Weight interlacing and Iwasawa theory

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Gan–Gross–Prasad conjecture

Basic setting:

- $K/\mathbf{Q}$ imaginary quadratic field.
- $V_n \subseteq V_{n+1}$ Hermitian spaces over K of dimensions n and $n+1$ respectively.

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smooth $\mathbb{C}$-coeff. irred. adm.

Local problem

Fix a place v of $\mathbf{Q}$. Given $\pi_{n+1,v} \in \mathrm{Irr}(\mathrm{U}(V_{n+1})_v)$, how does $\pi_{n+1,v}|_{\mathrm{U}(V_n)_v}$ decompose?

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Local problem

Fix a place v of $\mathbf{Q}$. Given $\pi_v = \pi_{n,v} \boxtimes \pi_{n+1,v} \in \mathrm{Irr}(G_v)$, compute $\boxed{\dim_{\mathbf{C}} \mathrm{Hom}_{H_v}(\pi_v, 1)}$.

aut. rep

Global problem

Given $\pi \in \mathcal{A}(G)$, is the automorphic period $\int_{[H]} \varphi(h)dh$ non-zero for any $\varphi \in \pi$?

$[H] = H(\mathbb{Q})\backslash H(\mathbb{A})$

Local answer

p-adic

Multiplicity one theorem (Aizenbud–Gourevitch–Rallis–Schiffmann, Sun–Zhu) archi. place

$\dim_{\mathbf{C}} \mathrm{Hom}_{H_v}(\pi_v, 1) \leq 1.$

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$\pi_{n,v} \boxtimes \pi_{n+1,v} \in \mathrm{Irr}(G_v)$

p-adic arch. place.

Local Gan–Gross–Prasad conjecture (Beuzart-Plessis, Xue)

As $\pi_{n,v}$ and $\pi_{n+1,v}$ run over the members of their Vogan *L*-packets, there exists a unique pair such that the above multiplicity space is 1-dimensional. This pair can be specified using certain local ε-factors.

Contains rep'n's from other inner forms

Local answer: archimedean case

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- $H_\infty = \mathrm{U}(V_n)_\infty \simeq \mathrm{U}(p, q)$, $p + q = n$.

Vogan L-packet contains rep^{n}. from

<u>all</u> $U(a,b)$ $a+b=n$

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- $H_\infty = \mathrm{U}(V_n)_\infty \simeq \mathrm{U}(p, q)$, $p + q = n$.
- Discrete series are indexed by Harish-Chandra parameters

$$(\underbrace{a_1 > \cdots > a_p}; b_1 > \cdots > b_q)$$

$a_i, b_j \in \mathbf{Z} + \frac{n-1}{2}$, $a_i \neq b_j$.

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e.g. Hol. disc. series

$$(a_1 > \cdots > a_p; \boxed{b_1 > \cdots > b_q})$$

$(b_1 > \cdots > b_q > a_1 > \cdots > a_p)$

  $a_i, b_j \in \mathbf{Z} + \frac{n-1}{2}$, $a_i \neq b_j$. Its infinitesimal character is $(a_1, \cdots, a_p, b_1, \cdots, b_q)$ (unordered).
- Vogan L-packets are indexed by infinitesimal characters

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- ~~Vogan L-packets~~ are indexed by infinitesimal characters
  - Size 2^n
  - $\binom{n}{p}$ representations on $\mathrm{U}(p, q)$
  - Elements indexed by characters of $(\mathbf{Z}/2\mathbf{Z})^n$

depends on choice of additive character $\mathbb{C}$

Local answer: archimedean case (cont.)

Two L-packets on $\mathrm{U}(V_n)$ and $\mathrm{U}(V_{n+1})$ with infinitesimal characters

$$(b_1 > \cdots > b_n), \quad (a_1 > \cdots > a_n) + 1$$

$$\mathbb{Z} + \frac{n-1}{2} \qquad \mathbb{Z} + \frac{n}{2}$$

Local answer: archimedean case (cont.)

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Distinguished pair from the local GGP conjecture described by *weight interlacing*, which can be drawn as a *tableau*.

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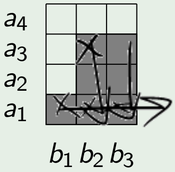
Examples

$a_1 > b_1 > a_2 > a_3 > b_2 > b_3 > a_4$ can be represented by



Shaded squares satisfy $a_i > b_j$.

Combinatorial recipe (Atobe)

1. Count the parity of the number of shaded squares in each row and column.
2. Flip the sign of every second row and every second column.
3. The negative signs of each row/column form the first part of the Harish-Chandra parameters.

