EULER SYSTEMS AND RELATIVE SATAKE ISOMORPHISM

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Abstract. We explain how the tame norm relation in Euler systems is a formal consequence of the unramified Plancherel formula in the relative Langlands program and a mild refinement of the generalized Cartan decomposition. This uniformly recovers many of the known Euler systems and produces a new split anticyclotomic Euler system in a case studied by Cornut.

Contents

1. Introduction

The Bloch–Kato conjecture suggests a deep connection between the arithmetic of motives and special values of L-functions. The method of Euler systems is a method of understanding this connection. An Euler system consists of a family of motivic classes satisfying two types of relations: "wild" ones for relations in a p-adic tower, and "tame" ones for relations at places away from p. Experiences suggest that the wild norm relations require quite special conditions on p, such as some form of ordinarity. On the other hand, tame norm relations seem to exist in much greater generalities, and they are already enough to deduce some cases of the Bloch–Kato conjecture, cf. [JNS24, LS24]. However, in the many known examples of Euler systems, the construction of classes satisfying the tame norm relations usually requires some *ad hoc* choices followed by extensive case-by-case calculations.

On the automorphic side of this picture, one way to get a handle on special values of L-functions is through period integrals. Correspondingly, all known constructions of Euler systems are based (perhaps implicitly) on motivic interpretations of period integrals. Recently, Ben-Zvi–Sakellaridis–Venkatesh proposed a relative Langlands program, which is a far-reaching framework for organizing these period integrals centered around spherical varieties and their generalizations [BZSV24]. It is natural to ask if there is an arithmetic analogue of this framework, which should organize the many constructions of motivic classes (or their realizations) in the literature.

In this paper, we examine Euler systems from the relative Langlands point of view. We explain that this gives an automatic way of producing tame norm relations. Our method is computation-free, and we uniformly recover many of the known examples, some of which are summarized in Table 1. More details can be found in §4.4, 4.5.

The idea of using spherical varieties in the construction of Euler systems goes back to Cornut [Cor18]. Unfortunately, there is a gap in his construction related to the definition of Hecke operators, as explained in [Sha23]. For this case, we produce a new split anticyclotomic Euler system and provide an independent proof of the main arithmetic application on Bloch-Kato conjecture.

1.1. **Simple case of main result.** We will state the application of our results for the pushforward of cycles, covering the first three rows of the above table. We construct split anticyclotomic Euler systems in the sense of Jetchev–Nekovář–Skinner [JNS24], which we will simply call JNS Euler systems.

Group	Spherical variety	Attribution
$\mathrm{U}(n) \times \mathrm{U}(n+1)$	$\mathrm{U}(n)\backslash\mathrm{U}(n) \times \mathrm{U}(n+1)$	Lai–Skinner [LS24]
$\mathrm{U}(2n)$	$\mathrm{U}(n) \times \mathrm{U}(n)\backslash\mathrm{U}(2n)$	Graham–Shah [GS23]
$\mathrm{SO}(2n+1)$	$\mathrm{U}(n)\backslash\mathrm{SO}(2n+1)$	Cornut [Cor18]
$\mathrm{GL}_2 \times \mathrm{GL}_2$	$\mathbf{G} \times^{\mathrm{GL}_2} \mathtt{std}$	Lei–Loeffler–Zerbes [LLZ14]
$\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$	$\uparrow$	Grossi [Gro20]
$\mathrm{GU}(2,1)$	$\uparrow$	Loeffler–Skinner–Zerbes [LSZ22a]
$\mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2$	$\mathbf{G} \times^{\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2} \mathtt{std}$	Hsu–Jin–Sakamoto [HJR20]

TABLE 1. Examples of spherical varieties giving rise to tame parts of Euler systems

Let E/F be a CM extension. Suppose $\mathbf{H} \hookrightarrow \mathbf{G}$ are reductive groups with Shimura varieties $\mathrm{Sh}_{\mathbf{H}} \hookrightarrow \mathrm{Sh}_{\mathbf{G}}$ defined over E such that the basic numerology

$$\dim \mathrm{Sh}_{\mathbf{G}} = 2 \dim \mathrm{Sh}_{\mathbf{H}} + 1$$

holds. After fixing level structures for $\mathbf{G}$ and $\mathbf{H}$, this defines a special cycle on $\mathrm{Sh}_{\mathbf{G}}$ in the arithmetic middle dimension, which we expect is related to the central derivative of an L-function.

We set up some notations for a JNS Euler system. Let $\mathscr{L}$ be the set of places of F which split in E, with an explicit finite set of exceptions. For each $\ell \in \mathscr{L}$, fix a place λ of E above ℓ, and let Frob_λ be the arithmetic Frobenius at λ. Let $\mathscr{R}$ be the set of square-free products of places in $\mathscr{L}$. For $\mathfrak{m} \in \mathscr{R}$, let $E[\mathfrak{m}]$ be the ring class field of conductor $\mathfrak{m}$, so it is associated to the order $\mathcal{O}_F + \mathfrak{m}\mathcal{O}_E$ by class field theory. Fix a minuscule representation V of the dual group $\check{G}$ of $\mathbf{G}$. For each ℓ, let $\mathcal{H}_{V,\ell}(X)$ be the Hecke polynomial of V at the place ℓ (cf. Example 3.12).

