1}$. This causes some differences in labelling from several references.

Definition 2.13. A *zonal spherical function* is a function $\omega \in C(K\backslash G/K)$ such that ω is an eigenfunction for $\mathcal{H}^\circ(G)$ and $\omega(1) = 1$.

These are the normalized spherical functions, so they can be labelled using elements $\chi \in \hat{T}$, identifying the W_T-orbits. Since $T/T^\circ \simeq X_*(T)$, we will equivalently view $\chi \in \hat{T}$ as unramified characters of T.

Theorem 2.14 (Macdonald, Casselman)**.** *Let $G = \mathbf{G}(F)$, where $\mathbf{G}$ is split and unramified, then*

(1) For all $\chi \in \hat{T}$, there exists a unique zonal spherical function ω_χ.
(2) Let χ be a regular unramified character, then for all $t \in X_*(T)^+$, we have

$$\omega_\chi(t) = \frac{\delta(t)^{\frac{1}{2}}}{Q(q^{-1})} \sum_{w \in W_T} \mathbf{c}(w\chi^{-1}) w\chi(t) \tag{ω}$$

where $Q(\xi)$ is the Poincaré polynomial of G, and $\mathbf{c}(\chi)$ is the Harish-Chandra $\mathbf{c}$-function.

(3) Let $d\chi$ be the probability measure on $\hat{T}^1$, then the Plancherel measure is

$$d\mu(\chi) = \frac{Q(q^{-1})}{\#W_T} \cdot |\mathbf{c}(\chi)|^{-2} \cdot d\chi \tag{$d\mu$}$$

Explicitly, for all $f, g \in C_c^\infty(K\backslash G/K)$, we have the Fourier inversion formula

$$f(x) = \frac{Q(q^{-1})}{\#W_T} \int_{\hat{T}^1} \hat{f}(\chi)\omega_\chi(x) \, |\mathbf{c}(\chi)|^{-2} \, d\chi \tag{Inversion}$$

and the Plancherel identity

$$\langle f, g\rangle_{K\backslash G/K} = \frac{Q(q^{-1})}{\#W_T} \int_{\hat{T}^1} \hat{f}(\chi)\overline{\hat{g}(\chi)} \, |\mathbf{c}(\chi)|^{-2} \, d\chi \tag{Plancherel}$$

We still need to explain $Q(\xi)$ and $\mathbf{c}$. The first one is easy,

$$Q(\xi) = \sum_{w\in W} \xi^{\ell(w)}$$

where $\ell(w)$ is the length of w, namely the minimal number of simple reflections required to decompose w. In particular, $Q(q) = \#(B\backslash G)(\mathbb{F}_q)$. It shows up in the volume of double cosets in the Cartan decomposition.

The Harish-Chandra $\mathbf{c}$-function is more interesting. By definition, for a regular χ,

$$\mathbf{c}(\chi) = \prod_{\alpha\in R^+} \mathbf{c}(\alpha, \chi), \quad \mathbf{c}(\alpha, \chi) := \frac{1 - q^{-1}\chi(\alpha^\vee)}{1 - \chi(\alpha^\vee)}$$

It satisfies the following properties, which follow from the definition

(1) $\mathbf{c}(\alpha, \chi) = \mathbf{c}(w\alpha, w\chi)$ for all $w \in W_T$.
(2) $\mathbf{c}(\chi^{-1}) = \mathbf{c}(w_0\chi)$, where w_0 is the longest element in W_T.
(3) $|\mathbf{c}(\chi)|^{-2}$ is an analytic, W_T-invariant function on $\hat{T}^1$.

