REVIEW OF LOCAL REPRESENTATION THEORY

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This is the notes for the second talk of a seminar on the moduli space of Langlands parameters. The goal here is to review the representation theory of $\mathrm{GL}_n(E)$ where E is an ℓ-adic field. Since the intention is to study p-adic variations, the flavour here will be more algebraic than analytic. The main references are [BZ76, BZ77] for field coefficients and [EH14] for more general coefficients. There is also a set of unpublished notes of Berstein's lectures.

1. Abstract representation theory

Let G be a second countable ℓ-adic group, meaning it contains a profinite open compact subgroup whose order is a power of ℓ. This is satisfied if G is the E-points of an algebraic group, when E is an ℓ-adic field. Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field k of characteristic not equal to ℓ. We can extend usual notion of smooth and admissible representations to representations over A.

Definition 1.1. A representation (π, V) of G is *smooth* if the stabilizer of every vector is open. It is *admissible* if it is smooth and V^H is finite dimensional for any open subgroup $H \subseteq G$.

From an analytic point of view, one subclass of smooth representations are particularly well-behaved.

Definition 1.2. A smooth representation (π, V) is *supercuspidal* if for all $v \in V$ and $\tilde{v} \in (V^*)^{\mathrm{sm}}$, the matrix coefficient $g \mapsto \langle \tilde{v}, \pi(g)v \rangle$ is compactly supported.

Proposition 1.3.
(1) Irreducible supercuspidal representations are admissible.
(2) The subcategory of all representations whose Jordan–Hölder series only contains supercuspidal representations is completely reducible.

Let $\mathrm{Sm}(G, A)$ be the category of smooth representations of G with coefficients in A. This is an abelian category, and we recall the following general definition.

Definition 1.4. The *Berstein centre* of an abelian category is the endomorphism ring of the identity functor. We will denote the centre of $\mathrm{Sm}(G, A)$ by $\mathfrak{Z}(G, A)$.

More concretely, one can show that $\mathfrak{Z}(G, A) = \varprojlim_K Z(\mathcal{H}(G, K))$, where $\mathcal{H}(G, K)$ is the Hecke algebra of K-biinvariant A-valued functions on G and the projective limit is taken over a neighbourhood basis of open compact subgroups of G.

By definition, an element of $\mathfrak{Z}(G, A)$ acts on all smooth representations of G in a functorial way. In particular, if V is irreducible, then by the version of Schur's lemma in this setting, $\mathfrak{Z}(G, A)$ acts on V by a character. This is the analogue of the infinitesimal character from Archimedean representation theory. In the next section, we will describe $\mathfrak{Z}(G, K)$ explicitly in the case $G = \mathrm{GL}_n(E)$ and K is an algebraically closed field of characteristic 0.

Definition 1.5. Let H be a closed subgroup of G. Let π be a smooth H-representation. Define

$$\mathrm{Ind}_H^G \pi = \{f : G \to \pi \mid f(hg) = hf(g) \text{ for all } h \in H,\ g \in G\}^{\mathrm{sm}}$$

where the action is $(\gamma \cdot f)(g) = f(g\gamma)$, and the superscript denotes the subspace of smooth vectors (those are the uniformly locally constant functions). The compact induction is the subspace

$$\text{c-ind}_H^G \pi = \{f \in \mathrm{Ind}_H^G \pi \mid f \text{ is compactly supported modulo } H\}$$

so if $H \backslash G$ is compact, then $\text{c-ind}_H^G = \mathrm{Ind}_H^G$.

Lemma 1.6 (Frobenius reciprocity). *Let $\sigma \in \mathrm{Sm}(H)$ and $\pi \in \mathrm{Sm}(G)$, then*
(1) $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma)$.
(2) The functors c-ind_H^G and Ind_H^G are exact.

