NORMS OF SINGULAR MODULI

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Abstract. The j-function is a special modular function on the upper half plane. Its values at imaginary quadratic points are called singular moduli. It is a classical result that they are always algebraic integers. In 1983, Zagier discovered an explicit formula for the prime factorizations of their norms. This talk will sketch his analytical proof of this result, after recalling the necessary background material.

This is the expanded version of the notes for my talk on the work of Gross and Zagier [GZ85] on the norm of differences of two singular moduli with relatively prime discriminants, given at the Princeton Graduate Students' Seminar.

0. Motivation

(1) This talk will explain the approximation

$$e^{\pi\sqrt{163}} \approx 744 + (2^6 \cdot 3 \cdot 5 \cdot 23 \cdot 29)^3$$

(2) The techniques used here are the same as the ones used in [GZ86] and [GKZ87] to relate the height pairing of Heegner points to central derivatives of L-functions, except we are working in the simpler level 1 case. The second paper is more relevant since we will be dealing with points of coprime discriminants.

1. Some modular functions

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half-plane. Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{R})^+$, define the weight-$k$ slash operator $f|_k\gamma$ by

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2}(cz+d)^{-k} f\Big(\frac{az+b}{cz+d}\Big)$$

We say that f transforms like a modular form of weight k if $f|_k\gamma = f$ for all $\gamma \in SL_2(\mathbb{Z})$. In this case, taking γ to be $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ shows that $f(z+1) = f(z)$, so if f is smooth, then it has a Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(y) e^{2\pi i n z} = \sum_{n\in\mathbb{Z}} a_n(y) q^n, \quad a_n(y) = \int_0^1 f(z) e^{-2\pi i n z} dx$$

where $y = \mathrm{Im}(z)$ and $q = e^{2\pi i z}$. If f is holomorphic, then $a_n(y)$ is just a constant. A modular form is a holomorphic function which transforms like a modular form and satisfies some growth conditions at infinity, namely $a_n = 0$ if $n < 0$.

The classical examples of modular forms are the Eisenstein series

$$G_k(z) = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(mz+n)^k} = \sum_{n=1}^{\infty}\frac{1}{n^k} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^k}$$

First suppose $k \geq 4$. The first sum converges absolutely. The slash operator changes the order of summation, so G_k is a modular form. Note in addition that G_k is 0 unless k is even. If $k = 2$, then G_2 is defined using the second expression. It is not modular, but we will see how it can be modified into a modular function. In either case, by applying Poisson summation to the inner sum, we get the q-expansion

$$G_k(z) = \zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n) q^n$$

Here, $\sigma_s(n) = \sum_{d|n} d^s$ is the divisor-sum function. Define E_k to be G_k normalized to have constant term 1, so by the classical formulae for zeta-values,

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where B_k are the Bernoulli numbers.

There is a good classification result for holomorphic modular forms. In particular, weight 2 holomorphic modular forms do not exist. But if we drop the holomorphic assumption, then a weight 2 form can be constructed via analytic continuation. Consider

$$G_2(z, s) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{y^s}{(mz+n)^2 \left|mz+n\right|^{2s}}$$

where s is a complex variable. If $\mathrm{Re}(s) > 2$, then the sum converges absolutely, giving a non-holomorphic modular form of weight 2. Its Fourier expansion can be computed as before and is given by

$$\zeta(2s+2)y^s - \frac{\sqrt{\pi}s\Gamma\big(s+\frac{1}{2}\big)\zeta(2s+1)}{\Gamma(s+2)} y^{-1-s} + \sum_{n\neq 0} a_n(y,s) e^{2\pi i n z}$$

with

$$a_n(y,s) = \frac{2\pi^{s+1} \left|n\right|^{-s-\frac{1}{2}}}{\Gamma(s+2)} \sigma_{s+1}(n) e^{2\pi n y} \Big(\frac{\partial}{\partial y} - 2\pi n\Big)\big(\sqrt{y} K_{s+\frac{1}{2}}(2\pi \left|n\right| y)\big)$$

where $K_\nu(x)$ is the (modified) Bessel function defined by

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t} \cosh(\nu t) dt$$

Remark (on normalization). This is the unique solution to the 2nd order linear ODE

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - (x^2+\nu^2)y = 0$$

on $(0,\infty)$ which satisfies $y(x) \sim \sqrt{\frac{\pi}{2}} x^{-1/2} e^{-x}$ as $x \to +\infty$.

The point is that all terms can be analytically continued to $s = 0$, giving a function

$$G_2^*(z) = \lim_{s\to 0^+} G_2(z,s)$$

which transforms like a modular form of weight 2 by the principle of analytic continuation. One checks using the q-expansion that $G_2^*(z) = G_2(z) - \frac{\pi}{2y}$. We will do something like this many times later.