Let $\chi \in \hat{T}^1$, so $\chi^{-1} = \bar{\chi}$, then

$$\begin{aligned}
|\mathbf{c}(\chi)|^2 &= \prod_{\alpha\in R^+} \mathbf{c}(\alpha, \chi)\overline{\mathbf{c}(\alpha, \chi)} \\
&= \prod_{\alpha\in R^+} \frac{1 - q^{-1}\chi(\alpha^\vee)}{1 - \chi(\alpha^\vee)} \cdot \frac{1 - q^{-1}\chi^{-1}(\alpha^\vee)}{1 - \chi^{-1}(\alpha^\vee)} \\
&= \prod_{\alpha\in R} \gamma(\chi(\alpha^\vee)), \quad \gamma(z) = \frac{1 - q^{-1}z^{-1}}{1 - z}
\end{aligned}$$

Here, γ is the gamma factor from Tate's thesis, up to an explicit constant. The product over all roots can be interpreted as taking the adjoint representation, and we can write $|\mathbf{c}(\chi)|^{-2} = \gamma(0, \chi, \mathrm{Ad})^{-1}$. The occurrence of the γ-factor is a general feature of the Plancherel formula for groups.

Remark 2.15. If χ is not regular, then $\mathbf{c}(\chi)$ has a pole. Macdonald's formula still makes sense by rational continuation, since we will prove that $\chi \mapsto \omega_\chi(t)$ for a fixed t is a rational function on $\hat{T}^1$.

Remark 2.16. One consequence of the formula is that the Plancherel measure is supported on $\hat{T}^1$, so the complementary series does not contribute to $L^2(K\backslash G/K)$. One way to understand this is their matrix coefficients are "very not square integrable". For example, the matrix coefficients for the trivial representation are the constant functions, which do not decay at all on $K\backslash G/K$, so it is unreasonable to expect the trivial representation as a component of $L^2(K\backslash G/K)$.

The precise interpretation is that a representation contributes to the Plancherel measure if and only if all of its matrix coefficients are in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. The proof of this depends on the asymptotic properties of the $\mathbf{c}$-function and harmonic analysis in $L^p(G)$. This phenomenon was first observed and proved for $\mathrm{SL}_2(\mathbb{R})$ in [KS60], and later generalized to all semisimple groups over local fields in [Cow78]. Note that this is very different from the abelian case, say $G = \mathbb{R}$, where the eigenfunctions are not integrable at all.

2.5. **Proof and example.** We now explain some aspects of the proof and make it explicit in the case $\mathbf{G} = \mathrm{SL}_2$. This is both for concreteness purpose and because many parts of the general picture reduces to a specific calculation on SL_2.

2.5.1. *Spherical functions.* Let $\chi \in \hat{T}$, and let $\pi_\chi \in \Pi^\circ_{\mathrm{sm}}(G)$ be the smooth representation labelled by χ. The space of invariants π_χ^K is 1-dimensional, so its dual is also 1-dimensional. Fix vectors $v^\circ \in \pi_\chi^K$ and $\tilde{v}^\circ \in \tilde{\pi}_\chi^K$ so that $\langle v^\circ, \tilde{v}^\circ \rangle = 1$, then the zonal spherical function ω_χ is just the matrix coefficient

$$\omega_\chi(g) = \langle \pi_\chi(g) v^\circ, \tilde{v}^\circ \rangle$$

This proves existence. The uniqueness holds since the G-invariant subspace of $C(G)$ generated by ω_χ is spherical and has character χ under the Satake isomorphism.

The function can be described more explicitly. Given an Iwasawa factorization $g = tnk$, define

$$\phi_\chi(g) = (\chi\delta^{\frac{1}{2}})(t)$$

Then a short calculation shows that

$$\omega_\chi(g) = \int_K \phi_\chi(kg)dk$$

satisfies all the defining properties of ω_χ. This gives an explicit description ω_χ.