Theorem 1.1 (Proposition 4.8+Corollary 4.6)**.** *Let $d = \dim \mathrm{Sh}_{\mathbf{G}}$. Suppose $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ is a spherical $\mathbf{G}$-variety, and there is a character $\nu : \mathbf{H} \to \mathrm{U}(1)$ satisfying the "combinatorially trivial" condition of Example 2.3 such that the induced Shimura datum on $\mathrm{U}(1)$ is non-trivial. Then there exists a collection of classes*

$$\left\{ z_{\mathfrak{m}} \in \mathrm{H}^{d+1}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}/E[\mathfrak{m}]}, \mathbb{Z}_p(d)) \,\middle|\, \mathfrak{m} \in \mathscr{R} \right\}$$

such that whenever $\mathfrak{m}, \mathfrak{m}\ell \in \mathscr{R}$, we have the tame norm relation

$$\mathrm{Tr}^{E[\mathfrak{m}\ell]}_{E[\mathfrak{m}]} z_{\mathfrak{m}\ell} = \mathcal{H}_{V,\ell}(\mathrm{Frob}_\lambda^{-1}) \cdot z_{\mathfrak{m}}.$$

Moreover z_1 is the image of the special cycle under the continuous étale cycle class map.

We can in fact replace $\mathbb{Z}_p$ by more general coefficient systems. Under some automorphic assumptions, a standard Abel–Jacobi map procedure described in §4.3 gives the required Euler system in Galois cohomology.

Corollary 1.2. *Suppose in addition that*

- $\mathbf{G}$ *is anisotropic modulo centre.*
- *Condition (C') of* [MS19] *holds for* $\mathbf{G}$.
- *Kottwitz's conjecture* [BR94, Conjecture 5.2] *holds for the middle degree cohomology of* $\mathrm{Sh}_{\mathbf{G}/\bar{E}}$.

Let π be a stable cohomological automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ distinguished by $\mathbf{X}$. Let ρ_π be the p-adic Galois representation attached to π and the Shimura cocharacter for $\mathbf{G}$.

Under the above set-up, there is a lattice T_π in ρ_π and a collection of Galois cohomology classes

$$\{c_{\mathfrak{m}} \in \mathrm{H}^1(E[\mathfrak{m}], T_\pi) \,|\, \mathfrak{m} \in \mathscr{R}\}$$

forming the tame part of a JNS Euler system. In other words, whenever $\mathfrak{m}, \mathfrak{m}\ell \in \mathscr{R}$, we have the tame norm relation

$$\mathrm{Tr}^{E[\mathfrak{m}\ell]}_{E[\mathfrak{m}]} c_{\mathfrak{m}\ell} = P_\lambda(\mathrm{Frob}_\lambda^{-1}) c_{\mathfrak{m}},$$

where $P_\lambda(X) = \det(1 - X\mathrm{Frob}_\lambda | \rho_\pi)$ is the characteristic polynomial of Frob_λ.

Remark 1.3. We briefly explain the roles of the conditions, which are not too serious thanks to a large body of work in the area.

- The anisotropic modulo centre is a simplifying condition so that the Shimura variety is compact and we can directly apply the above cited works.

– Condition (C') is a collection of statements related to Arthur's conjecture for $\mathbf{G}$, If $\mathbf{G}$ is a unitary or orthogonal group, then condition (C') is known (cf. the discussion after Remark 1.6 in [MS19]). Their result is only used to modify our classes $z_{\mathfrak{m}}$ to be null-homologous. In the unitary case, [LL21, Proposition 6.9] also suffices for this purpose.
– Kottwitz's conjecture is used to show that ρ_π actually contributes to the cohomology of $\mathrm{Sh}_{\mathbf{G}/\bar{E}}$. In all of our cases, the Shimura variety is of abelian type, and what we need follows from the work of Kisin–Shin–Zhu [KSZ21].

By combining our construction with the results of [JNS24], we obtain the following implication.

Theorem 1.4. *With notations as in the previous corollary, suppose the Galois representation* $\rho_\pi : \mathrm{Gal}_E \to \mathrm{GL}(V_\pi)$ *satisfies the following conditions.*

(1) ρ_π *is absolutely irreducible.*
(2) There exists $\sigma \in \mathrm{Gal}_{E[1](\mu_{p^\infty})}$ *such that* $\dim V_\pi/(\sigma-1)V_\pi = 1$.
(3) There exists $\gamma \in \mathrm{Gal}_{E[1](\mu_{p^\infty})}$ *such that* $V_\pi/(\gamma-1)V_\pi = 0$.