Define the modular discriminant $\Delta = \frac{E_4^3 - E_6^2}{1728}$. It follows easily from the Fourier expansions and elementary number theory that Δ has an integral q-expansion. It is a classical fact that Δ has a product expansion

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

which can be proven using the G_2^* introduced earlier. From this, it follows that Δ is nowhere vanishing on $\mathbb{H}$. Finally, define j-invariant to be $j = E_4^3/\Delta$. This is a holomorphic modular function of weight 0 with a simple pole at infinity. In fact, the quotient $SL_2(\mathbb{Z})\backslash\mathbb{H}$ has a complex structure whose compactification by adding ∞ is isomorphic to $\mathbb{P}^1$ (one need to be careful around the points $i, \rho = \frac{1+i\sqrt{3}}{2}$ and ∞, which have non-trivial stabilizers, but we will not go into details). The j-function gives an explicit isomorphism $SL_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{P}^1$. It follows that any meromorphic function f on $\mathbb{H}$ invariant under $SL_2(\mathbb{Z})$ is a rational function of j. In particular, if the only pole of f is at ∞, then f is a polynomial in j. By comparing the q-expansions, the coefficients of this polynomial lies in the ring generated by the principal part of f at ∞.

The q-expansion of the j-function is

$$j(z) = \frac{1}{q} + 744 + 196884q + \cdots = \frac{1}{q} + \sum_{n\geq 0} c_n q^n$$

It is clear that c_n are positive integers. The invariance of j under $SL_2(\mathbb{Z})$ means we have very precise information about its behaviour near the real line. Using techniques such as the Hardy-Littlewood circle method, one can prove an asymptotics

$$c_n \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$$

and extract an explicit bound on c_n from the proof. More precise analysis done by [BP05] shows that c_n is always less than the above asymptotics if $n \geq 1$. The upshot is that j is computable to arbitrary precision. If we try to compute $j\left(\frac{1+i\sqrt{163}}{2}\right)$, we find that

$$j\left(\frac{1+i\sqrt{163}}{2}\right) = -e^{\pi\sqrt{163}} + 744 + \epsilon$$

where $\epsilon \leq 10^{-12}$. We will see next that the left hand side is an integer, which explains why $e^{\pi\sqrt{163}}$ is almost an integer.

2. Singular moduli

Let τ be an imaginary quadratic point in $\mathbb{H}$, then τ satisfies $a\tau^2 + b\tau + c = 0$ for some integers a, b, c with $\gcd(a, b, c) = 1$. Its discriminant is $d = b^2 - 4ac$. By rearranging the equation, we can find a matrix $M \in GL_2(\mathbb{Z})^+$ such that $M\tau = \tau$. We are therefore led to investigating the function $j(M\tau) = j|_0 M(\tau)$.

Proposition 1. *The function $j|_0 M$ is integral over $\mathbb{Z}[j]$.*

Proof. Let $m = \det M$, and let S_m be the set of matrices with determinant m. The double coset space $SL_2(\mathbb{Z})\backslash S_m/SL_2(\mathbb{Z})$ is represented by

$$\{M_1, \cdots, M_h\} = \left\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : ad = \det M,\ 0 \leq b < d\right\}$$

Consider the expression

$$\prod_{i=1}^{h}(x - j|_0 M_i) = x^h + P_{h-1}x^{h-1} + \cdots + P_0$$

where $P_0, \cdots, P_{h-1}$ are symmetric polynomials of $\{j|_0 M_i\}_i$. By construction, they are $SL_2(\mathbb{Z})$-invariant. Since $j|_0\gamma = j$ for all $\gamma \in SL_2(\mathbb{Z})$, it follows that j is a root of this polynomial. Observe that

$$j|_0\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = j\left(\frac{az+b}{d}\right) = \sum_{n=-1}^{\infty} c_n e^{2\pi i bn/d} q^{an/d} \in \mathbb{Z}[\mu_d][[q^{-1/d}, q^{1/d}]]$$

It follows that P_i are all in the above ring. But since they are modular of weight 0, we can only have integral powers of d. Furthermore, by applying a transformation of the form $\zeta_d \mapsto \zeta_d^r$ with $\gcd(r, d) = 1$, we see that the coefficients of the q-expansions of P_i are invariant. Therefore, P_i have integral q-expansions at infinity, so they are integral polynomials of j. □

Corollary 2. *If τ is imaginary quadratic, then $j(\tau)$ is an algebraic integer. They are called* singular moduli.

This proof combined with the fact that j can be computed to arbitrary precision gives an algorithm to compute the exact values of singular moduli (though it is not very efficient).