The relation between the two constructions above is *parabolic induction.* Since $B = T \ltimes N$, we can extend the character $\chi\delta^{\frac{1}{2}}$ to B by making it trivial on N. The parabolic induction of χ has the underlying vector space given by

$$\mathrm{Ind}_T^G \chi := \{f : G \to \mathbb{C} \mid f(bg) = (\chi\delta^{\frac{1}{2}})(b)f(g) \text{ for all } b \in B,\ g \in G\}^{\mathrm{sm}}$$

with G acting by right translation. The action is smooth by definition. The subspace of K-invariant vectors is 1-dimensional by the Iwasawa decomposition, and it is generated by ϕ_χ. It follows that $\mathrm{Ind}_T^G \chi$ has a unique unramified subquotient.

It is a general fact that Ind_T^G is irreducible if and only if χ is regular: $\chi \neq w\chi$ for all non-trivial $w \in W_T$. However, we are taking a spherical matrix coefficient, so we only care about the K-invariant vectors. Taking the linear form $\int_K dk$ recovers the integral formula for ω_χ.

Remark 2.17. The only place where the Satake isomorphism was used was the statement that all zonal spherical functions have the form ω_χ. This is essentially the injectivity part.

2.5.2. *Intertwining operator.* We now derive Macdonald's formula for ω_χ, following the approach of Casselman. The first piece of input is the asymptotic expansion of a general matrix coefficient due to Jacquet and Casselman.

Theorem 2.18. *Let* $\pi = \mathrm{Ind}_T^G \chi$*.*

(1) *For each* $w \in W_T$*, there is a* B*-equivariant linear form* $\Omega_w : \pi \to \mathbb{C}(\delta^{\frac{1}{2}} w\chi)$ *such that the direct sum* $\bigoplus_{w\in W_T} \Omega_w : \pi|_B \to \bigoplus_{w \in W_T} \mathbb{C}(\delta^{\frac{1}{2}} w\chi)$ *identifies the right hand side with the* N*-coinvariants of* $\pi|_B$*.*

(2) *If* $|\chi| \ll 1$*, then the following integral converges*

$$\Omega_w(f) = \int_{N\cap w^{-1}Nw\backslash N} f(wn)dn$$

and defines by analytic continuation a normalized operator Ω_w *for all regular* χ*.*

(3) *For all* $v \otimes \tilde{v} \in \pi \otimes \tilde{\pi}$*, there exists* $\varepsilon > 0$ *such that whenever* $t \in T$ *satisfies* $|\alpha(t)| < \varepsilon$ *for all* $\alpha \in R^+$*, we have*

$$\langle \pi(t)v, \tilde{v}\rangle = \delta(t)^{\frac{1}{2}} \cdot Q(q^{-1})^{-1} \sum_{w\in W_T} \Omega_w(v)\widetilde{\Omega}_w(\tilde{v}) w\chi(t)$$

About the proof. The construction $(\pi|_B)_N$ is called the *Jacquet functor*. Both part (1) and part (2) can be motivated by Mackey theory applied to decomposing $\mathrm{Res}_B^G \mathrm{Ind}_B^G \chi$. Moreover, by Frobenius reciprocity, Ω_w is equivalent to an *intertwining operator*

$$T_w : \mathrm{Ind}_T^G \chi \to \mathrm{Ind}_T^G w\chi$$

The analytic continuation of the intertwining operator, as well as its zeroes and poles is very important in local harmonic analysis. An example for SL_2 will be done later.

Finally, part (3) is a general fact relating matrix coefficients of π with the matrix coefficients of its Jacquet module. The only additional input is to compute the normalization factor $Q(q^{-1})^{-1}$. This is done by computing both sides at a specific function with a small support. $\square$

When taking v and $\tilde{v}$ to be spherical, part (1) shows that the asymptotic expansion of part (3) is actually exact on all $t \in X_*(T)^+$. As a result, all we need to compute is the intertwining operator. By a formal calculation, there is a product decomposition

$$T_{w_1 w_2} = T_{w_1} T_{w_2} \quad \text{if } \ell(w_1 w_2) = \ell(w_1)\ell(w_2)$$

This is the Gindikin–Karpelevič formula, which is very natural when expressed expressed in this representation theoretic language. The following example is found in [Kna03].

