Transfer Principle for the Fundamental Lemma

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September 21, 2020

Background

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Decompose the representation $R = L^2(G(\mathbf{Q})\backslash G(\mathbb{A}))$ with the right regular $G(\mathbb{A})$-action.

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If $G = \mathrm{SL}_2$, then this contains the study of modular forms and Maass forms.

The main tool is the trace formula.

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Geometric expansion: $R(f)$ is an integral operator, and

$$\begin{aligned}\operatorname{Tr} R(f) &= \int_{G(\mathbf{Q})\backslash G(\mathbb{A})} \sum_{\gamma \in G(\mathbf{Q})} f(x^{-1}\gamma x) dx \\ &= \sum_{\{\gamma\}} \mathrm{vol}(G(\mathbf{Q})_\gamma \backslash G(\mathbb{A})_\gamma) \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(x^{-1}\gamma x) dx\end{aligned}$$

where the sum is over conjugacy classes in $G(\mathbf{Q})$, and a subscript γ indicates the centralizer of γ in that group.

Comparison gives

$$\sum_{\pi} m(\pi) \operatorname{Tr} \pi(f) = \sum_{\{\gamma\}} \mathrm{vol}(G(\mathbf{Q})_\gamma \backslash G(\mathbb{A})_\gamma) \mathcal{O}(G, \gamma, f)$$

where $\mathcal{O}(G, \gamma, f)$ is the orbit integral

$$\begin{aligned} \mathcal{O}(G, \gamma, f) &= \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(x^{-1}\gamma x) dx \\ &= \prod_v \int_{G(\mathbf{Q}_v)_\gamma \backslash G(\mathbf{Q}_v)} f_v(x^{-1}\gamma x) dx \\ &= \prod_v \mathcal{O}(G, \gamma, f_v) \end{aligned}$$

Stabilization

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Group terms together so that they are stably invariant.

- Geometric side: stable orbit integral

$$SO(G, \gamma, f_v) = \sum_{\gamma' \sim \gamma} \mathcal{O}(G, \gamma', f_v)$$

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$$S\mathcal{O}(G, \gamma, f_v) = \sum_{\gamma' \sim \gamma} \mathcal{O}(G, \gamma', f_v)$$

  Conjugacy classes in a stable conjugacy class are classified by the cohomology set $H^1(F, G_\gamma)$.
- Spectral side: L-packets...

Endoscopy

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Theorem-like object

- There is a natural finite abelian group structure on $H^1(F, G_\gamma)$.
- A character κ determines a reductive group H, and there is a map ν from semisimple stable conjugacy classes of H to semisimple stable conjugacy classes of G.
- Given $f_G \in C_c^\infty(G(\mathbf{Q}_p))$, there exists $f_H \in C_c^\infty(H(\mathbf{Q}_p))$ such that for all regular semisimple $\gamma \in H(\mathbf{Q}_p)$,

$$\Delta(\gamma)\mathcal{O}_\kappa(G, \nu(\gamma), f_G) = S\mathcal{O}(H, \gamma, f_H)$$

where $\mathcal{O}_\kappa(\gamma) = \sum_{\gamma' \sim \gamma} \kappa(\gamma')\mathcal{O}(\gamma')$.

Fundamental lemma: vague statement

Suppose G extends to a smooth group scheme over $\mathbf{Z}_p$. One version of the fundamental lemma is

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We will now state in more explicit terms a Lie algebra variant of this statement.

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$$\mathcal{O}_\kappa(G, \gamma_0) = \sum_{\gamma \sim \gamma_0} \kappa(\gamma) \int_{G(F)_\gamma \backslash G(F)} 1_{\mathfrak{g}(\mathcal{O}_F)}((\mathrm{Ad}\, g^{-1})\gamma) dg$$

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and $S\mathcal{O} = \mathcal{O}_1$, then for a specified r depending on G, κ, γ,

$$\mathrm{FL}: \quad \mathcal{O}_\kappa(G, \gamma_0) = q^r S\mathcal{O}(H, \gamma)$$

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For p sufficiently large depending on G, if the fundamental lemma holds for $F = \mathbf{F}_p((t))$, then it holds for $F = \mathbf{Q}_p$.

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Consequences

- Hales: We get fundamental lemma for $F = \mathbf{Q}_p$ for *all* p.
- Waldspurger: The group version also holds.

Principle

Theorem (Ax–Kochen–Eršov)

Let φ be a first order sentence in the language of valued fields, then for p sufficiently large, φ holds in $\mathbf{F}_p((t))$ if and only if φ holds for $\mathbf{Q}_p$.

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We need to extend this to account for integration.

Wishful Thinking

Let K be a Henselian local field. Suppose $f : K^n \to \mathbf{Q}$ is a "first-order definable" function, then $\int_{K^n} f(\underline{x})d\underline{x}$ only depends on the residue field of K, provided it has a sufficiently large characteristics.

Main ideas

- If p is large enough relative to a fixed set of formulae, then operations over $\mathbf{Q}_p$ and $\mathbf{F}_p((t))$ both look like $k((t))$ for any field k of characteristic 0. Model theory makes this precise.