Proof. Part (1) is the standard Frobenius reciprocity given by the bijections

$$(\alpha : \pi \to \operatorname{Ind}_H^G \sigma) \mapsto (\alpha' : \pi|_H \to \sigma,\ x \mapsto \alpha(x)(1))$$
$$(\beta : \pi|_H \to \sigma) \mapsto (\beta' : \pi \to \operatorname{Ind}_H^G \sigma,\ x \mapsto (g \mapsto \beta(gx)))$$

For part (2), define an auxiliary functor $\iota^! : \mathrm{Sm}(G) \to \mathrm{Sm}(H)$ as follows: $\iota^!\pi$ consists of the H-smooth vectors of the distribution space $\mathcal{H}(G)^* \otimes_{\mathcal{H}(G)} \pi$. We have a map $\operatorname{Hom}_H(\sigma, \iota^!\pi) \simeq \operatorname{Hom}_G(\operatorname{Ind}_H^G \sigma, \pi)$,

$$(\alpha : \sigma \to \iota^!\pi) \mapsto (\alpha' : \operatorname{Ind}_H^G \sigma \to \pi,\ f' \otimes v \mapsto \pi(f')\alpha(v))$$

It is easy to see that this is bijective, so Ind_H^G has a right adjoint. This proves the exactness of Ind_H^G. The claim about c-ind_H^G follows. □

Not much can be said about admissibility in this generality, but in the cases we are interested in, there will be deep results showing that admissibility is preserved by these functors.

2. Representation theory of $\mathrm{GL}_n(E)$

The theme of this section is the study of induced representations based on the two papers [BZ77] and [Zel80]. We will give a relatively complete picture of how the smooth representations are built up from cuspidal representations.

In this section, E is an ℓ-adic local field, $G = \mathrm{GL}_n(E)$, and K is an algebraically closed field of characteristic 0. Fix a choice of $\ell^{\frac{1}{2}} \in K$. The fact that G is not compact causes some issues. We define a subgroup $G^\circ := \{g \in G \,|\, v(\deg g) = 0\}$ with compact centre. We call a representation of G supercuspidal if its restriction to G° is supercuspidal in the previous sense.

2.1. Parabolic induction and Jacquet functor.

Let $P \subseteq G$ be a parabolic subgroup, with Levi decomposition $P = MN$. Let δ be its modulus character. We can modify induction and restriction in this setting to a pair of functors

$$i_{M,G} : \mathrm{Sm}(M) \to \mathrm{Sm}(G), \quad J_{G,M} : \mathrm{Sm}(G) \to \mathrm{Sm}(M)$$

The induction $i_{M,G}(\sigma)$ consists of all uniformly locally constant functions $f : G \to \sigma$ such that

$$f(mng) = \sigma(m)\delta(m)^{\frac{1}{2}} f(g) \quad \text{for all } m \in M,\ n \in N,\ g \in G$$

with G acting from the right. The Jacquet functor $J_{G,M}$ is the composite of restricting to P, taking N-coinvariants, then twisting by $\delta(m)^{-\frac{1}{2}}$. The twisting by $\delta^{\frac{1}{2}}$ preserves contragredient, and this is where the choice of $\ell^{\frac{1}{2}}$ comes into play.

Since G/P is compact, we have the adjunction

$$\operatorname{Hom}_M(J_{G,M}\pi, \sigma) \simeq \operatorname{Hom}_G(\pi, i_{M,G}\sigma)$$

The results of the previous section show that both functors are exact. We now state a fundamental finiteness property of the two functors.

Theorem 2.1 (Jacquet)**.** *Both $i_{M,G}$ and $J_{G,M}$ preserves admissibility. The functor $J_{G,M}$ also sends finitely generated representations to finitely generated representations.*

The Jacquet functor was first used to characterize supercuspidal representations as those which do not arise from parabolic inductions.

Definition 2.2. A representation π is *cuspidal* if $J_{G,M}(\pi) = 0$ for all parabolics $P \subseteq G$.