Then

$$c_1 \neq 0 \implies \dim \mathrm{H}^1_f(E, \rho_\pi) = 1$$

We also have a version of the above results for the pushforward of Eisenstein classes (Example 4.3), which recovers the final four entries in the table. However, they depend on numerical consequences of the results of Sakellaridis–Wang [SW22] which are not yet available in the mixed characteristics setting.

Remark 1.5. Since we are not using wild norm relations, there is no specific hypothesis on the prime p in any of the above results.

1.2. **Idea of proof.** The theorem above is a combination of the following two steps which are completely different in nature:

(1) Construction of a "motivic theta series" (Definition 4.1).
(2) Local harmonic analysis on spherical varieties (Proposition 3.13).

We now explain each item in turn.

1.2.1. *Motivic theta series.* By definition, this is a $\mathbf{G}(\mathbb{A}^{p\infty})$-equivariant map between an adèlic function space and certain "motivic classes". In the settings considered in this paper, the continuous étale cohomology group plays the role of this space of motivic classes. For arithmetic applications, it is important to have integral coefficients on the target cohomology group.

For the pushforward construction, versions of this map with *rational* coefficients have appeared, for example in [LSZ22a, Definition 9.2.3] and [GS23, Proposition 9.14]. Our first main idea is that by considering the correct function space, there is a natural integral refinement of this construction.

Observation 1.6 (Definition 4.1, Propositions 4.8, 4.11)**.** In many cases, including the ones cited above, a more natural statement is that there is an *integral*, $\mathbf{G}(\mathbb{A}^{p\infty})$-equivariant map

$$C_c^\infty(\mathbf{X}(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \to \{\text{Integral motivic classes}\},$$

where $\mathbf{X}$ is a spherical variety.

In the cycles case, we take $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$, so we are interested in the function space $C_c^\infty(\mathbf{H}\backslash\mathbf{G}, \mathbb{Z}_p)$. In previous works, the source space is often taken to be a space of coinvariants $C_c^\infty(\mathbf{G}, \mathbb{Z}_p)_{\mathbf{H}}$. These two spaces are equal after tensoring with $\mathbb{Q}_p$, but they are *different* integrally. In other words, our observation is that $C_c^\infty(\mathbf{X}, \mathbb{Z}_p)$ is the more natural integral structure. The cases involving Eisenstein classes can be treated similarly, taking $\mathbf{X}$ to be the vector bundle

$$\mathbf{X} = \mathbf{G} \times^{\mathrm{GL}_2} \mathtt{std} = \mathrm{GL}_2\backslash(\mathbf{G} \times \mathtt{std}),$$

instead of the previously used coinvariant space $C_c^\infty(\mathbf{G} \times \mathtt{std}, \mathbb{Z}_p)_{\mathrm{GL}_2}$. Here, $\mathtt{std}$ is the standard two-dimensional representation of GL_2.

This observation was already used by the third named author in [LS24] to bypass some Iwasawa-theoretic arguments needed previously to deal with torsions.

1.2.2. *Local harmonic analysis.* Once such a motivic theta series exist, the problem of tame norm relations is entirely reduced to constructing test vectors satisfying norm relations in $C_c^\infty(X, \mathbb{Z}_p)$, where $X = \mathbf{X}(F_\ell)$.

The first question to understand is the origin of the field extension $E[\mathfrak{m}]$. Inspired by [Loe21, GS23], we introduce an augmented group $\widetilde{\mathbf{G}} = \mathbf{G} \times \mathbf{T}$, where $\mathbf{T}$ is a one-dimensional torus that is supposed to parameterize character twists. By considering the desiderata of such an augmentation, we are led to the definition of a combinatorially trivial $\mathbf{T}$-bundle (Definition 2.2). This ensures that π is $\mathbf{X}$-distinguished implies that $\pi \times \chi$ is $\widetilde{\mathbf{X}}$-distinguished for any $\chi \in \hat{\mathbf{T}}$, giving rise to a family of character twists.

Using this bundle, we can define two level structures: for $i = 0, 1$, let $K^i = \mathbf{G}(\mathcal{O}) \times \mathbf{T}^i$, where $\mathbf{T}^0 = \mathbf{T}(\mathcal{O})$ and $\mathbf{T}^1$ is the subset which is congruent to 1 modulo the uniformizer. The existence of tame norm relations is reduced to the following question: given a Hecke operator $\mathcal{H}_\ell$ and a "basic element" Φ_0,

$$(\dagger) \qquad \text{is it true that } \mathcal{H}_\ell \cdot \Phi_0 \in \mathrm{Tr}_{K^0}^{K^1} C_c^\infty(X, \mathbb{Z}_p)^{K^1}?$$