Example 3. Suppose we want to compute $j\left(\frac{1+i\sqrt{7}}{2}\right)$. We first note that

$$\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} \frac{1+i\sqrt{7}}{2} = \frac{1+i\sqrt{7}}{2}$$

According to the proof, we need to consider the polynomial

$$\begin{aligned}\left(x - j|_0\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)\right) \cdot \left(x - j|_0\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)\right) \cdot \left(x - j|_0\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) &= (x - j(2z))\left(x - j\left(\frac{z+1}{2}\right)\right)\left(x - j\left(\frac{z}{2}\right)\right) \\ &= x^3 - P_2(z)x^2 + P_1(z)x - P_0(z)\end{aligned}$$

One can check using the q-expansion that

$$\begin{aligned}
P_2 &= j^2 - 1488j + 16200\\
P_1 &= 1488j^2 + 40773375j + 8748\,000\,000\\
P_0 &= -j^3 + 162000j^2 - 8748\,000\,000j + 157\,464\,000\,000\,000
\end{aligned}$$

It follows that $j\left(\frac{1+i\sqrt{7}}{2}\right)$ is the root of the polynomial $\Psi_2(x,x)$, where

$$\begin{aligned}
\Psi_2(x,j) = x^3 - (j^2 - 1488j + 16200)x^2 + (1488j^2 + 40773375j + 8748\,000\,000)x\\
- (-j^3 + 162000j^2 - 8748\,000\,000j + 157\,464\,000\,000\,000)
\end{aligned}$$

One can factor $\Psi_2(x,x) = -(x-8000)(x+3375)^2(x-1728)$. By numerical computation, $j\left(\frac{1+i\sqrt{7}}{2}\right)$ is close to -3375, so it must be -3375. The other roots are $j(i) = 1728$ and $j(i\sqrt{2}) = 8000$.

Remark. The exponent is related to the number of matrices of the given determinant (in this case 2) fixes the point, but we need to mod out the contribution from the automorphism group in $SL_2(\mathbb{Z})$.

Remark. It is a general fact that $\Psi_n(x,y) = \Psi_n(y,x)$.

Here are some more examples

$$\begin{aligned}
&j(\rho) = 0,\ j\left(\frac{1+i\sqrt{67}}{2}\right) = -2^{15}3^3 5^3 11^3,\ j\left(\frac{1+i\sqrt{163}}{2}\right) = -2^{18}3^3 5^3 23^3 29^3\\
&j(i\sqrt{5}) = 2^6\sqrt{5}^3\left(\frac{13+5\sqrt{5}}{2}\right)^3,\ j(i\sqrt{6}) = 2^6 3^3(1+\sqrt{2})^2(5+2\sqrt{2})^3\\
&j\left(\frac{1+i\sqrt{67}}{2}\right) - j\left(\frac{1+i\sqrt{163}}{2}\right) = 2^{15}3^7 5^3 7^2\cdot 13\cdot 139\cdot 331
\end{aligned}$$

One can say a lot more about the fields of definition of $j(\tau)$ and the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on them. One consequence is that if $\mathbb{Q}(\sqrt{d})$ has class number 1, then $j(\tau)$ is an actual integer, as seen above.

Note that the numbers are very smooth. This is made explicit in the following theorem describing their factorizations, which is the main theorem of the talk.

Theorem 4 (Gross-Zagier)**.** *Let d_1 and d_2 be coprime fundamental discriminants, then*

$$\Bigg(\prod_{\substack{\tau_1,\tau_2\in SL_2(\mathbb{Z})\backslash\mathbb{H}\\ \mathrm{disc}(\tau_i)=d_i}} (j(\tau_1)-j(\tau_2))\Bigg)^{\frac{8}{w_1w_2}} = \pm \prod_{\substack{x^2<D\\ x^2\equiv D \pmod 4}} \prod_{n\mid\frac{D-x^2}{4}} n^{-\epsilon(n)}$$

where the left product is over pairs (τ_1,τ_2) of $SL_2(\mathbb{Z})$-orbits of the given discriminants, w_i is the number of roots of unities in $\mathbb{Q}(\sqrt{d_i})$, and ϵ is defined on primes by

$$\epsilon(\ell) = \begin{cases} 0 & \text{if } \left(\frac{d_1d_2}{\ell}\right) = -1\\ \left(\frac{d_1}{\ell}\right) & \text{if} \ell\nmid d_1\\ \left(\frac{d_2}{\ell}\right) & \text{if } \ell\nmid d_2\end{cases}$$

and extended multiplicatively.

Corollary 5. *If $\ell | j(\tau_1)-j(\tau_2)$, then $\left(\frac{d_i}{\ell}\right) = -1$ for $i=1,2$, and $\ell \leq \frac{d_1d_2}{4}$.*

Remark. (1) ϵ is well-defined.

(2) d is a fundamental discriminant if it is the discriminant of a quadratic field. This corresponds to the maximal order case and is assumed for simplicity. There are similar formulae for norms of $\Phi_m(j(\tau_1),j(\tau_2))$ for $m\geq 1$, which allows one to generalize the formula to situations where d_1 and d_2 may not be coprime or fundamental (see section 4 of [GZ85]).

(3) If $d_1,d_2 < -4$, then the left hand side is the sqaure of the norm of the algebraic integer $j(\tau_1)-j(\tau_2)$. Otherwise, we may have taken a square or cube root.

(4) If one were to define j as an isomorphism from $X_0(1)$ to $\mathbb{P}^1$, then it is natural to specify that it has a simple pole of residue 1 at infinity. This still leaves us with a translational degree of freedom, which is eliminated by taking differences.