Theorem 2.3 (Jacquet)**.** *If π is an irreducible smooth representation of G, then π is cuspidal if and only if it is supercuspidal. This implies the admissibility of π.*

Moreover, for any irreducible smooth representation π, there exists a parabolic subgroup $P = MN \subseteq G$ and a supercuspidal representation σ of M such that $\pi \hookrightarrow i_{G,M}(\sigma)$. In particular, π is admissible.

The proof of the above two theorems can be found in Chapter 3 of [BZ76].

Remark 2.4. The fact that cuspidal and supercuspidal representations are the same is a characteristic 0 phenomenon. For mod-p representations, supercuspidal implies cuspidal, but not the other way around.

2.2. **Cuspidal data and inertial type.** Using the Jacquet functor, we see that the structure of $\mathrm{Sm}(G)$ should be governed by cuspidal representations. This will be made precise later, but the immediate goal now is to describe the data that show up.

Definition 2.5. A *cuspidal datum* for G consists of a pair (M, σ), where M is a Levi subgroup of G, and σ is an irreducible cuspidal representation of M. If π is an irreducible subrepresentation of $i_{M,G}\sigma$, then we say (M, σ) is the *cuspidal support* of π.

Theorem 2.6 (Bernstein–Zelevinsky)**.** *The cuspidal support is well-defined, i.e. given a smooth irreducible π, there exists a unique (M, σ) up to conjugation such that $\pi \hookrightarrow i_{M,G}\sigma$.*

We can describe this theorem more explicitly. The parabolic subgroups of G correspond to partitions $n = n_1 + \cdots + n_r$ for some r, so a cuspidal datum with this Levi subgroup is a tuple $(\pi_1, \cdots, \pi_r)$, where each π_i is a cuspidal representation of $\mathrm{GL}_{n_i}(E)$. The equivalence relation means the tuple $(n_1, \cdots, n_r)$ is actually an un-ordered multiset.

The existence part of the theorem is clear from the definition of cuspidal representations and the transitivity of Jacquet functors. For uniqueness, we need what's often called the "basic geometric lemma", which is in essence a non-semisimple geometric version of Mackey theory, applied to the setting of two parabolic subgroups in G. After this, it is not too hard to prove the following decomposition theorem.

Let Ω denote the set of all cuspidal data for G up to equivalence, so we have a well-defined map $\mathrm{Irr}(G) \to \Omega$, which can be shown to have finite non-empty fibres. We will give Ω the structure of an algebraic variety. If G were compact, then arguing similar to the classical Peter–Weyl theory shows that Ω is discrete. Even though $G = \mathrm{GL}_n(E)$ is not compact, we can build up its representation theory by restricting the subgroup G° which has compact centre. The quotient G/G° is isomorphic to $\mathbf{Z}$, and it contributes a continuous part to each component of Ω.

To start making this precise, let $\Psi(G)$ denote the set of unramified characters of G. Its members have the form $g \mapsto z^{v(\det g)}$ for some $z \in K^\times$, so $\Psi(G) \simeq K^\times$. This set acts on $\mathrm{Sm}(G)$ by twisting. It is easy to show that if π, π' are irreducible smooth, then they are in the same orbit if and only if $\pi|_{G^\circ} \simeq \pi'|_{G^\circ}$. Each such orbit is isomorphic to a quotient of $\Psi(G)$. The stabilizer is a finite subgroup, so we get the structure of an algebraic variety on each orbit. The result is an algebraic variety $\Omega_{n,\mathrm{cusp}}$ parametrizing cuspidal representations of G, with infinitely many connected components corresponding to inertial types.

Let M be a Levi subgroup of G, then $M \simeq \mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_r}(E)$. Let $\Omega(M) = \prod_{i=1}^r \Omega_{n_i,\mathrm{cusp}}$. This is almost a subset of Ω. The difference is that the Weyl group for M acts on $\Omega(M)$. This is a finite group, so the quotient by it is still a variety. In summary, we have the following proposition.