This is *stronger* than just requiring the function $\mathcal{H}_\ell \cdot \Phi_0$ to take values in $\mathbb{Z}_p$. Indeed, in the extreme case where the K^0-orbit of a point $x \in X$ coincides with its K^1-orbit, we need the value at x to be divisible by the index $[K^0 : K^1] = \ell - 1$. We give a necessary condition for when this kind of additional divisibility requirements can occur, in terms of the geometry of $\mathbf{X}$ (Proposition 3.4). This is done by extending the proof of the generalized Cartan decomposition due to Gaitsgory–Nadler [GN10, Theorem 8.2.9] and Sakellaridis [Sak12, Theorem 2.3.8],

To verify these divisibility conditions, we need to compute $\mathcal{H}_\ell \cdot \Phi_0$. Since $\mathcal{H}_\ell$ is described using its Satake image, we use works on the *relative* Satake isomorphism by Sakellaridis [Sak13] and Sakellaridis–Wang [SW22]. Our second main idea is the following.

Observation 1.7 (Proposition 3.9)**.** The structure of the inverse relative Satake transform implies additional divisibility properties.

In the cases considered in this paper, the formula for the inverse relative Satake isomorphism gives the values of the function $\mathcal{H}_\ell \cdot \Phi_0$ as polynomials in ℓ. Now view ℓ as a formal variable. Instead of computing this polynomial, we specialize at $\ell = 1$ and show by an anti-symmetry argument that the result is 0 at certain points in X (along "walls of type T"). This proves its divisibility by $\ell - 1$ at those points. This additional divisibility is a new phenomenon in the relative setting and not present in the classical Satake isomorphism (Remark 3.10).

Combining these two computations, we note that this automatic divisibility is stronger than the geometric requirements, so the answer to (†) is yes, and we obtain the tame norm relation!

1.3. **Related works.** In our language, the gap in Cornut's construction [Cor18] (see [Sha23]) is a difference between functions on $\mathbf{X}$ and distributions on $\mathbf{X}$. In this paper, we establish tame norm relations at split primes. We believe our framework also works in the original non-split setting studied by Cornut and we plan to pursue such a generalization in a future paper (see §1.4.1 below).

In the other works cited in Table 1, the Euler system is constructed using the zeta integral method first developed by Loeffler–Skinner–Zerbes [LSZ22b]. In this method, one writes down a carefully chosen candidate for the test vector in $C_c^\infty(X, \mathbb{Z}_p)^{K^1}$ and verifies that its trace is equal to $\mathcal{H}_\ell \cdot \Phi_0$ by an explicit, often intricate, zeta integral computation. It also relies crucially on local multiplicity one. Our method avoids both the computations and the local multiplicity one hypothesis.

In certain spherical settings where classes are obtained by a pushforward construction, Loeffler gave a construction of wild norm relations [Loe21]. He also considered more general cases where a mirabolic subgroup has an open orbit (Definition 4.1.1 of *op. cit.*). In many such cases, we reinterpret the mirabolic subgroup as the point stabilizer of a naturally occurring affine *inhomogeneous* spherical variety. However, in some examples such as $(\mathrm{GSp}_4, \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2)$, only a parabolic subgroup of $\mathbf{G}$ has an open orbit. Our method does not handle this case. This is an Eisenstein degeneration of the spherical pair $(\mathrm{SO}_4 \times \mathrm{SO}_5, \mathrm{SO}_4)$, in a precise sense explained in [LR24], so it would be interesting to understand how this can be interpreted as an operation on spherical varieties.

During the preparation of this work, the preprint [Sha24] was posted. Shah considered a similar local question as (†), phrased using double cosets on the group $\mathbf{G}$. At this point, the *classical* Satake isomorphism is used in *op. cit.*, leading to complicated expressions involving Kazhdan–Lusztig polynomials, which need to be computed on a case-by-case basis. However, his method can treat certain non-spherical cases, which at the present falls outside the conjectural framework of [BZSV24].

1.4. **Further works.**

1.4.1. *Removing conditions.* To make our results unconditional, we would need to have the function-level results of [SW22] for mixed characteristic local fields. Such a statement should follow from motivic integration techniques, along the lines of [CHL11]. The third named author plans to write a short note on this matter in the future, though it would certainly be more desirable to have a sheaf-level statement and proof.

It would also be useful to have a non-split version of our results. Indeed, in certain twisted settings, even the construction of a split anticyclotomic Euler system requires non-split groups. The necessary local harmonic analysis results should be within reach of current methods. For example, [SW22] only assumes quasi-split, and [CZ23] handles many symmetric cases.

1.4.2. *Speculations.* One byproduct of our observation of automatic divisibility (Proposition 3.9) is that there should be some modification to the local unramified conjecture [BZSV24, Conjecture 7.5.1] when the coefficient ring is not a field, cf. §9.4 of *op. cit.* We hope this provides a useful piece of phenomenon in the future refinement of this framework.