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- If p is large enough relative to a fixed set of formulae, then operations over $\mathbf{Q}_p$ and $\mathbf{F}_p((t))$ both look like $k((t))$ for any field k of characteristic 0. Model theory makes this precise.
- Model-theoretic techniques can be applied to break up $k((t))^n$ into well-understood pieces or "cells".
- Define formal volume/integration over cells valued in some formal ring with a symbol $\mathbb{L}$ to stand for p.

Specification

Definition

The *Denef–Pas language* has three sorts: valued field, residue field, and valuation group. It contains

- Rings over $\mathbf{Z}[[t]]$: $(0, 1, +, -, \times, \mathbf{Z}[[t]])$ in the valued field variables.
- Rings over $\mathbf{Q}$ in the residue field variables.
- Ordered group: $(0, +, -, <)$ in the valuation group variables.
- Unary function ord from valued field to valuation group.
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A model of this is a triple of set (K, k, Z) with interpretations for above symbols.

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Three models: $(k((t)), k, \mathbf{Z})$, $(\mathbf{Q}_p, \mathbf{F}_p, \mathbf{Z})$, $(\mathbf{F}_p((t)), \mathbf{F}_p, \mathbf{Z})$

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- $\mathbf{Q}_p$ is a $\mathbf{Z}[[t]]$-algebra by sending t to p.
- $\mathrm{ac}(x)$ is the leading term of a formal Laurent series resp. p-adic expansion.
- $\mathbf{Q}$ in the residue field language is interpreted in $\mathbf{F}_p$ as follows: if $r \in \mathbf{Q}$ is not defined in $\mathbf{F}_p$, then interpret it as 0.

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Can view φ as a functor

$$X_\varphi : \{k \text{ field of char } 0\} \to \{\text{subsets of } k((t))^m \times k^n \times \mathbf{Z}^r\}$$

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  for $K = \mathbf{Q}_p$ or $\mathbf{F}_p((t))$ if p is sufficiently large.
- If $f : X_\varphi \to X_{\varphi'}$ is a definable morphism, then it specializes to algebraic maps

$$f_K : X_{K,\varphi} \to X_{K,\varphi'}$$

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Forms a ring $\mathcal{C}(X_\varphi)$.

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Corollary of Ax–Kochen–Eršov

This is independent of a presentation of f.

Motivic integration

Formalism:

- For each φ, construct a subset $I\mathcal{C}(X_\varphi)$ of integrable Functions.
- For each morphism $f : X_\varphi \to X_{\varphi'}$, construct a pushforward $f_! : I\mathcal{C}(X_\varphi) \to I\mathcal{C}(X_{\varphi'})$.
- Let $*$ be the one-point functor. Given φ and $f \in I\mathcal{C}(X_\varphi)$, define $\mu(f) = p_!(f)$, where $p : X_\varphi \to *$.

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Specialization:

- Integrable Functions f should have integrable specialization $f_K : X_{K,\varphi} \to \mathbf{Q}$.
- Functions on $*$ specializes to rational numbers, and we need

$$\mu(f)_K = \int_{X_{K,\varphi}} f_K(\underline{x})d\underline{x}$$

Transfer

Theorem (Cluckers–Loeser)

Let f be a function on X_φ, then for p sufficiently large, $f_{\mathbf{Q}_p} \equiv 0$ if and only if $f_{\mathbf{F}_p((t))} \equiv 0$.

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This combined with the previous slide gives a transfer principle for theorems involving integrals in this framework.

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Hard part: Fubini-type theorems for motivic integrals

Work left

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Easy data

- Field extensions of a fixed degree n are represented by a minimal polynomial and an n-tuple of numbers.
- There is a uniform way of describing unramified extensions and the Frobenius action.
- Groups are Zariski closed subsets of matrix groups, classified by root data plus Frobenius action.
- Same for Lie algebras
- Endoscopic data are expressed in terms of root data.
- Semisimplicity corresponds to diagonalizability over a bounded field extension.
- The map ν is algebraic if we represent stable conjugacy classes by their characteristic polynomials (and generalizations).

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  Key point: motivic integration specializes to the Serre canonical measure, which must be the G-invariant measure up to a factor which can be computed.

Galois cohomology

Endoscopy requires the identification

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i.e. for each pair of non-negative integers a, b, a formula $D_{a,b}(t, t')$ stating that

1. t is a cocyle in $Z^1(G, T)$, t' is a cocyle in $Z^1(G, X^*(T))$.
2. $\mathrm{inv}(t \smile t') = \frac{a}{b}$

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Recall the construction of the invariant map:

$$H^2(E/F, E^\times) \xleftarrow{\text{inf}} H^2(F_r/F, F_r^\times)$$
$$\xrightarrow{\text{val}} H^2(F_r/F, \mathbf{Z}) \simeq \mathrm{Hom}(\mathrm{Gal}(F_r/F), \mathbf{Q}/\mathbf{Z}) \simeq \mathbf{Q}/\mathbf{Z}$$

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Choose an explicit cocycle α representing $\frac{a}{b} \in H^2(F_r/F, \mathbf{Z})$. Given $c \in H^2(E/F, E^\times)$, $\mathrm{inv}(c) = \frac{a}{b}$ is equivalent to

$$(\exists c', b_1, b_2)(\inf(c') = cb_1 \wedge \mathrm{val}(c') = \alpha b_2)$$

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