Proposition 2.7. *Let $\{M\}$ be a set of representatives of conjugacy classes of Levi subgroups of G (which can be indexed by unordered partitions of n), then*

$$\Omega = \bigsqcup_{\{M\}} \Omega(M)/W(M)$$

Once again, each connected component can be interpreted as an inertial type. In fact, the pre-image of each connected component under $\mathrm{Irr}(G) \to \Omega$ correspond to a representation of the inertia group I_K under the local Langlands correspondence.

2.3. **Decomposition of category.** Using the variety Ω introduced in the previous section, we can now describe $\mathrm{Sm}(G)$ as a direct product of subcategories indexed by cuspidal representations on $\mathrm{GL}_{n'}(E)$ for $n' < n$. This is the main result of [BZ77].

Theorem 2.8. *Let ω be a connected component of Ω. Define the full subcategory $\mathrm{Sm}_\omega(G) \subseteq \mathrm{Sm}(G)$ to be the set of smooth representations whose Jordan–Hölder factors all have cuspidal support in ω. We have a direct product decomposition of categories*

$$\mathrm{Sm}(G) \simeq \prod_{\omega \in \pi_0(\Omega)} \mathrm{Sm}_\omega(G)$$

More explicitly, each $V \in \mathrm{Sm}(G)$ has a functorial decomposition $V = \bigoplus_\omega V_\omega$, where $V_\omega \in \mathrm{Sm}_\omega(G)$.

Corollary 2.9. $\mathfrak{Z}(G) \simeq \prod_\omega \mathfrak{Z}(\mathrm{Sm}_\omega(G))$

2.4. **Description of Bernstein centre.** We will now describe some results on the internal structure of each category $\mathrm{Sm}_\omega(G)$. In this section, we compute its Bernstein centre.

Let $f \in \mathfrak{Z}(G)$ and $\pi \in \mathrm{Irr}(G)$, then Schur's lemma shows that f acts on π by a scalar multiple $\theta_\pi(f)$. This gives us an algebra homomorphism from $\mathfrak{Z}(G)$ to the ring of functions on the set $\mathrm{Irr}(G)$. It is not hard to show that this is injective. We can take the ω-component of each side. The following result is the main theorem of [Ber84].

Theorem 2.10. *Let $\mathcal{O}(\omega)$ be the ring of regular functions on the algebraic variety ω, then the above map induces an isomorphism $\theta_\omega : \mathfrak{Z}(\mathrm{Sm}_\omega(G)) \xrightarrow{\sim} \mathcal{O}(\omega)$.*

We now give some ideas of where the difficulty lies. Let (M, σ) be a cuspidal datum in ω, then we can consider the category $\mathrm{Sm}_\sigma(M)$. Its irreducibles are just unramified twists of σ, so it is not hard to compute that $\mathfrak{Z}(\mathrm{Sm}_\sigma(M)) \simeq \mathcal{O}(\omega_M)$, where ω_M is the variety of cuspidal representations constructed before. Let W be the Weyl group of M in G, then $\omega = \omega_M/W$, so we need to show

$$\mathfrak{Z}(\mathrm{Sm}_\omega(G)) \simeq \mathfrak{Z}(\mathrm{Sm}_\sigma(M))^W$$

It's reasonable to compare the two categories using induction, which requires considering intertwining operators between (M, σ) and (M^w, σ^w) for $w \in W$. The proof ends up showing that $i_{M,G}\sigma$ is generically irreducible as σ varies over ω_M, so the induced representations of two such data in ω are "generically isomorphic". This is enough to construct enough elements in $\mathfrak{Z}(\mathrm{Sm}_\omega(G))$.