Perhaps the deepest question raised by our work is to understand a space of the form

$$\mathrm{Hom}_{\mathbf{G}(\mathbb{A})}(\mathrm{Fun}(\mathbf{X}(\mathbb{A}), \mathbb{Z}), \{\text{Integral motivic classes}\})$$

in larger generality. Indeed, replacing "integral motivic classes" with "automorphic functions", then this space (though not its two constituent pieces) has a conjectural dual description in the relative Langlands program. In addition to the pushforward of cycles or motivic classes described above, the arithmetic theta lifting (cf. [Liu11, LL21, Dis24]) should also be part of this framework. It would also be interesting to understand its relation with the recent boundary class construction of Skinner–Sangiovanni-Vincentelli [SV24].

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2. Spherical varieties

Let F be a field of characteristic 0, not necessarily algebraically closed. Unless otherwise specified, everything in this section will be defined over F.

Let $\mathbf{G}$ be a split reductive group with a Borel subgroup $\mathbf{B}$. Let $\mathbf{X}$ be a variety with a right $\mathbf{G}$-action. Recall that $\mathbf{X}$ is *spherical* if $\mathbf{B} \subseteq \mathbf{G}$ acting on $\mathbf{X}$ has an open orbit. We make the following assumptions

(1) $\mathbf{X}$ is smooth, affine, connected.
(2) $\mathbf{X}$ has no root of type N (cf. §2.1.2).
(3) Every $\mathbf{B}$-orbit on $\mathbf{X}_{/\overline{F}}$ contains an F-point.
(4) $\mathbf{X}$ has an invariant $\mathbf{G}$-measure.

The assumptions imply that there is a unique open $\mathbf{B}$-orbit even on the level of F-points. We denote this orbit by $\mathring{\mathbf{X}}$. Fix once and for all a point $x_0 \in \mathring{\mathbf{X}}(F)$. Let $\mathbf{H}$ be the stabilizer of x_0. Let $\mathbf{X}^\bullet$ be the open $\mathbf{G}$-orbit in $\mathbf{X}$, so $\mathbf{X}^\bullet \simeq \mathbf{H}\backslash\mathbf{G}$, and it contains $\mathring{\mathbf{X}}$ as an open dense subset.

Remark 2.1. Assumptions (1) and (2) roughly correspond to the assumptions imposed in [BZSV24] in the case of polarized Hamiltonian varieties. There have been progresses towards the unramified Plancherel formula without some of these hypothesis, for example [SW22] for certain singular varieties, and [CZ23] for certain varieties with roots of type N. It would be interesting to see if the behaviour observed in this paper still holds in these settings.

Assumption (3) is purely a matter of convenience for the present paper to rule out spherical roots of type T non-split, which requires a separate analysis. In a future work, we plan to remove it and moreover consider the case when $\mathbf{G}$ is not split.

Assumption (4) is included also for simplicity of notation. It will hold if $\mathbf{H}$ is reductive or if $\mathbf{X}$ is of the form $\mathbf{G} \times^{\mathbf{H}'} V$, where $\mathbf{H}'$ is reductive and V is a linear representation of $\mathbf{H}'$. These are the two cases needed for the Euler system constructions in the paper. In general, one may assume there is an $\mathbf{G}$-eigenmeasure after a trivial modification [Sak08, §3.8], and our formulae in §3.3 need to be modified by the corresponding character in a well-understood way.

2.1. **Structure theory.** We now recall some general results from the theory of spherical varieties, stating only what is needed for the present work. We refer to [Sak13, SV17] and references found therein for a more systematic development with proofs.

2.1.1. *Notations.* Let $\mathbf{P}(\mathbf{X})$ be the subgroup of $\mathbf{G}$ fixing the open orbit. It contains the Borel subgroup, so it is a parabolic subgroup. Choose a good Levi subgroup $\mathbf{L}(\mathbf{X})$ as in [Sak13, §2.1], and let $\mathbf{A}$ be a maximal torus of $\mathbf{G}$ contained in $\mathbf{B} \cap \mathbf{L}(\mathbf{X})$. Define the torus

$$\mathbf{A}_X = \mathbf{A}/(\mathbf{A} \cap \mathbf{H})$$

Write Λ_X for its cocharacter lattice, and let $\mathfrak{a}_X = \Lambda_X \otimes_{\mathbb{Z}} \mathbb{Q}$. Similarly define $\mathfrak{a}$, then we have a quotient map $\mathfrak{a} \twoheadrightarrow \mathfrak{a}_X$. There is a natural cone $\mathcal{V} \subseteq \mathfrak{a}_X$ containing the image of the *negative* Weyl chamber. Let $\Lambda_X^+ = \mathcal{V} \cap \Lambda_X$. Its elements will be called $\mathbf{X}$-anti-dominant.