Example 2.19. For the group $G = \mathrm{SL}_3(\mathbb{R})$, the integral is

$$\iiint_{\mathbb{R}^3} (1 + x^2 + z^2)^{-a}(1 + y^2 + (xy - z)^2)^{-b} dxdydz$$

By completing the square on the second term and changing the variable $y \to y \cdot \frac{\sqrt{1+x^2+z^2}}{1+x^2}$, the integral simplifies. With another change of variable, it becomes

$$\iiint_{\mathbb{R}^3} (1 + x^2)^{-a}(1 + y^2)^{-b}(1 + z^2)^{-a-b+\frac{1}{2}} dxdydz$$

which factors and can be evaluated in terms of the Γ-function.

The product formula expresses $\Omega_w(v^\circ)$ as a product over simple reflections in w. Each term is a computation on a rank 1 group, which in the split case is just SL_2. The result is a term in the product of $\mathbf{c}(w\chi)$. The dual term $\tilde{\Omega}_w(\tilde{v})$ consists of the other terms. Together, this gives Macdonald's formula (ω).

2.5.3. *Plancherel formula.* We will now prove the inner product formula, which formally implies the other results. By linearity, it suffices to prove that for all $\lambda, \mu \in X_*(T)^+$, we have the identity

$$\frac{Q(q^{-1})}{\# W_T} \int_{\hat{T}^1} \hat{c}_\lambda(\chi)\overline{\hat{c}_\mu(\chi)}\, |\mathbf{c}(\chi)|^{-2}\, d\chi = \begin{cases} 0 & \lambda \neq \mu \\ \mu_G(K\lambda K) & \lambda = \mu \end{cases}$$

where c_λ, c_μ are the indicator functions we have seen before.

We first derive another expression for the spherical transform. Let $f \in C_c^\infty(K\backslash G/K)$, then

$$\begin{aligned} \hat{f}(\chi) &= \int_{K\backslash G/K} f(x)\overline{\omega_\chi(x)} dx \\ &= \int_{K\backslash G/K} f(x) \int_K \overline{\phi_\chi(kx)} dk dx \\ &= \int_G f(g)\overline{\phi_\chi(g)} dg \end{aligned}$$

The Iwasawa decomposition gives an integration formula

$$\int_G f(g) dg = \int_T \int_N \int_K f(tnk) dk dn dt$$

This is not entirely straightforward since the Borel $B = TN$ is not unimodular, and one has to be careful with measures. Applying this, we get

$$\begin{aligned} \int_G f(g)\overline{\phi_\chi(g)} dg &= \int_T \int_N \int_K f(tnk)\overline{\phi_\chi(tnk)} dk dn dt \\ &= \int_T \overline{\chi(t)}\delta(t)^{\frac{1}{2}} \int_N f(tn) dn dt \\ &= \int_T \mathcal{S}f(t)\overline{\chi(t)} dt \end{aligned}$$

This is exactly the Fourier transform of $\mathcal{S}f$ evaluated at χ.

Proposition 2.20. $\hat{f}(\chi) = \mathcal{F}\mathcal{S}f(\chi)$.

Remark 2.21. This proposition suggests that we should relate harmonic analysis on the torus T with harmonic analysis on G. While we do not explicitly pursue it here, there is such a relation between the Plancherel measures, with the intervening coefficients obtained from asymptotic expansions of matrix coefficients. This and Macdonald's formula explains the role of the Harish-Chandra $\mathbf{c}$-function in the Plancherel formula.

There are various technical complications since the intertwining operators do not converge on the unitary axis. We will see this for $G = \mathrm{SL}_2$. For the group case, this is carefully done in Waldspurger's proof of the full Plancherel formula [Wal03]. In the general spherical setting, this is the content of [SV17, Part 3].