3. Sketch of proof

Fix coprime fundamental discriminants $d_1, d_2 < 0$. We are trying to prove $T = -S$, where

$$T = \frac{4}{w_1 w_2} \sum_{\mathrm{disc}(\tau_i)=d_i} \log |j(\tau_1) - j(\tau_2)|^2$$

$$S = \sum_{\substack{x^2<D \\ x^2 \equiv D \pmod 4}} \sum_{n \mid \frac{x^2-D}{4}} \epsilon(n) \log n$$

(1) **Re-write S.**

Let $D = d_1 d_2$, and let $K = \mathbb{Q}(\sqrt{D})$, then its ring of integers consists of numbers of the form $\frac{1}{2}(a + b\sqrt{D})$, where $a^2 \equiv b^2 D \pmod 4$, and its different is $\mathfrak{d} = (\sqrt{D})$. Denote by $'$ the non-trivial automorphism of K. Let χ be the genus character of the narrow class group corresponding to the extension $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})/\mathbb{Q}(\sqrt{D})$, which is unramified at all finite places. Then one can verify that $S = a_1$, where

$$a_n = \sum_{\substack{v \in \mathfrak{d}^{-1} \\ v \gg 0 \\ \mathrm{Tr}(v)=n}} \sum_{\mathfrak{n} \mid v\mathfrak{d}} \chi(\mathfrak{n}) \log \mathbf{N}\mathfrak{n}$$

This step is elementary and corresponds to the bijection $(x, n) \leftrightarrow (v = \frac{x+\sqrt{D}}{2\sqrt{D}}, \mathfrak{n})$, where $\mathbf{N}\mathfrak{n} = n$.

This can be further re-written as

$$a_n = \sum_{\substack{v \in \mathfrak{d}^{-1} \\ v \gg 0 \\ \mathrm{Tr}(v)=n}} \frac{\partial}{\partial s}\Big|_{s=0} \sigma_{s,\chi}(v\mathfrak{d})$$

where $\sigma_{s,\chi}$ is a divisor sum function

$$\sigma_{s,\chi}(\mathfrak{a}) = \sum_{\mathfrak{n} \mid \mathfrak{a}} \chi(\mathfrak{n}) (\mathbf{N}\mathfrak{n})^s$$

(2) **Relate a_n to something which transforms as a modular form of weight 2.**

We do this using Hilbert modular forms, which are higher dimensional generalizations of classical modular forms. More precisely, they are functions $f : \mathbb{H}^2 \to \mathbb{C}$ such that

$$f(\gamma z) = (cz_1 + d)^k (c' z_2 + d')^k f(z)$$

for $\gamma \in SL_2(\mathcal{O}_K)$ (we only consider the parallel weight case). Since they are invariant under translation by the lattice $\mathcal{O}_K$, they have Fourier expansions of the form

$$f(z_1, z_2) = \sum_{v \in \mathfrak{d}^{-1}} c_v e^{2\pi i (v z_1 + v' z_2)}$$

where c_v may depend on y and y'. The inverse different $\mathfrak{d}^{-1}$ shows up since by definition, it is the dual lattice to $\mathcal{O}_K$ under the trace form. The transformation properties imply that $f(z, z)$ transforms like a modular form of weight $2k$ with Fourier expansion

$$f(z, z) = c_0 + \sum_{n=1}^{\infty} c_n e^{2\pi i n z}, \quad c_n = \sum_{\substack{v \in \mathfrak{d}^{-1} \\ v \gg 0 \\ \mathrm{Tr}(v)=n}} c_v$$

So we want to find an f such that $c_v(f) = \frac{\partial}{\partial s}\big|_{s=0} \sigma_{1+s,\chi}(v\mathfrak{d})$. This is the derivative of a divisor sum, so we are led to consider the Eisenstein series

$$E_s(z_1, z_2) = \sum_{\mathfrak{a} \in \mathrm{Cl}_K} \chi(\mathfrak{a}) (\mathbf{N}\mathfrak{a})^{1+2s} \sideset{}{'}\sum_{(m,n) \in \mathfrak{a}^2/\mathcal{O}_K^\times} \frac{y_1^s y_2^s}{(mz_1 + n)(m'z_2 + n') |mz_1 + n|^{2s} |m'z_2 + n'|^{2s}}$$

which converges absolutely if $\mathrm{Re}(s)$ is sufficiently large. This is a non-holomorphic Eisenstein series of parallel weight 1 (we specify the value at each cusp, which are in bijection with ideal classes of $\mathcal{O}_K$). In particular, one need to verify that the expression being summed does not depend on the choice of $\mathfrak{a}$ in an (wide) ideal class. It is also standard to work out its Fourier expansion

$$\begin{aligned} E_s(z_1,z_2) =& L(1+2s,\chi)y_1^s y_2^s + D^{-\frac{1}{2}}L(s,\chi)\Phi_s(0)^2 y_1^{-s}y_2^{-s} \\ &+ \sum_{\substack{v\in\mathfrak{d}^{-1}\\ v\neq 0}} D^{-\frac{1}{2}}\Phi_s(vy_1)\Phi_s(v'y_2)y_1^{-s}y_2^{-s}\sigma_{-2s,\chi}(v\mathfrak{d})e^{2\pi i(vx_1+v'x_2)}\end{aligned}$$

