SOME GENERIC NON-VANISHING RESULTS FOR L-VALUES

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We look at some results of Rohrlich and Cornut–Vatsal on the generic order of vanishing of $L\big(\frac{1}{2}, f \times \chi\big)$ as χ varies over families of finite order characters.

1. Rohrlich's Results

This section concerns the following two theorems.

Theorem 1.1. [Roh89] *Let π be a unitary cuspidal automorphic form on GL_2, then there exists infinitely many Dirichlet characters χ such that $L\big(\frac{1}{2}, \pi \times \chi\big) \neq 0$. The same holds if we specify the primes dividing the conductor of χ.*

Theorem 1.2. [Roh84] *Let f be a weight 2 modular form. Given a fixed finite set of primes P away from the level of f, for all but finitely many Dirichlet characters χ unramified away from P, $L(f \times \chi, 1) \neq 0$.*

The idea of proof for both of them is to study the average $\frac{1}{|X|}\sum_{\chi \in X} L(s_0, \pi \times \chi)$ for X a suitable family of characters. Using analytic methods, we show that this average is non-zero, which is enough for the first theorem. For the second one, we in addition choose X to be a Galois orbit of a character and use the algebraicity of the central value.

The first theorem holds with $\frac{1}{2}$ replaced by any complex number and GL_2 replaced by GL_1. The proof is essentially the same. It also holds with $\mathbf{Q}$ replaced by any number field, but this requires some new ideas in choosing the correct family and is the main innovation in [Roh89]. In the second theorem, we can replace by any even weight modular form, since Shimura's algebraicity theorem still holds. We can also allow P to contain primes dividing the level of f. The only change in the proof is a more complicated expression relating the root numbers of f and $f \times \chi$. There are also p-adic approaches of deducing the "all but finitely many" result from "infinitely many" result, cf. [Gre83].

1.1. Approximate functional equation.

The basic tool we will use is the approximate functional equation, which we will derive roughly following [Har02]. Let $L(s, \pi)$ be the usual automorphic L-function without the factor at infinity, normalized so that it has centre $\frac{1}{2}$. It has a Dirichlet series $L(s, \pi) = \sum_{n=1}^{\infty} a_n n^{-s}$. Let $\tilde{\pi}$ be the contragredient of π, then $L(s, \tilde{\pi}) = \sum_{n=1}^{\infty} \overline{a_n} n^{-s}$. Finally, if χ is a Dirichlet character, then $L(s, \pi \times \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$.

Let $L(s, \pi_\infty)$ be the archimedean factor. Since π_∞ is unitary, we have two cases

$$L(s, \pi_\infty) = \begin{cases} \Gamma_{\mathbf{R}}(s+\mu)\Gamma_{\mathbf{R}}(s-\mu) & 0 \leq \mathrm{Re}(\mu) < \frac{1}{2} \\ \Gamma_{\mathbf{R}}(s + \frac{k-1}{2})\Gamma_{\mathbf{R}}(s + \frac{k+1}{2}) & \pi_\infty \text{ is a discrete series} \end{cases}$$

In this notation, the Selberg $\frac{1}{4}$-conjecture states that $\mathrm{Re}(\mu) = 0$. If π is the automorphic representation associated to a cusp form f of weight k, then $a_n = a_f(n) n^{-\frac{k-1}{2}}$, so the Ramanujan conjecture now states that $|a_p| \leq 2$.

We have a functional equation

$$L(s, \pi_\infty) L(s, \pi) = \epsilon(\pi) N(\pi)^{\frac{1}{2}-s} L(1-s, \tilde{\pi}_\infty) L(1-s, \tilde{\pi})$$

where N is the conductor of π, and $\epsilon(\pi)$ is a complex number of modulus 1. We symmetrize the Γ-factors by setting

$$F(s, \pi_\infty) = \frac{1}{2} + \frac{1}{2} \cdot \frac{L(\frac{1}{2}+s, \pi_\infty)}{L(\frac{1}{2}-s, \tilde{\pi}_\infty)}$$

then $\overline{F(s, \pi_\infty)} = F(\bar{s}, \tilde{\pi}_\infty)$, and the functional equation can be rewritten as

$$F(s, \pi_\infty) L\Big(\frac{1}{2}+s, \pi\Big) = \epsilon(\pi) N(\pi)^{-s} F(-s, \tilde{\pi}_\infty) L\Big(\frac{1}{2}-s, \tilde{\pi}\Big)$$