The left hand side of the identity we need to prove is an $L^2(\hat{T}^1, d\chi)$ inner product between the functions

$$f_\lambda(\chi) = \hat{c}_\lambda(\chi), \quad g_\mu(\chi) = \frac{Q(q^{-1})}{\#W_T}\hat{c}_\mu(\chi)\left|\mathbf{c}(\chi)\right|^{-2}$$

By the usual Plancherel identity and the above proposition,

$$\langle f_\lambda, g_\mu\rangle_{L^2(\hat{T}^1)} = \langle \mathcal{S}c_\lambda, \mathcal{F}^{-1}g_\mu\rangle_{L^2(X_*(T))}$$

We now need to compute $\mathcal{F}^{-1}g_\mu$, which amounts to decomposing $g_\mu(\chi)$ into a Laurent series in χ.

From now on, we assume χ is unitary. Let $v_\mu = \mu_G(K\mu K)$, then by definition, $\hat{c}_\mu(\chi) = v_\mu\omega_{\bar{\chi}}(\mu) = v_\mu\omega_{\chi^{-1}}(\mu)$. Further assuming χ is regular, Macdonald's formula implies

$$\begin{aligned} g_\mu(\chi) &= \frac{v_\mu\delta(\mu)^{\frac{1}{2}}}{\#W_T}\sum_{w\in W_T}\frac{\mathbf{c}(w\chi^{-1})}{\left|\mathbf{c}(\chi)\right|^2}\langle w\chi,\mu\rangle \\ &= \frac{v_\mu\delta(\mu)^{\frac{1}{2}}}{\#W_T}\sum_{w\in W_T}\mathbf{c}(w\chi)^{-1}\langle w\chi,\mu\rangle \end{aligned}$$

Recall the definition of the $\mathbf{c}$-function,

$$\mathbf{c}(\chi)^{-1} = \prod_{\alpha\in R^+}\frac{1-\chi(\alpha^\vee)}{1-q^{-1}\chi(\alpha^\vee)} = \prod_{\alpha\in R^+}\Big(1+(q-1)\sum_{n>0}q^{-n}\langle\chi, n\alpha^\vee\rangle\Big)$$

Expanding the product, we see that $\mathbf{c}(\chi)^{-1} = 1 + \sum_{\lambda>0}a_\lambda\langle\chi,\lambda\rangle$ for some coefficients a_λ. As a result,

$$g_\mu(\chi) = \frac{v_\mu\delta(\mu)^{\frac{1}{2}}}{\#W_T}\left(\sum_{w\in W_T}\langle\chi, w\mu\rangle + \sum_{w\in W_T}\sum_{\lambda>0}a_\lambda\langle\chi, w(\mu+\lambda)\rangle\right)$$

It follows that $\mathcal{F}^{-1}g_\mu$ is supported on the W_T-orbit of cocharacters which are $\geq\mu$. On the other hand, $\mathcal{S}c_\lambda$ is supported on W_T-orbits of characters which are $\leq\lambda$. If $\mu\not\leq\lambda$, then the supports are disjoint, so the inner product is 0. If $\mu>\lambda$, then we obtain the same result by switching λ and μ. Finally, if $\lambda=\mu$, then

$$\langle\mathcal{S}c_\lambda, \mathcal{F}^{-1}g_\lambda\rangle_{L^2(X_*(T))} = \delta(\lambda)^{-\frac{1}{2}}\cdot\frac{v_\lambda\delta(\lambda)^{\frac{1}{2}}}{\#W_T}\sum_{w\in W_T}\mathbf{1}[w\lambda] = v_\lambda$$

This concludes the proof.

2.5.4. *Example:* SL_2. For $G = \mathrm{SL}_2(F)$, we have $\hat{T} = \mathbb{C}^\times$. The complex number z determines the unramified character $\mathrm{diag}(t, t^{-1}) \mapsto z^{2\nu(t)}$. It is often more convenient to write $z = q^{\frac{s}{2}}$, so the character becomes

$$\chi_s : \left(\begin{smallmatrix} t & \\ & t^{-1}\end{smallmatrix}\right) \mapsto |t|^s$$

The Weyl group has order two, with the non-trivial element w acting by $s\mapsto -s$. The character χ_s is unitary when $\mathrm{Re}(s) = 0$. Finally, the character δ sends $\mathrm{diag}(t,t^{-1})$ to $|t|^2$.