Let $\mathcal{V}^\perp$ be the negative-dual cone to $\mathcal{V}$ in $X^*(\mathbf{A}_X) \otimes_{\mathbb{Z}} \mathbb{Q}$, then $\mathcal{V}^\perp$ is strictly convex. In [Sak13, §6.1], Sakellaridis defined a based root system Φ_X whose set of simple roots Δ_X lie on extremal rays of $\mathcal{V}^\perp$ intersected with $X^*(\mathbf{A}_X)$. Elements of Δ_X are called (normalized) *spherical roots* of $\mathbf{X}$. There is a canonical embedding $X^*(\mathbf{A}_X) \hookrightarrow X^*(\mathbf{A})$, allowing us to view spherical roots as characters of $\mathbf{A}$. The Weyl group of Φ_X is the *little Weyl group*, denoted by W_X. It is canonically contained in the Weyl group of $\mathbf{G}$, which we will denote by W.

Knop and Schalke defined a dual group $\check{G}_X$ whose coroot system is $(X^*(\mathbf{A}_X), \Phi_X)$ [KS17]. It is a subgroup of the Langlands dual group $\check{G}$ of $\mathbf{G}$. We have an equality $\check{G}_X = \check{G}$ only if for all simple roots α, the pair $(\mathring{\mathbf{X}}, \alpha)$ is of type T in the classification below. We call $\mathbf{X}$ with this property *strongly tempered*.

2.1.2. *Classification of roots.* Let α be a simple root of $\mathbf{G}$. Let $\mathbf{P}_\alpha$ be its associated standard parabolic subgroup, with radical $\mathcal{R}(\mathbf{P}_\alpha)$. Let $\mathbf{Y}$ be a $\mathbf{B}$-orbit in $\mathbf{X}$. The geometric quotient

$$\mathbf{Y}\mathbf{P}_\alpha/\mathcal{R}(\mathbf{P}_\alpha)$$

is a homogeneous spherical variety of PGL_2. There are four cases.

- Type G: $* = \mathrm{PGL}_2 \backslash \mathrm{PGL}_2$.
- Type U: $\mathbf{S}\backslash \mathrm{PGL}_2$, where $\mathbf{S}$ is the subgroup of a Borel subgroup which contains the unipotent radical.
- Type T: $\mathbf{T}\backslash \mathrm{PGL}_2$, where $\mathbf{T}$ is a maximal torus.
- Type N: $\mathbf{N}(\mathbf{T})\backslash \mathrm{PGL}_2$, where $\mathbf{N}(\mathbf{T})$ is the normalizer of the torus. This case is excluded by our assumption.

We say the pair $(\mathbf{Y}, \alpha)$ is of type G, U, or T according to this classification.

Somewhat confusingly, there is a related but separate classification of spherical roots. Note that it is also standard terminology that the spherical roots only refer to the *simple* roots of the spherical root system Φ_X. Let γ be a spherical root. Under our standing assumptions, it has one of the two types.

- Type T: This happens exactly if γ is a root of $\mathbf{G}$.
- Type G: In all other cases, $\gamma = \alpha + \beta$, where α, β are orthogonal roots of $\mathbf{G}$, and they are simple roots in some choice of basis.

More details, as well as a proof of the above dichotomy, can be found in [Sak13, §6.2]. The ample examples there should illustrate the classification. We simply note that by the description in *loc. cit.*, if there is a pair $(\mathbf{Y}, \alpha)$ of maximal rank of type T, then there is a spherical root of type T.

2.2. **Equivariant bundles.** For our intended application, it is necessary to consider certain torus bundles over $\mathbf{X}$.

Definition 2.2. Let $\widetilde{\mathbf{X}} \to \mathbf{X}$ be an $\mathbf{G}$-equivariant $\mathbf{T}$-bundle, where $\mathbf{T}$ is a torus. It is *combinatorially trivial* if its restriction to $\mathring{\mathbf{X}}$ is trivial as a $\mathbf{B}$-equivariant bundle.

In particular, $\widetilde{\mathbf{X}}$ is a spherical variety for the group $\widetilde{\mathbf{G}} := \mathbf{G} \times \mathbf{T}$, and the stabilizer of any lift of x_0 is isomorphic to $\mathbf{H}$ by projection to the first factor. It is immediate that

$$\mathbf{A}_{\widetilde{X}} = \mathbf{A}_X \times \mathbf{T}$$

as quotients of the maximal torus $\mathbf{A}\times\mathbf{T}$. As a result, all of the combinatorial data introduced in the previous subsection are either unchanged or change in a trivial way. This explains our terminology. In particular, we note that the classification of spherical roots is unchanged.