Let u and σ be positive real numbers. Formally by the residue theorem, we have

$$\begin{aligned} L\Big(\frac{1}{2},\pi\Big) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} L\Big(\frac{1}{2}+s,\pi\Big)F(s,\pi_\infty)u^{-s}\frac{ds}{s} - \frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} L\Big(\frac{1}{2}+s,\pi\Big)F(s,\pi_\infty)u^{-s}\frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} L\Big(\frac{1}{2}+s,\pi\Big)F(s,\pi_\infty)u^{-s}\frac{ds}{s} + \frac{\epsilon(\pi)}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} L\Big(\frac{1}{2}+s,\tilde{\pi}\Big)F(s,\tilde{\pi}_\infty)\Big(\frac{1}{uN(\pi)}\Big)^{-s}\frac{ds}{s} \end{aligned}$$

Define a function

$$I(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s,\pi_\infty)x^{-s}\frac{ds}{s}$$

It's easy to check that the similarly defined integral using $\tilde{\pi}_\infty$ is just $\bar{I}(x)$. We can now shift each contour so σ is sufficiently large, then we can expand the L-value into its Dirichlet series and integrate term by term. This gives the approximate functional equation, with a parameter u

$$L\Big(\frac{1}{2},\pi\Big) = \sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}I(nu) + \epsilon(\pi)\sum_{n=1}^{\infty}\frac{\overline{a_n}}{\sqrt{n}}\bar{I}\Big(\frac{n}{uN(\pi)}\Big)$$

For the above manipulation to work, we need to know that $F(s,\pi_\infty)$ is entire in the region $\mathrm{Re}(s) \geq -\sigma$, and that everything has moderate growth in vertical strips. The holomorphicity hold automatically if π_∞ is a discrete series. Otherwise we need $\sigma < \frac{1}{2}-\mu$. Stirling's approximation gives estimates for $F(s,\pi_\infty)$ in vertical strips, and convexity bound is enough for $L(s,\pi)$. From these, we have the estimates

$$I(x) = F(0,\pi_\infty) + O_\sigma(x^\sigma), \quad I(x) = O_\sigma(x^{-N})$$

as $x \to 0$ and $x \to \infty$ respectively, for any $\sigma < \frac{1}{2}-\mu$ and $N > 0$. Both constants can be made explicit. In particular, $|I(x)|$ is bounded.

1.2. **Averaging.** Let X be a finite set of Dirichlet characters. We want to study

$$L_{\mathrm{av}}(\pi) := \frac{1}{|X|}\sum_{\chi\in X} L\Big(\frac{1}{2},\pi\times\chi\Big)$$

The goal is to bound this away from zero for suitably chosen X. Twisting by a finite order character does not change the archimedean local factor, so using the approximate functional equation, we can rewrite the average as

$$L_{\mathrm{av}}(\pi) = \sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}\Big(\frac{1}{|X|}\sum_{\chi\in X}\chi(n)\Big)I(nu) + \sum_{n=1}^{\infty}\frac{\overline{a_n}}{\sqrt{n}}\Big(\frac{1}{|X|}\sum_{\chi\in X}\bar{\chi}(n)\epsilon(\pi\times\chi)\Big)\bar{I}\Big(\frac{n}{uN(\pi\times\chi)}\Big)$$

Before continuing, we recall how ϵ and N changes under twists. Let q be the conductor of χ and N be the conductor of π. In either of the two theorems, we may assume $(q,N)=1$. Under this assumption,

$$N(\pi\times\chi) = Nq^2, \quad \epsilon(\pi\times\chi) = \epsilon(\pi)\chi_\pi(q)\chi(N)\frac{G(\chi)^2}{q}$$

where χ_π is the central character of π, and $G(\chi) = \sum_{(k,q)=1}\chi(k)e^{2\pi ik/q}$ is the Gauss sum.