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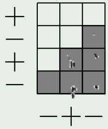

$\overset{(1)}{\Longrightarrow} \quad (-,+,-),(-,+,-,+)$

$\overset{(2)}{\Longrightarrow} \quad (-,-,-),(-,-,-,-)$

$\overset{(3)}{\Longrightarrow} \quad \mathrm{U}(3,0) \times \boxed{\mathrm{U}(4,0)}$

classical Branching law

Global answer

$U(V_n) \times U(W_{n+1})$

Theorem (W. Zhang, Beuzart-Plessis–Liu–Zhang–Zhu)

Given (tempered stable) cuspidal $\pi = \bigotimes'_v \pi_v \in \mathcal{A}(G)$, the period integral $\int_{[H]} \varphi(h)dh$ is non-zero for some $\varphi \in \pi$ if and only if

$H = U(V_n)$

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- π_v is the distinguished element of the L-packet in the sense of the local GGP conjecture; $\int_{[H]} \in \mathrm{Hom}_H(\pi, \mathbb{1})$

$$= \underbrace{\bigotimes \mathrm{Hom}_{H_v}(\pi_v, \mathbb{1})}_{\neq 0}$$

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- π_v is the distinguished element of the L-packet in the sense of the local GGP conjecture; and
- $L(\frac{1}{2}, \pi) \neq 0$.

Rankin-Selberg

$GL_n(K) \times GL_{n+1}(K)$

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Moreover, there is a formula of the form

$$L\left(\frac{1}{2}, \pi\right) = \left| \int_{[H]} \varphi(h)dh \right|^2 \cdot (\text{local factors}) \cdot \overset{\text{Adjoint + abelian}}{\underbrace{(\text{other } L\text{-values})}_{\neq 0}}$$

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$\pi = \pi_f \otimes \pi_\infty$

$[H] = H(\mathbb{Q}) \backslash H(\mathbb{A})$

$H(\mathbb{A}_f) \times H(\mathbb{R})$

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dim depends on signature of V_n

$L(1, As^{\pm}(\pi_n))$

$L(1, As^{\mp} \pi_{n+1})$

This gives an integral representation for certain Rankin–Selberg L-functions over K. The archimedean shape of the integral depends on weight interlacing.

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Fix a prime p which is ordinary for $\pi = \pi_n^\vee \boxtimes \pi_{n+1}$ and splits in K.

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Theorem (Morel, Skinner) $\rho_n \cong \rho_n^c(-)$

There is a conjugate self-dual Galois representation $\rho_n : \mathrm{Gal}_K \to \mathrm{GL}_n(\bar{\mathbf{Q}}_p)$ attached to π_n such that $\rho_n|_{\mathrm{Gal}_{K_\mathfrak{p}}}$ is upper triangular for $\mathfrak{p}|p$.

ord $\Rightarrow$

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Choose a basis $\{v_1, \cdots, v_n\}$ for ρ_n so that $\rho_n|_{\mathrm{Gal}_{K_\mathfrak{p}}}$ is upper triangular. By conjugate self-duality, $\rho_n|_{\mathrm{Gal}_{K_{\bar{\mathfrak{p}}}}}$ is lower triangular with respect to the dual basis.

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Let $\rho = \rho_n^\vee \otimes \rho_{n+1}$. $(-)$ It is conjugate self-dual of weight -1 with a basis $\{v_i^\vee \otimes w_j\}$. $1 \leq i \leq n$

$1 \leq j \leq n+1$

Panchishkin condition

The Hodge–Tate weights of ρ can be computed from the infinitesimal character of π_∞.

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At $\mathfrak{p}$, we have a Panchishkin subspace

$$\rho_\square = \langle v_i^\vee \otimes w_j \mid a_i > b_j \rangle \subseteq \rho$$

$W \subseteq V$ is Panchishkin if W has negative H-T wts and V/W has ≥ 0 HT-wts

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So weight interlacing determines the correct local conditions at p.

Selmer group

If $\square$ and $\triangle$ are two weight interlacing relations, we can use them to define a Selmer group

$$H^1_{\square,\triangle}(K,\rho)$$

with unramified local conditions away from p, Greenberg condition defined using $\rho_\square$ at $\mathfrak{p}$, and Greenberg condition defined using $\rho^c_\triangle$ at $\bar{\mathfrak{p}}$.