Example 2.3. If $\mathbf{X}=\mathbf{H}\backslash\mathbf{G}$ is homogeneous, then the datum of an equivariant $\mathbf{T}$-bundle over $\mathbf{X}$ is equivalent to a character $\nu:\mathbf{H}\to\mathbf{T}$ by the recipe

$$\widetilde{\mathbf{X}}=\widetilde{\mathbf{H}}\backslash(\mathbf{G}\times\mathbf{T}),\quad \widetilde{\mathbf{H}}=\{(h,\nu(h))\,|\,h\in\mathbf{H}\}$$

It is combinatorially trivial if and only if $\nu|_{\mathbf{H}\cap\mathbf{A}}=1$, where recall that $\mathbf{A}$ is a maximal torus in the Borel subgroup $\mathbf{B}$ such that $\mathbf{HB}$ is open in $\mathbf{G}$. This is the condition imposed by Loeffler for wild norm relations [Loe21, §4.6].

Observe that if $\mathbf{H}$ contains a maximal unipotent subgroup of $\mathbf{G}$ (so $\mathbf{X}$ is horospherical), then any combinatorially trivial bundle is trivial, since ν has to be trivial on the unipotent part.

Remark 2.4. Combinatorially trivial in particular implies that

$$\check{G}_{\widetilde{X}}=\check{G}_X\times\check{\mathbf{T}}.$$

On the representation theory side, this means if π is $\mathbf{X}$-distinguished, then $\pi\times\chi$ is $\widetilde{\mathbf{X}}$-distinguished for any character χ of the torus. So we are looking a family of character twists for π, exactly what is needed for Euler system constructions. If $\mathbf{T}=\mathbb{G}_m$, this is related to having an "s-variable" in the L-function.

The existence of such a bundle is unfortunately quite restrictive, and it rules out very interesting cases, including the triple product case $\mathbf{G}=\mathrm{SO}_3\times\mathrm{SO}_4$, $\mathbf{H}=\mathrm{SO}_3$.

Proposition 2.5. *Suppose $\mathbf{X}$ is homogeneous and has only spherical roots of type G, then every combinatorially trivial $\mathbf{T}$-bundle over $\mathbf{X}$ is trivial.*

Proof. Let $\mathbf{Y}$ be a $\mathbf{B}$-orbit in $\mathbf{X}$ of maximal rank. Let α be a simple root of $\mathbf{G}$ with associated parabolic $\mathbf{P}_\alpha$. The pair $(\mathbf{Y},\alpha)$ cannot have type T, since otherwise there would be a spherical root of type T. It follows that the pair is of type G or U. In particular, $\mathbf{YP}_\alpha$ is either $\mathbf{Y}$ or the disjoint union of an open $\mathbf{B}$-orbit and a closed $\mathbf{B}$-orbit. Moreover, all $\mathbf{B}$-orbits are of maximal rank, so all pairs $(\mathbf{Y},\alpha)$ are not of type T.

Let $\widetilde{\mathbf{X}}$ be any equivariant $\mathbb{G}_m$-bundle. Suppose it trivializes as a $\mathbf{B}$-equivariant bundle over $\mathbf{Y}$ and α is a root such that $\mathbf{Y}$ is the open orbit in $\mathbf{YP}_\alpha$. Let $\mathbf{H}$ be the stabilizer of a point in $\mathbf{Y}$, so the bundle structure becomes a map $\nu:\mathbf{H}\to\mathbf{T}$. Consider the short exact sequence of groups

$$1\to\mathbf{H}\cap\mathcal{R}(\mathbf{P}_\alpha)\to\mathbf{H}\cap\mathbf{P}_\alpha\to\big((\mathbf{H}\cap\mathbf{P}_\alpha)\mathcal{R}(\mathbf{P}_\alpha)\big)/\mathcal{R}(\mathbf{P}_\alpha)\to 1$$

Since $\mathbf{P}_\alpha\subseteq\mathbf{B}$, the restriction of ν to the first term is trivial, so it descends to a character $\bar{\nu}$ on the quotient. On the other hand, this quotient is the stabilizer group in the PGL_2-spherical variety $\mathbf{YP}_\alpha/\mathcal{R}(\mathbf{P}_\alpha)$, which is of type U. But then $\bar{\nu}$ is trivial by Example 2.3. Therefore, $\nu|_{\mathbf{H}\cap\mathbf{P}_\alpha}=1$. In other words, $\widetilde{\mathbf{X}}$ trivializes as a $\mathbf{B}_\alpha$-bundle over $\mathbf{YP}_\alpha$.

Suppose that $\widetilde{\mathbf{X}}$ is combinatorially trivial, then we can construct a $\mathbf{B}$-equivariant section over $\mathring{\mathbf{X}}$. By the above discussion, this section extends to $\mathring{\mathbf{X}}\mathbf{P}_\alpha$ for any simple root α. We may continue this process starting from the closed orbits in $\mathring{\mathbf{X}}\mathbf{P}_\alpha$. Since $\mathbf{X}$ is homogeneous, all $\mathbf{B}$-orbits are reached this way. Moreover, all of the sections glue since the $\mathbf{B}$-equivariant sections on a $\mathbf{B}$-orbit are unique up to a constant multiple. Therefore, we have shown that $\widetilde{\mathbf{X}}$ is trivial as a $\mathbf{B}$-equivariant bundle.