Example 2.11. Consider the component ω_{ur} of unramified representations. It consists of cuspidal data of the form (T, χ), where T is the diagonal maximal torus, and χ is an unramified character.. From the description of ω, the ring of regular functions on ω_{ur} is $K[T_i^{\pm 1} \mid 1 \le i \le n]^{S_n}$. The theorem in this case is essentially equivalent to the Satake isomorphism.

2.5. **Bernstein–Zelevinsky classification.** Let (M, σ) be a cuspidal datum. We mentioned in the previous section that $i_{M,G}\sigma$ is generically irreducible as σ varies in the cuspidal component. The Bernstein–Zelevinsky classification goes further and pins down exactly where it is reducible. The reference for this section is [Kud94] and [EH14, Section 4].

To simplify notation, if for $i = 1, \cdots, r$, σ_i is a representation of $\mathrm{GL}_i(E)$, then we let $(\sigma_1, \cdots, \sigma_r)$ denote the representation $\sigma_1 \otimes \cdots \otimes \sigma_r$ of the appropriate block-diagonal Levi subgroup in $\mathrm{GL}_{n_1+\cdots+n_r}(E)$, and we write $I(\sigma_1, \cdots, \sigma_r)$ to mean its parabolic induction to G.

Theorem 2.12. *The induction is reducible if and only if there exists i, j such that $n_i = n_j$ and $\sigma_i = \sigma_j(1)$.*

Therefore, a prominent player in the theory are *segments*, which takes the form $(\sigma, \sigma(1), \cdots, \sigma(m-1))$, where σ is cuspidal.

Theorem 2.13. *The induction $I(\sigma, \sigma(1), \cdots, \sigma(m-1))$ has a unique irreducible quotient, which we call generalized Steinberg representations and denote by $\mathrm{St}_{\sigma,m}$. It is generic and square integrable. Moreover, every square integrable representation has this form.*

Now to understand $I(\sigma_1, \cdots, \sigma_r)$, there are two steps. First we group the representations into segments, and take an induction step to end up with a sequence of generalized Steinberg representations. There are multiple ways to do this, yielding different constituents of $I(\sigma_1, \cdots, \sigma_r)$. We then take a second induction. To describe this induction, make the following definition

Definition 2.14. Two segments $\Delta = (\sigma, \cdots, \sigma(r-1))$ and $\Delta' = (\sigma', \cdots \sigma'(s-1))$ are *linked* if $\Delta \cup \Delta'$ is also a segment, and neither contains the other. We say Δ *precedes* Δ' if they are linked and $\sigma' = \sigma(k)$ for some $k > 0$.

A sequence of segments $\Delta_1, \cdots, \Delta_r$ is in good order if for all $i < j$, Δ_i does not precede Δ_j.

Theorem 2.15. *Let $\Delta_1, \cdots, \Delta_r$ be a collection of segments. We will also use the same symbols to denote their associated generalized Steinberg representations.*

- *If no two segments are linked, then $I(\Delta_1, \cdots, \Delta_r)$ is irreducible.*
- *The irreducible subrepresentations of $I(\Delta_1, \cdots, \Delta_r)$ are generic.*
- *If $\Delta_1, \cdots, \Delta_r$ is in a good order, then $I(\Delta_1, \cdots, \Delta_r)$ has a unique irreducible quotient, which we denote by $Q(\Delta_1, \cdots, \Delta_r)$.*
- *The representation $Q(\Delta_1, \cdots, \Delta_r)$ is independent of the good ordering chosen.*

We can therefore label an irreducible representation by a list of segments. Given a cuspidal representation $(\sigma_1, \cdots, \sigma_r)$, the Jordan–Hölder components of its induction correspond to different ways of combining the representations into segments. This can be made very precise by introducing a partial order on segments, as indicated by the following theorem.