Greenberg condition:

$$\mathrm{res}_{\mathfrak{p}}\, c \in \ker\left(H^1(G_{\mathbb{Q}_p}, \rho) \to H^1(G_{\mathbb{Q}_p}, \rho/\rho_\square) \right)$$

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If $\square$ is the actual weight interlacing of π_∞, then $\underline{H^1_{\square,\square}(K,\rho)}$ is the Bloch–Kato Selmer group for ρ.

p-adic deformations

$\boxed{p \text{ split}}$ $(U_n)_p = GL_n(\mathbb{Q}_p)$

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Using pseudo-representations, can form big Galois representation $\boldsymbol{\rho}$ valued in an ordinary big Hecke algebra $\mathbb{I}$, so we can define big Selmer groups

$$H^1_{\square,\triangle}(K, \boldsymbol{\rho})$$

with ordinary filtration

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Key point

The family $\boldsymbol{\rho}$ covers all weight interlacing relations, so $H^1_{\square,\square}(K,\boldsymbol{\rho})$ should have arithmetic significance for all possible $\square$.

Iwasawa main conjecture

Given a weight interlacing relation $\Box$, it determines an integral representation of L-functions and a self-dual Selmer condition.

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Expectation (Iwasawa main conjecture)

For any $\square$,

1. There should be a p-adic L-function $\mathcal{L}^{\square}$ interpolating the algebraic part of the central L-values at all specializations of ρ which has archimedean weight interlacing $\square$.
2. $\mathrm{char}_{\mathbb{I}}(H^2_{\square}(K,\rho)) = (\mathcal{L}^{\square})$.
3. Neither side of the above equality is zero.

$H^2_{\square}$ torsion

$\mathcal{L}^{\square} \neq 0$

Iwasawa main conjecture

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Warning

Part (3) cannot be true for all $\square$ for sign reasons.

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$\varepsilon = +1$ Coherent case: expectation from last slide should hold, so have coherent main conjecture

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$$\mathrm{char}_{\mathbb{I}}\, H^2_{\Box}(K, \rho) \stackrel{?}{=} (\mathcal{L}^{\Box}) \stackrel{?}{\neq} 0$$

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Expect an incoherent main conjecture (Perrin-Riou)

Incoherent main conjecture

Conjecture

The compact Selmer group $H^1_{\Box}(K, \rho)$ has rank 1, and it contains a special class $\mathbf{z}$ such that

interpolate "special cycles"

$$\mathrm{char}_{\mathbb{I}}\left(\frac{H^1_{\Box}(K,\rho)}{\mathbb{I}\cdot \mathbf{z}}\right)^2 = \mathrm{char}_{\mathbb{I}}\, H^2_{\Box}(K,\rho)_{\mathrm{tors}}$$

Wall crossing

What happens when a pair of weight orders are reversed?

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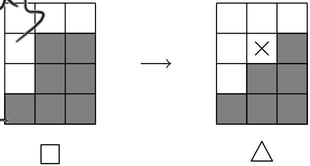
$(a_1 > b_1 > b_2 > a_2 > a_3$

$(b_1 > a_1) >$



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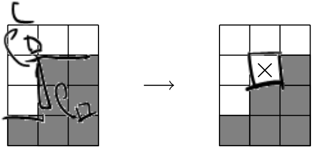

$\square$ $\qquad$ $\triangle$

non-zero res

- Interlacing relations $\square$ and $\triangle$ have opposite signs.
- Selmer condition <u>decreases</u> at $\mathfrak{p}$ and <u>increases</u> at $\bar{\mathfrak{p}}$.

Lemma

Suppose $\square$ is the coherent interlacing relation. If $\mathbf{z}_{\triangle}$ exists, then the incoherent main conjecture for $\triangle$ is equivalent to the coherent main conjecture for $\square$.

Upshot

- For a given Hida family $\boldsymbol{\rho}$, we have $\binom{2n+1}{n}$ different main conjectures, indexed by weight interlacing relations.

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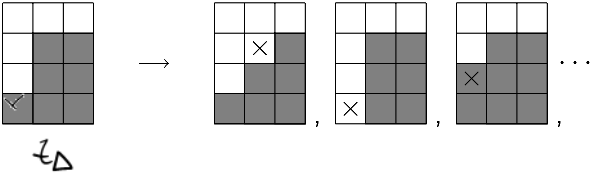
- Each such special class also implies results on <u>several</u> coherent main conjectures



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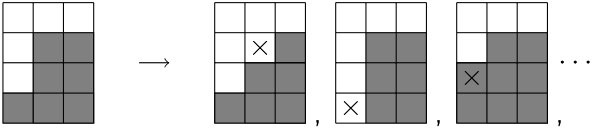

- Need explicit reciprocity laws and arithmetic interpretation of integral representations to relate motivic p-adic L-functions to analytic p-adic L-functions.

defined using interpolations

Coleman map of special class

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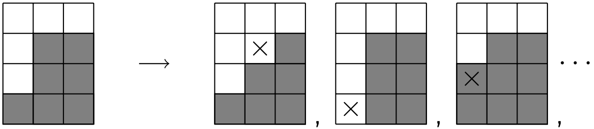

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- cf. Loeffler–Zerbes $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ (Orthogonal (Gross-Prasad conj.)