where

$$\Phi_s(t) = \int_{-\infty}^{\infty}\frac{e^{-2\pi ixt}}{(x+i)(x^2+1)^s}dx$$

One can check that each term can be analytically continued to $s=0$, and that the series actually converges. The analytic continuation of $\Phi_s(t)$ is done by deforming the path of integration to a contour from $i\infty$ to i in the $\mathrm{Re}(z)<0$ region and then back in the $\mathrm{Re}(z)>0$ region. This also give the bound

$$|\Phi_s(t)| \ll |t|^{O(1)}e^{-2\pi|t|},\ |t|\to\infty$$

locally uniformly in s.

Remark. Using this expansion, it is easy to see that $E_0(z_1,z_2)=0$.

Let $F(z)=\frac{\sqrt{D}}{8\pi^2}\frac{\partial}{\partial s}\big|_{s=0}E_s(z,z)$, then F is a non-holomorphic modular form of weight 2, and a computation gives the q-expansion of F

$$F(z)=\frac{\sqrt{D}}{2\pi^2}(L(1,\chi)\log y + C_\chi) + \sum_{\substack{v\in\mathfrak{d}^{-1}\\ v\gg 0}}\frac{\partial}{\partial s}\Big|_{s=0}\sigma_{s,\chi}(v\mathfrak{d})e^{2\pi i\,\mathrm{Tr}(v)z} - \sum_{\substack{v\in\mathfrak{d}^{-1}\\ v>0>v'}}\sigma_{0,\chi}(v\mathfrak{d})\Phi(|v'|\,y)e^{2\pi i\,\mathrm{Tr}(v)z}$$

where

$$C_\chi = L'(1,\chi)+\left(\frac{1}{2}\log D-\log\pi-\gamma\right)L(1,\chi),\quad \Phi(t)=\frac{i}{2\pi}e^{-2\pi t}\frac{\partial}{\partial s}\Big|_{s=0}\Phi_s(-t)=\int_1^\infty e^{-4\pi tu}\frac{du}{u}$$

The coefficient of $e^{2\pi iz}$ is almost what we want.

(3) **Apply holomorphic projector.**

The space of functions of sufficiently slow growth at infinity which transform like modular forms of weight k carries an inner product defined by

$$(f,g)=\int_{SL_2(\mathbb{Z})\backslash\mathbb{H}}\overline{f(z)}g(z)y^k\frac{dxdy}{y^2}$$

This is non-degenerate on the space of cusp forms of weight k (modular forms for which $a_0=0$). Given a non-holomorphic modular form $f=\sum_{n\geq 0}a_n(y)q^n$, the functional $(f,-)$ on $S_k(SL_2(\mathbb{Z}))$ is therefore represented by a holomorphic cusp form $g=\sum_{n\geq 0}b_nq^n$. By pairing with special test functions (namely the Poincaré series), one can evaluate b_n in terms of $a_n(y)$. This operation is called holomorphic projection.

But in weight 2, there are no holomorphic modular forms, so holomorphic projection should give 0. The formulae for b_n also requires modifications because the Poincaré series are no longer absolutely convergent. Therefore, we need to use analytic continuation. The result is

Theorem 6 (Proposition 7.3 of [GZ85], Proposition 6.2 of [GZ86])**.** *Let $F(z)$ be a function on $\mathbb{H}$ which transforms like a weight 2 modular form. Suppose it satisfies the growth condition $F(z)=A\log y+B+O(y^{-\epsilon})$ for some $\epsilon>0$. Let $\sum_{n\in\mathbb{Z}}a_n(y)e^{2\pi inz}$ be its Fourier expansion, then for $n\geq 1$,*

$$\begin{aligned}\lim_{s\to 0}&\left(4\pi n\int_0^\infty a_n(y)e^{-4\pi ny}y^s dy+\frac{24A\sigma_1(n)}{s}\right)\\ &= 24A\left(\left(2\frac{\zeta'}{\zeta}(2)+1+\log(4n^2)\right)\sigma_1(n)-\sum_{d|n}d\log d\right)-24B\sigma_1(n)\end{aligned}$$

Proof. Consider the non-holomorphic weight 2 Poincaré series

$$P_{2,s}^{(m)} = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \big(y^s \cdot e^{2\pi i m z}\big)|_2 \gamma$$

Its Fourier coefficients can be computed using the usual methods:

$$a_n(P_{2,s}^{(m)}) = \delta_{mn} y^s + \sum_{c>0} c^{-2s+2} K(m,n,c) y^s \int_{-\infty+iy}^{\infty+iy} e\Big(-\frac{m}{c^2 z} - nz\Big) z^{-2} \, |z|^{-2s} \, dz$$

where $K(m,n,c)$ is the Kloosterman sum

$$K(m,n,c) = \sum_{\substack{a,d \in (\mathbb{Z}/c\mathbb{Z})^\times \\ ad=1}} e\Big(\frac{md+na}{c}\Big)$$

The classical Weil estimate gives $|K(m,n,c)| \ll_\epsilon c^{\frac{1}{2}+\epsilon}$ uniformly in n if $m > 0$ [Sar90]. It follows that if $m > 0$, the $P_{2,s}^{(m)}$ can be analytically continued to $s = 0$ to a holomorphic form of weight 2, which must then be identically 0 (if $m = 0$, we get the non-holomorphic E_2).