To deal with the coefficients a_n, the Ramanujan conjecture would give $|a_n| \ll_\epsilon n^\epsilon$ for all $\epsilon > 0$. This is not known for cuspidal Maass forms, so we will use the following estimates

$$\sum_{n\leq x}|a_n|^2 \ll_\epsilon x^{1+\epsilon}, \quad |a_n| \ll_\epsilon n^{\frac{1}{4}+\epsilon}$$

for all $\epsilon > 0$. The first comes from the Rankin–Selberg method, and it gives $|a_n| \ll_\epsilon n^{\frac{1}{2}+\epsilon}$, which we will see later is not good enough. The second sharper estimate for the individual terms comes from symmetric square functoriality. An easy corollary of the results is that

$$\frac{|a_n|}{\sqrt{n}} \ll_\epsilon n^{-\frac{1}{4}+\epsilon}, \quad \sum_{n\leq x}\frac{|a_n|}{\sqrt{n}} \ll_\epsilon x^{\frac{1}{2}+\epsilon}$$

We now suppose X consist entirely of characters with conductor q and study the asymptotic behaviour of $L_{\text{av}}(\pi)$ as $q \to \infty$. Let $u = q^{-\gamma}$ for some $\gamma > 0$. This determines the lengths of the two sums. Define

$$\chi_{\text{av}}(n) = \frac{1}{|X|} \sum_{\chi \in X} \chi(n), \quad \tilde{\chi}_{\text{av}}(n) = \frac{1}{|X|} \sum_{\chi \in X} \bar{\chi}(n)\epsilon(\pi \times \chi)$$

The main term of the expression for $L_{\text{av}}(\pi)$ will be $I(0) \neq 0$. The error has four components

$$\begin{aligned} |L_{\text{av}}(\pi) - I(0)| &\ll \sum_{2 \le n \le q^a} |\chi_{\text{av}}(n)| \frac{|a_n|}{\sqrt{n}} |I(nu)| + \sum_{1 \le n \le q^b} |\tilde{\chi}_{\text{av}}(n)| \frac{|a_n|}{\sqrt{n}} \left| I\Big(\frac{n}{uNq^2}\Big) \right| + \sum\nolimits_{\text{tail}} + |I(0) - I(u)| \\ &\ll_\epsilon \sum_{2 \le n \le q^a} |\chi_{\text{av}}(n)| \, n^{-\frac{1}{4}+\epsilon} + \max_n |\tilde{\chi}_{\text{av}}(n)| \, q^{b(\frac{1}{2}+\epsilon)} + \sum\nolimits_{\text{tail}} + q^{-\sigma\gamma} \end{aligned}$$

where $a, b > 0$ are cut-off exponents to be chosen later, and σ is a sufficiently small real number depending on π_∞ (which can be made absolute using progresses on the Selberg conjecture). The final term is $o(1)$, and the parameters we have left are a, b, γ, ϵ.

We use the trivial estimates $|\chi_{\text{av}}(n)|, |\tilde{\chi}_{\text{av}}(n)| \le 1$ for the tail. Recall that $I(x)$ decays faster than any polynomial in x, so for any $N > 0$, we have

$$\begin{aligned} \sum\nolimits_{\text{tail}} &\ll_N u^{-N} \sum_{n > q^a} n^{-\frac{1}{4}-N+\epsilon} + q^{2N} u^N \sum_{n > q^b} n^{-\frac{1}{4}-N+\epsilon} \\ &\ll_N q^{\frac{3}{4}a + (\gamma - a)N + a\epsilon} + q^{\frac{3}{4}b + (2-\gamma-b)N + b\epsilon} \end{aligned}$$

This tends to zero for some choice of N if $a > \gamma$ and $b > 2 - \gamma$.

Now we are left with estimating the character average $\chi_{\text{av}}(n)$ and the twisted character average $\tilde{\chi}_{\text{av}}(n)$. First observe that $\chi_{\text{av}}(n) = 1$ whenever $n \equiv 1 \pmod{q}$, so the first main term cannot be too long. For it to decay with q, we also need that $\chi_{\text{av}}(n) = 0$ for small n. The second main term also cannot be too long, so we expect a, b, γ all close to 1. The key feature there is that $\tilde{\chi}_{\text{av}}(n)$ actually tends to 0 uniformly in n as $q \to \infty$, using additional cancellation from Kloosterman sums.