$\sim O(5) \times O(4))$

$n = 1$

$H = \mathrm{U}(V_1) = (K^\times)^{\mathrm{Norm}=1}$, $G \approx (K^\times)^{\mathrm{Norm}=1} \times \mathrm{GL}_2$ U(2)

$\pi \approx \underline{\chi} \times \underline{f}$, χ anticyclotomic character of weight $(\ell, -\ell)$, f modular form of weight k.

Infinitesimal characters: $\chi \rightsquigarrow (\ell)$, $f \rightsquigarrow \left(\frac{k-1}{2}, -\frac{k-1}{2}\right)$.

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Ordinary filtration

$$0 \to T_f^- \to T_f \to T_f^+ \to 0$$

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$$0 \to T_f^- \to T_f \to T_f^+ \to 0$$

Selmer conditions:

$\ker\left(H^1(T_f) \to H^1(T_f^+)\right)$

$\mathtt{rel}$: no condition

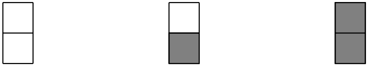
$\mathtt{ord}$: ordinary Selmer condition

$\mathtt{str}$: strict, i.e. localization $= 0$

Weight interlacing

$\varepsilon = \underline{\varepsilon_f}\, \varepsilon_\infty$

Three weight interlacing relations



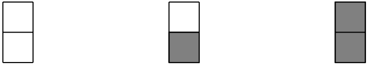
	$\ell > \frac{k-1}{2}$	$\lvert\ell\rvert < \frac{k-1}{2}$	$\ell < -\frac{k-1}{2}$
ε_∞	$+1$	-1	$+1$
Selmer condition	(str,rel)	(ord,ord)	(rel,str)

Selmer group

Greenberg Selmer groups

Weight interlacing

Three weight interlacing relations



	$\ell > \frac{k-1}{2}$	$\lvert\ell\rvert < \frac{k-1}{2}$	$\ell < -\frac{k-1}{2}$
ε_∞	$+1$	-1	$+1$
Selmer condition	(`str`,`rel`)	(`ord`,`ord`)	(`rel`,`str`)

The coherence of each case depends on the finite part ε_f. This is related to the Heegner hypothesis.

Heegner points

Heegner hypothesis

$\ell | N_f \implies \ell$ splits in K

This implies $\varepsilon_f = +1$.

Heegner points

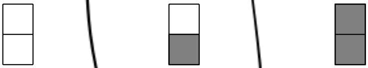
Heegner hypothesis

$\ell | N_f \implies \ell$ splits in K

This implies $\varepsilon_f = +1$.

$\varepsilon = -1$ $\qquad$ $\varepsilon = +1$

$\varepsilon + 1$ $\qquad$ BDP $\qquad$ ℓ



$\ell > \frac{k-1}{2}$ $\qquad$ $|\ell| < \frac{k-1}{2}$ $\qquad$ $\ell < -\frac{k-1}{2}$

Heegner point

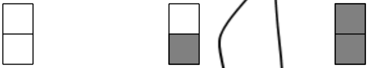
- Special class is big Heegner point $\boxed{\mathbf{z}_{\mathrm{Heeg}}}$ (Howard).
- Incoherent main conjecture is Perrin-Riou's main conjecture.
- Two coherent main conjectures are BDP main conjectures, related by a complex conjugation.

No Heegner points

Now suppose $\varepsilon_f = -1$.

No Heegner points

Now suppose $\varepsilon_f = -1$.



$\ell > \frac{k-1}{2}$ $\qquad |\ell| < \frac{k-1}{2}$ $\qquad \ell < -\frac{k-1}{2}$

$\varepsilon = -1$ $\qquad \varepsilon = +1$ $\qquad \varepsilon = -1$

- Special class constructed in recent work of Kim Tuan Do.
- Construction not geometric, splits up Selmer classes from the diagonal cycle on a triple product.
- Coherent main conjecture interpolates rank 0 BSD for modular forms twisted by finite order anticyclotomic characters.

$n = 2$

There are 10 interlacing relations, or 6 up to complex conjugation.