If $A = B = 0$, then the integral defining $(F, P_{2,s}^{(m)})$ converges absolutely when $\mathrm{Re}(s) > 1$. The integral itself can be computed using the standard unfolding trick. The formula follows from analytically continuing the result to $s = 0$. In the general case, we need to subtract off non-holomorphic forms with known behavior at infinity. Recall that

$$G_2(z,s) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{y^s}{(mz+n)^2 \, |mz+n|^{2s}}$$

Define $E_2(z) = \frac{6}{\pi^2} G_2(z,0)$ and $F_2(z) = \frac{6}{\pi^2} \frac{\partial}{\partial s}\big|_{s=0} G_2(z,s)$. Then one can check that

$$E_2(z) = 1 + O\big(y^{-1}\big), \quad F_2(z) = \log y + O\big(y^{-1}\log y\big)$$

So $F^* = F - AF_2 - BE_2$ satisfies $F^*(z) = O(y^{-\epsilon})$. We can apply the special case to this and recover the required formula using the Fourier expansion of $G_2(z,s)$ computed earlier. $\square$

We apply this theorem to $F(z)$ above to get finally

$$S = \lim_{s\to 0} \Bigg(\sum_{\substack{n > \sqrt{D} \\ n \equiv D \pmod 2}} \Big(\sum_{d | \frac{n^2-D}{4}} \epsilon(d) \Big) \Psi_s\Big(\frac{n-\sqrt{D}}{\sqrt{D}}\Big) - \frac{12\sqrt{D}}{\pi^2} L(1,\chi) s^{-1} \Bigg)$$

$$+ \frac{12\sqrt{D}}{\pi^2} L(1,\chi) \Big(2\frac{\zeta'}{\zeta}(2) + 1 + \gamma + \log \frac{4\pi}{\sqrt{D}} \Big) - \frac{12\sqrt{D}}{\pi^2} L'(1,\chi)$$

where

$$\Psi_s(t) = 4\pi \int_0^\infty \Phi(ty) e^{-4\pi y} y^s \, dy$$

The sum comes from re-writing the extra terms involving $v > 0 > v'$ in elementary terms.

(4) **Archimedean height on modular curves**

On a smooth compact Riemann surface X with a fixed point ∞, consider functions $G(x,y)$ which satisfy the next two properties

(i) $G(x,y)$ is symmetric.
(ii) For a fixed $y_0 \in X - \{\infty\}$, $x \mapsto G(x, y_0)$ is harmonic on $X - \{\infty, y_0\}$. It has a log-pole at y and ∞ of residues 1 and -1 respectively.

Such a function is unique up to an additive constant if it exists, and it is called a Green's function. We can then define a height pairing between degree zero divisors with disjoint supports by setting

$$\langle D_1, D_2 \rangle = \Big\langle \sum_i a_i(x_i), \sum_j b_j(y_j) \Big\rangle = \sum_{i,j} G(x_i, y_j)$$

This is well-defined and is supposed to describe $-\log($"distance between D_1 and D_2"$)$. If X is the complex analytification of an arithmetic surface, then this gives the Néron local height at complex places. See [Gro86] for more details.

Now specializing to the case $X = \mathbb{P}^1$ with ∞ marked, an obvious Green's function is given by

$$G(z_1, z_2) = \log |z_1 - z_2|^2$$

On the other hand, we had an isomorphism $j : SL_2(\mathbb{Z})\backslash\mathbb{H} \xrightarrow{\sim} \mathbb{P}^1$. In Chapter 2 of [GZ86], it was explained how to construct a Green's function on a modular curve. Roughly speaking, we try to average over some harmonic function g on $\mathbb{H}$ satisfying the necessary boundary conditions. It turns out that this does not converge, so we consider instead solutions to $\Delta g_s = s(1-s)g_s$, hoping they analytically continue to $s = 1$. This actually works after some tweaking. Define the following:

$$Q_{s-1}(t) = \int_0^\infty (t + \sqrt{t^2-1}\cosh x)^{-s} dx \quad \text{(Legendre function of the 2nd type)}$$

$$g_s(z_1, z_2) = -2Q_{s-1}\Big(1 + \frac{|z_1 - z_2|^2}{2y_1y_2}\Big) \quad \text{(Argument is } \cosh(d(z_1, z_2))\text{, so invariant under } SL_2(\mathbb{R}))$$

$$G_s(z_1, z_2) = \frac{1}{2}\sum_{\gamma \in SL_2(\mathbb{Z})} g_s(z_1, \gamma z_2) \quad \text{(Converges absolutely and locally uniformly on } \mathrm{Re}(s) > 1)$$

$$E_0(z, s) = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{y^s}{|mz+n|^{2s}} \quad \text{(Eisenstein series of weight 0)}$$