1.3. **Second term.** In this section, we reduce the estimation of $\tilde{\chi}_{\text{av}}$ to the estimation of χ_{av}. This is independent of the choice of X. Given n coprime to q, recall that

$$\tilde{\chi}_{\text{av}}(n) = \frac{1}{|X|} \sum_{\chi \in X} \bar{\chi}(n)\epsilon(\pi \times \chi) = \frac{\epsilon(\pi)\chi_\pi(q)}{q\,|X|} \sum_{\chi \in X} \chi(n') G(\chi)^2$$

where n' is chosen such that $nn' \equiv N \pmod{q}$.

We expand the Gauss sum

$$G(\chi)^2 = \sum_{1 \le j,k \le q} \chi(jk) e^{\frac{2\pi i (j+k)}{q}} = \sum_{c=1}^{q} \chi(c) K(1, c; q)$$

where $K(1, c; q)$ is the Kloosterman sum $\sum_{k=1}^{q} e^{\frac{2\pi i}{q}(k + ck^{-1})}$. Averaging over X, we get that

$$|\tilde{\chi}_{\text{av}}(n)| \le \frac{1}{q} \sum_{c=1}^{q} |\chi_{\text{av}}(n'c)| \cdot |K(1, c; q)|$$

The Weil bound gives $|K(1, c; q)| \ll_\epsilon q^{\frac{1}{2}+\epsilon}$. In the next section, we will show that $\sum_{c=1}^{q} |\chi_{\text{av}}(c)| \ll_\epsilon q^\epsilon$ for some choices of X. In both cases, ϵ is any positive real number, so combining them gives the estimate

$$|\tilde{\chi}_{\text{av}}(n)| \ll_\epsilon q^{-\frac{1}{2}+\epsilon}$$

Therefore, the second main term is bounded by $q^{\frac{1}{2}(b-1)+\epsilon}$. This imposes the condition $b < 1$, which combined with $2 - \gamma < b$ implies $\gamma > 1$.

1.4. **First term.** In what follows, fix P to be a finite set of primes not containing any bad prime for π and restrict the prime divisors of q to P. Further fix a number field F. Let χ be a character of conductor q. We will take the family

$$X = \{\chi^\sigma \mid \sigma \in \mathrm{Gal}(F(\chi)/F)\}$$

When applied to the second theorem, F will be the field generated by the Fourier coefficients of f, and this ensures $L(f \times \chi, 1) = 0 \implies L(f \times \chi^\sigma, 1) = 0$. For application to the first theorem, we can just take $F = \mathbf{Q}$, and in fact the larger family of all characters of conductor q work, with a simpler proof.

The first term has length q^a, where $a > \gamma > 1$. We split it into a full period and the remainder term.

$$\sum_{2 \le n \le q^a} |\chi_{\mathrm{av}}(n)| \, n^{-\frac{1}{4}+\epsilon} = \sum_{2 \le n \le q} |\chi_{\mathrm{av}}(n)| \, n^{-\frac{1}{4}+\epsilon} + \sum_{q < n < q^a} |\chi_{\mathrm{av}}(n)| \, n^{-\frac{1}{4}+\epsilon}$$

Call them $\sum_I$ and $\sum_{II}$. To estimate $\sum_{II}$, we group it by periods

$$\sum\nolimits_{II} \le q^{-\frac{1}{4}+\epsilon} \sum_{q < n < q^a} |\chi_{\mathrm{av}}(n)| \le q^{-\frac{1}{4}+\epsilon+(a-1)} \sum_{n=1}^{q} |\chi_{\mathrm{av}}(n)|$$

Suppose the sum is $O_\epsilon(q^\epsilon)$ for any $\epsilon > 0$, then for $a < 1 + \frac{1}{4}$, we have $\sum_{II}$ to tend to 0 as $q \to \infty$. For $\sum_I$, we need to deal with small n terms, and we will prove that $\chi_{\mathrm{av}}(n) = 0$ if $2 \le n \le q^\delta$ for some constant δ depending only on P, so $\sum_I \ll_\epsilon q^{-\frac{1}{4}\delta+\epsilon} \sum_{n=1}^{q} |\chi_{\mathrm{av}}(n)| \ll_{\epsilon'} q^{-\frac{1}{4}\delta+\epsilon'}$ by the same lemma. Therefore, we are reduced to proving the following lemma in algebraic number theory.