On the induced representation $\mathrm{Ind}_K^G\chi_s$, the two linear forms are $\Omega_1 : f\mapsto f(1)$ and

$$\Omega_{-1} : f \mapsto \int_N f(wn)dn = \int_F f\big(\left(\begin{smallmatrix} & -1 \\ 1 & x\end{smallmatrix}\right)\big)dx$$

To compute the value of f, we need to know the Iwasawa factorization. In this case, it can be done explicitly.

$$\begin{pmatrix} & -1 \\ 1 & x\end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ & x\end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix}$$

In particular, if $f = \phi_s$ is the normalized right K-invariant vector, then

$$\begin{aligned}\Omega_{-1}\phi_s &= 1 + \int_{F-\mathcal{O}} |x|^{-(s+1)}\,dx \\ &= 1 + \sum_{n\geq 1} q^{-(s+1)n}(q^n - q^{n-1}) \\ &= 1 + (1-q^{-1})\sum_{n\geq 1} q^{-sn} = \frac{1-q^{-1}q^{-s}}{1-q^{-s}}\end{aligned}$$

This manipulation only makes sense if $\mathrm{Re}(s) > 0$, so it just fails to be defined on the unitary axis. However, the analytic continuation gives the required expression.

Macdonald's formula now reads

$$\omega_s(\mathrm{diag}(t, t^{-1})) = \frac{|t|}{1+q^{-1}}\left(\frac{1-q^{s-1}}{1-q^s}\,|t|^s + \frac{1-q^{-(s+1)}}{1-q^{-s}}\,|t|^{-s}\right)$$

On the other hand, by our computation earlier,

$$(\mathcal{F}\mathcal{S}c_n)(q^s) = q^n\sum_{|k|\leq n} q^{sk} - q^{n-1}\sum_{|k|\leq n-1} q^{sk} = q^{2n}(1+q^{-1})\,\omega_s(\mathrm{diag}(\varpi^n, \varpi^{-n}))$$

This verifies the above computations in a different way.

Remark 2.22. Let $G = \mathrm{SL}_2(\mathbb{R})$, then the analogous construction would take $K = \mathrm{SO}_2$, so $K\backslash G/K$ can be identified with the line segment $[0,\infty)$. The infinitesimal version of $\mathcal{H}^\circ$ is a space of linear differential operators (centre of $\mathcal{U}\mathfrak{g}$), generated by the hyperbolic Laplacian $Df = f''(r) + 2\coth(r)f'(r)$, so the zonal spherical functions are solutions to the differential equation $Df = \lambda f$. By studying the singularity at ∞, Harish-Chandra was able to obtain asymptotic expansions for f analogous to Macdonald's formula [HC58a]. The coefficients are expressed as certain integrals analogous to Ω_w, though he was unable to evaluate them explicitly beyond the rank 1 case. Assuming certain asymptotic properties, he proved the Plancherel formula [HC58b]. These conjectures were later resolved in part thanks to the Gindinkin–Karpelevič formula.

Intertwining operators did not explicitly appear in this part of Harish-Chandra's work, but they were already part of harmonic analysis under different names. Again for $G = \mathrm{SL}_2(\mathbb{R})$, the intertwining operator turns out to be the integral operator

$$Tf = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^s}\,dy$$

If $s = 1$, this is essentially the Hilbert transform. It is easy to obtain analytic continuation if f is smooth, but the general L^2-case requires Fourier analysis. Boundedness properties on L^p are directly related to the construction of the complementary series, initially observed by Kunze–Stein [KS60].

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