Then finally,

$$G(z_1, z_2) = \lim_{s\to 1}\left(G_s(z_1, z_2) + \frac{24}{\pi}E_0(z_1, s) + \frac{24}{\pi}E_0(z_2, s) - \frac{12}{s-1}\right)$$

is also a Green's function on $SL_2(\mathbb{Z})\backslash\mathbb{H}$. The Eisenstein series are there to make sure it is harmonic. We therefore have the identity

$$\log|j(z_1) - j(z_2)|^2 = \lim_{s\to 1}\left(G_s(z_1, z_2) + \frac{24}{\pi}E(z_1, s) + \frac{24}{\pi}E(z_2, s) - \frac{12}{s-1}\right) + C$$

where C is a constant determined by studying the limit as $y_1 \to \infty$, which gives

$$C = 24\Big(-\frac{\zeta'}{\zeta}(2) + \log 2 - \gamma + 1\Big)$$

(5) **Evaluate T by summing**

Suppose z is imaginary quadratic, then $E_0(z, s)$ can be identified with a partial Dedekind ζ-function for $\mathbb{Q}(z)$. More precisely, if τ has discriminant d which is fundamental, then

$$E(\tau, s) = w\,|d|^{-\frac{s}{2}}\,2^{s-1}\zeta_{K,[\mathfrak{a}]}(s)$$

where $K = \mathbb{Q}(\tau)$, w is the number of roots of unities in K, and $[\mathfrak{a}]$ is the ideal class of ideals homothetic to $\mathbb{Z} + \tau\mathbb{Z}$. Summing over all τ such that $\mathrm{disc}(\tau) = d$ gives $\zeta_K(s)$ multiplied by the same factor.

To compute the summed version of the first term, unfold to get

$$\frac{4}{w_1w_2}\sum_{\mathrm{disc}(\tau_i)=d_i} G_s(\tau_1, \tau_2) = \sum_{\substack{(\tau_1,\tau_2)\in SL_2(\mathbb{Z})\backslash\mathbb{H}^2 \\ \mathrm{disc}(\tau_i)=d_i}} g_s(\tau_1, \tau_2)$$

The map $(a, b, c) \mapsto \tau = \frac{-b+\sqrt{d}}{2a}$ gives a bijection between $\tau \in \mathbb{H}$ with discriminant d and triples of integers (a, b, c) such that $b^2 - 4ac = d$. When we represent τ_1 and τ_2 in this way, we see that

$$1 + \frac{|\tau_1 - \tau_2|^2}{2\tau_1\tau_2} = \frac{n}{\sqrt{D}}, \text{ with } n = 2a_1c_2 + 2a_2c_1 - b_1b_2 > \sqrt{D}$$

It is also clear that $n \equiv D \pmod 2$. Therefore, the above sum can be re-written as

$$-2 \sum_{\substack{n > \sqrt{D} \\ n \equiv D \pmod 2}} \rho(n) Q_s\Big(\frac{n}{\sqrt{D}}\Big)$$

where

$$\rho(n) = \frac{1}{2}\#\big(\{(a_1,b_1,c_1,a_2,b_2,c_2) \in \mathbb{Z}^6 : b_i^2 - 4a_ic_i = d_i, b_1b_2 - 2a_1c_2 - 2a_2c_1 = -n\}/SL_2(\mathbb{Z})\big)$$

The factor of $\frac{1}{2}$ is to account for negatively definite forms.

Finally, we reach another formula

$$T = \lim_{s\to 1}\Bigg(\sum_{\substack{n > \sqrt{D} \\ n \equiv D \pmod 2}} \rho(n) Q_s\Big(\frac{n}{\sqrt{D}}\Big) + \frac{24}{\pi}\Big(h_2'\Big|\frac{d_1}{2}\Big|^{s/2}\zeta_{K_1}(s) + h_1'\Big|\frac{d_2}{2}\Big|^{s/2}\zeta_{K_2}(s)\Big) - \frac{12h_1'h_2'}{s-1}\Bigg) + h_1'h_2'C$$

where $h_i' = 2h_i/w_i$.

(6) **Compare the formulae**

The inputs necessary are

(i) The function $\Psi_s(\lambda) - \frac{2\Gamma(2s+2)}{(4\pi)^s\Gamma(s+2)}Q_s(1+2\lambda)$ is $O(\lambda^{-s-2})$ as $\lambda \to \infty$ and tends to 0 as $s \to 0$.

(ii) $\rho(n) = \sum_{d|\frac{n^2-D}{4}} \epsilon(d)$.

(iii) $L(s,\chi) = L(s,\chi_1)L(s,\chi_2)$, where χ_i is the character associated to $K_i/\mathbb{Q}$.