Lemma 1.3. *Let P be a finite set of primes. Let X be the set of characters of conductor q, with the prime divisors of q lying in P. Then*

(1) There exists $\delta > 0$ depending only on P such that $\chi_{\mathrm{av}}(n) = 0$ if $2 \le n \le q^\delta$.
(2) For all $\epsilon > 0$, $\sum_{n=1}^{q} |\chi_{\mathrm{av}}(n)| \ll_\epsilon q^\epsilon$.

Proof. Factor $\chi = \prod_{p \in P} \chi_p$, where χ_p has conductor p^a. We can extend χ_p to a character of $\mathbf{Z}_p^\times$ which is trivial on $1 + p^a\mathbf{Z}_p$. Moreover for notational simplicity, suppose $p \ne 2$. Given n, we can write $n = \xi(1+p)^u$ where $u \in \mathbf{Z}_p$ and ξ is a $(p-1)$-th root of unity. Then $\chi_p((1+p)^u)$ is a p-power root of unity. Let p^μ be its exact order, so $\chi_p((1+p)^{up^\mu}) = 1$, which implies $v_p(u) \ge a - (\mu+1)$. It follows that $v_p(n - \xi) \ge a - \mu$.

We will show that if μ is too large, then $\chi_{\mathrm{av}}(n) = 0$. Let $\chi(n) = \zeta \in F(\chi)$, then $\chi_{\mathrm{av}}(n) = \mathrm{Tr}_{F(\chi)/F}\, \zeta$. Consider the sub-extension $F(\zeta)/F(\zeta^p)$. By considering the cyclotomic $\mathbf{Z}_p$-tower over F, we see that there exists a constant $C_{p,F}$ such that if the p-part of the order of ζ is at least $p^{C_{p,F}}$, then this extension is non-trivial. But then $\mathrm{Tr}_{F(\zeta)/F(\zeta^p)}\, \zeta = 0$, so $\chi_{\mathrm{av}}(n) = 0$. Now suppose $\mu > v_p(\ell - 1)$ for all $\ell \in P$, then the p-part of the order of ζ comes entirely from $\chi_p(n)$. If in addition $\mu > C_{p,F}$, then the above argument shows that $\chi_{\mathrm{av}}(n) = 0$. This proves the assertion.

Therefore, we get a constant C depending on P and F such that if $\chi_{\mathrm{av}}(n) \ne 0$, then there exists a root of unity $\xi \in \mathbf{Z}_p$ such that $v_p(n - \xi) \ge a - C$. But $\xi^{p-1} - 1 = 0$, so we can use basic results in Diophantine approximation to bound n from below. Explicitly, if $n \ne 1$, then

$$|n - \xi|_p \ge \left|n^{p-1} - 1\right|_p \ge \left|n^{p-1} - 1\right|^{-1} \ge n^{-p}$$

Doing this for each $p \in P$ shows that if $\chi_{\mathrm{av}}(n) \ne 0$, then

$$\prod_{p \in P} |n - \xi|_p \le \prod_{p \in P} p^{-a_p + C} = D_F q^{-1}, \quad D_F = \prod_{p \in P} p^C$$

and on the other hand

$$\prod_{p \in P} |n - \xi|_p \ge \prod_{p \in P} n^{-p} = n^{-\gamma}, \quad \gamma = \sum_{p \in P} p$$

Combining those two inequalities shows that any $\delta < \gamma^{-1}$ works for the first statement.

For the second statement, we need to count the number of $n \le q$ such that n is close to a root of unity in $\prod_{p \in P} \mathbf{Z}_p$. There are $\prod_{p \in P}(p-1)$ roots of unities. Around each of them, there are $\prod_{p \in P} p^C$ possibilities, so we see that the total number is in fact bounded above by a constant depending only on F and P, and independent of q. □

2. Cornut–Vatsal's work

(To be added...)

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