Theorem 2.16. *Let $\mathcal{S}$ be a tuple of segments. Let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by replacing a pair (Δ, Δ') of linked segments by $(\Delta \cap \Delta', \Delta \cup \Delta')$, then $\mathrm{Hom}_G(I(\mathcal{S}'), I(\mathcal{S}))$ is 1-dimensional, and the non-zero maps are injective.*

After a finite number of times of applying this operation, no two segments will be linked, and we will obtain an irreducible representation, which is necessarily generic by the theorem. This is the unique generic component of $I(\mathcal{S})$.

3. Generic local Langlands correspondence

For applications to families, the local Langlands correspondence we have stated is not the best choice because there is no reasonable way to preserve functoriality. The goal of this section is to describe Breuil–Schneider's modification which makes it functorial, but at the cost of replacing $\mathrm{Irr}(G)$ by a collection of possibly reducible *generic* representations. We follow the treatment of [EH14].

3.1. **Classical local Langlands correspondence.** We first review the construction of the classical local Langlands correspondence. To start with, we state some general results on Weil–Deligne representations over a general coefficient ring.

Definition 3.1. Let A be a complete local Noetherian ring with residue characteristic $p \neq \ell$. Let R be a subring of $\mathrm{Frac}(A)$ containing $A[\frac{1}{p}]$. A *Weil–Deligne representations* over R is a pair (ρ', N), where $\rho' : W_E \to \mathrm{GL}_n(R)$ is smooth, and N is a nilpotent endomorphism of $\mathrm{End}(R^n)$ satisfying the compatibility relation $\rho'(w)N\rho'(w)^{-1} = \|w\|\, N$. It is *Frobenius-semisimple* if $\rho' \otimes_R \overline{\mathrm{Frac}(A)}$ is semisimple.

As usual, the data of a Weil–Deligne representation is equivalent to a continuous Galois representation $\mathrm{Gal}_E \to \mathrm{GL}_n(R)$, where now the topology R is the adic topology. This identification is compatible with base change in R. We can also define $(\rho', N)^{\mathrm{ss}}$ to be the Frobenius-semisimplification of $(\rho', N) \otimes_R \mathrm{Frac}(R)$. One can check that the result is still defined over R.

In our cases, R will often be a DVR $\mathcal{O}$ with residue field K containing $\mathbf{Q}_p$. The Frobenius-semisimplification defined in the previous paragraph is compatible with reduction. The goal is to compare the local Langlands correspondence for $(\rho', N) \otimes_{\mathcal{O}} \mathrm{Frac}(\mathcal{O})$ and the reduction $(\rho', N) \otimes_{\mathcal{O}} K$. In this section, we review the correspondence for algebraically closed fields of characteristic 0, and comment on why it is not satisfactory for the intended application.

Theorem 3.2 (Classical local Langlands correspondence)**.** *Let $K = \bar{\mathbf{Q}}_p$, then there is a bijective correspondence between smooth irreducible representations of $\mathrm{GL}_n(E)$ and Frobenius-semisimple Weil–Deligne representations (ρ', N).*

The Weil–Deligne representations (ρ', N) where $N = 0$ and ρ' is irreducible should correspond to supercuspidal representations, and this part was first shown by Harris–Taylor and Henniart. We now describe how to deduce the general result from this part using the Bernstein–Zelevinsky classification.

First we extend the result from cuspidal to square integrable representations. Let $\rho : W_E \to \mathrm{GL}_n(K)$ be irreducible, and let m be a positive integer. We can define an nm-dimensional Weil–Deligne representation $\mathrm{Sp}_{\rho,m} = (\rho', N)$ as follows:

$$\rho' = \bigoplus_{i=0}^{m-1} V_i, \text{ where } V_i = \rho(i), \quad N : V_i \xrightarrow{\sim} V_{i+1} \text{ for } 0 \leq i \leq m-2$$

We call it the special representation. If ρ is attached to σ under the cuspidal correspondence, then we attach $\mathrm{Sp}_{\rho,m}$ with the generalized Steinberg representation $\mathrm{St}_{\sigma,m}$, which we recall is generic and square-integrable.