(iv) $L(1,\chi_i) = \pi h_i'/\sqrt{|d_i|}$.

Of these, the first is standard asymptotic analysis of the functions present. The second can be deduced from genus theory, and it will be explained. The third is standard L-function formalism. The fourth is the analytic class number formula. Using these, the rest is just a computation.

We now explain step (ii) in detail. First observe that the right hand side is

$$\sum_{d|\frac{n^2-D}{4}} \epsilon(d) = \sum_{\mathfrak{n}|\mu} \chi(\mathfrak{n}) = r(\mu)$$

where $\mu = \frac{n-\sqrt{D}}{2}$, $\mathfrak{n}$ and χ are as before, and r is the number of integral ideals of $L = \mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ whose norm to K is $\mu\mathcal{O}_K$. Using the correspondence between quadratic forms and ideal classes, the left hand side can be manipulated into the following expression

$$\rho(n) = \sum_{[\mathfrak{a}_i]\in C_{K_i}} \#\{\lambda \in \mathfrak{a}_1\mathfrak{a}_2/U_{K_1}U_{K_2} : \mathbf{N}_{L/K}\lambda = \mathbf{N}\mathfrak{a}_1\mathbf{N}\mathfrak{a}_2\mu\}$$

where $K_i = \mathbb{Q}(\sqrt{d_i})$, C_{K_i} is the class group of K_i, U_{K_i} is the group of units of $\mathcal{O}_{K_i}$, and $\mathfrak{a}_1\mathfrak{a}_2$ is the set of $\mathbb{Z}$-linear combinations of $\alpha_1\alpha_2$ for $\alpha_i \in \mathfrak{a}_i$.

Let U_K^+ denote the totally positive units of K, and C_K^+ denote the narrow class group. We want to define maps $\delta_i : U_K^+/\mathbf{N}_{L/K}U_L \to C_{K_i}$. Consider the following exact sequence arising from Galois cohomology of cyclic extensions

$$1 \to H^1(L/K, U_L) \to I_L^{\sigma=1}/K^\times \to C_L^{\sigma=1} \to H^2(L/K, U_L) \to H^2(L/K, L^\times)$$

where I_L are the ideals of L, and $\sigma \in \mathrm{Gal}(L/K)$ is the non-trivial element. Observe that $\mathbf{N}_{L/K} : C_L \to C_K$ is surjective since L/K is ramified at infinity. But $\mathbf{N}_{L/K} : C_L \to C_K^+$ cannot be surjective χ is non-trivial (by class field theory). Hence $C_K^+ \neq C_K$, i.e. K has a totally positive fundamental unit ϵ. By the Hasse norm theorem, $\epsilon \in \mathbf{N}_{L/K}L^\times$. Furthermore, L/K is unramified at finite places, so $I_L^{\sigma=1} = I_K$. The relevant piece of the sequence simplifies to

$$I_K \to C_L^{\sigma=1} \to U_K^+/\mathbf{N}_{L/K}U_L \to 1$$

Let δ_i be a section of the second arrow followed by norm down to C_{K_1} and C_{K_2}. If $\mathfrak{a}$ is an ideal in K, then $\mathbf{N}_{L/K_i}(\mathfrak{a}L)$ can be generated by integers, so it is principal. Therefore, δ_i is well-defined.

Now consider the following diagram

$$0 \to U_K^+/\mathbf{N}_{L/K}U_L \xrightarrow{(\delta_1,\delta_2)} C_{K_1}\times C_{K_2} \to C_L \xrightarrow{\mathbf{N}_{L/K}} C_K^+ \xrightarrow{\chi} \{\pm 1\} \to 0$$

Observe that if $u \in U_K^+$, then by construction, $\delta_1(u)L \cdot \delta_2(u)L = \mathfrak{a}^{1+\sigma_1}\mathfrak{a}^{1+\sigma_2}$ for some $\mathfrak{a} \in C_L^{\sigma=1}$ (these are called ambiguous ideals). But then $(1+\sigma_1+1+\sigma_2)\mathfrak{a} = \mathbf{N}_{L/\mathbb{Q}}\mathfrak{a}$, so $\delta_1(u)L \cdot \delta_2(u)L$ is principal. From this, it is clear that the diagram is a complex.

The complex is exact at C_K^+ by the definition of χ. By a careful study of the maps $C_{K_i} \to C_L$ using genus theory, one can show that the complex is exact at $C_{K_1} \times C_{K_2}$. Let $Q = [U_L : \mu_L U_K] = [\mathbf{N}_{L/K}U_L : U_K^2]$, then it follows from the analytic class number formula that $h_L = \frac{1}{2}Qh_1h_2h_K$. We have $[U_K : U_K^+] = 2$ and $h_K^+ = 2h_K$, so the Euler characteristic of the complex is 1. Since exactness is proven at every second term, this proves the complex is exact. Using the exact sequence, one can show that $\rho(n) = r(\mu)$ by sending $(\mathfrak{a}_1, \mathfrak{a}_2, \lambda)$ to $\lambda\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}$.

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