Every indecomposable Weil–Deligne representation has this form for a unique choice of (ρ, m) up to isomorphism. Therefore, every Frobenius-semisimple Weil–Deligne representations is a direct sum of special representations. We attach to it the Langlands quotient of the corresponding induced representation. By the Bernstein–Zelevinsky classification, this is bijective. The combinatorics of linking segments corresponds to the choices for N given ρ'.

Example 3.3. The trivial representation has parameter $\rho' = \bigoplus_{i=0}^{n} |\cdot|^{\frac{n-2i-1}{2}}$, $N = 0$. The Steinberg representation has the same ρ', but N is the nilpotent matrix with entirely 1 on the off-diagonal.

There are two issues with the correspondence as defined:

(1) It depends on a choice of $\ell^{\frac{1}{2}}$, and some of the representations are not rational.
(2) It does not commute with specialization. For example the family of unramified principal series specializes to character at certain points on the parameter side, but not on the representation side.

These will be remedied essentially by replacing the irreducible representations by the full indecomposable but reducible induction.

3.2. **Modification and properties.** Let K be an arbitrary extension of $\mathbf{Q}_p$. We first define a correspondence $(\rho', N) \mapsto \pi(\rho', N)$, where (ρ', N) is a Weil–Deligne representation over K, and $\pi(\rho', N)$ is an indecomposable admissible representation of $\mathrm{GL}_n(E)$ defined over K. It can be characterized by the following three compatibility properties:

(1) Compatibility with character twists.
(2) Compatibility with field extension in K.
(3) Compatibility with a modified local Langlands correspondence.

We describe the third point in detail, which deals with the case $K = \bar{\mathbf{Q}}_p$. As before, (ρ', N) can be written a direct sum of indecomposable representations $\mathrm{Sp}_{\rho_i, n_i}$. Under the classical local Langlands correspondence, each piece corresponds to a generalized Steinberg representation St_{π_i, n_i}, where π_i is cuspidal. We now define

$$\pi(\rho', N) = i_{Q,G}(\mathrm{St}_{\pi_i, n_i} \otimes \cdots \otimes \mathrm{St}_{\pi_r, n_r}) \otimes |\cdot|^{-\frac{n-1}{2}}$$

The components are ordered in a way that the unique generic component of the induced representation is a subrepresentation. This is equivalent to certain non-linking conditions on the segments. The twist makes the resulting representation rational and independent of the choice of $\ell^{\frac{1}{2}}$.

Definition 3.4. If ρ is a continuous Galois representation, then we define $\pi(\rho)$ to be the representation attached to $(\rho', N)^{\mathrm{ss}}$.

We end the talk with the following list of properties, which is the main result of [EH14, Section 4], with parts of the work dealing with rationality already done in [BS07, Lemma 4.2].

Theorem 3.5. *(1) The generic local Langlands correspondence exists and is unique.*
(2) $\pi(\rho)$ is an essentially AIG representation (its socle is absolutely irreducible and generic, no other components are generic, and it is generated by finite length submodules).
(3) Compatibility with specialization: Let $\mathcal{O}$ be a DVR with uniformizer ϖ, $\mathcal{K} = \mathrm{Frac}(\mathcal{O})$, and $K = \mathcal{O}/\varpi\mathcal{O}$. Let $\rho : \mathrm{Gal}_E \to \mathrm{GL}_n(\mathcal{O})$ be a continuous Galois representation. Up to homothety there exists a unique separated $\mathcal{O}$-lattice $\pi(\rho)$ in $\pi(\rho \otimes_{\mathcal{O}} \mathcal{K})$ such that $\pi(\rho)/\varpi\pi(\rho)$ is AIG. Moreover, $\pi(\rho)/\varpi\pi(\rho)$ embeds into $\pi(\rho/\varpi\rho)$, and there is an exact criterion on the monodromy operator to decide if the embedding is an isomorphism.

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