In particular, $\widetilde{\mathbf{X}}$ is trivial as a line bundle, so the only $\mathbf{G}$-equivariance structure on it comes from a character $\chi:\mathbf{G}\to\mathbf{T}$. Its restriction to $\mathbf{B}$ is trivial by the discussion above, so χ itself must be trivial. □

3. Local computations

We now specialize to the case where F is a local field, with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. Let $q=\#\mathbb{F}$. Let ϖ be a uniformizer. For any of the varieties denoted by bold letters, we will use the normal font to denote its F-points, so for example $X=\mathbf{X}(F)$.

The group G has a natural smooth left action on the function space $C^\infty(X,\mathbb{C})$. In this section, we recall its spectral decomposition, following the works of Sakellaridis and his collaborators. The goal is to observe certain automatic divisibility properties and match them with a corresponding phenomenon in geometry.

3.1. **Assumptions.** For all results in this section, we need to impose some "good reduction" hypotheses, which we will specify. In the global setting, they hold for all but finitely many places.

Assumption 3.1. Both $\mathbf{G}$ and $\mathbf{X}$ extend to smooth schemes over $\operatorname{Spec}\mathcal{O}$, which we denote by the same letter. Moreover, all statements of [Sak12, Proposition 2.3.5] hold. In particular,

(1) $\mathbf{G}$ is reductive and $\mathbf{X}$ is affine.
(2) The chosen base point x_0 belongs to $\mathbf{X}(\mathcal{O})$.
(3) A local structure theorem for $\mathbf{X}$ and its compactifications hold.

We will write $K = \mathbf{G}(\mathcal{O})$. By hypothesis, this is a hyperspecial maximal compact subgroup of G.

3.2. **Generalized Cartan decomposition.** Under the assumptions made above, we will state a generalized Cartan decomposition due to Gaitsgory–Nadler in the equal characteristics case [GN10, Theorem 8.2.9] and adpoted to the mixed characteristics case by Sakellaridis [Sak12, Theorem 2.3.8]. Recall that $\Lambda_X \simeq \mathbf{A_X}(F)/\mathbf{A_X}(\mathcal{O})$ contains an $\mathbf{X}$-anti-dominant monoid Λ_X^+.

Proposition 3.2 (Genralized Cartan decomposition). *For each $\check{\lambda} \in \Lambda_X^+$, fix a representation $x_{\check{\lambda}} \in A_X$, viewed as an element of X by the orbit map through x_0. Then there is a disjoint union decomposition*

$$X^\bullet = \bigsqcup_{\check{\lambda}\in\Lambda_X^+} x_{\check{\lambda}} K.$$

3.2.1. *Representation of functions.* Since $\mathbf{X} - \mathbf{X}^\bullet$ is Zariski closed in $\mathbf{X}$, a function $\phi \in C_c^\infty(X, \mathbb{C})$ is uniquely determined by its restriction to $X^\bullet$, where it no longer has to be compactly supported. By the generalized Cartan decomposition, we have a canonical isomorphism $C^\infty(X^\bullet, \mathbb{C})^K \xrightarrow{\sim} \mathbb{C}[[\Lambda_X^+]]$ defined by

$$\phi \mapsto \sum_{\check{\lambda}\in\Lambda_X^+} \phi(x_{\check{\lambda}}) e^{\check{\lambda}} \in \mathbb{C}[[\Lambda_X^+]] \tag{1}$$

where $e^{\check{\lambda}}$ is a formal symbol representing the monoid element $\check{\lambda}$. We will use such a formal power series to represent functions in $C_c^\infty(X, \mathbb{C})$ in the future.

3.2.2. *Interaction with bundle.* Let $\widetilde{\mathbf{X}} \to \mathbf{X}$ be a combinatorially trivial $\mathbb{G}_m$-bundle defined over $\operatorname{Spec}\mathcal{O}$. Let

$$J_0 = \mathcal{O}^\times, \quad J_1 = \{x \in F^\times \mid x \equiv 1 \pmod{\varpi}\}.$$

The maximal compact subgroup $K^0 := \widetilde{\mathbf{G}}(\mathcal{O})$ is equal to $\mathbf{G}(\mathcal{O}) \times J_0$. For global reasons, we are also interested in the subgroup

$$K^1 := \mathbf{G}(\mathcal{O}) \times J_1 \subseteq K^0.$$

Clearly, K^1 is a normal subgroup of K^0 whose quotient is $\mathbb{F}^\times$, which has size $q-1$. The main result of this subsection is to describe the space $\widetilde{X}/K^1$.

Definition 3.3. A coroot $\check{\lambda}$ is said to *lie on a wall of type T* if there is a spherical root γ of type T such that $\langle \check{\lambda}, \gamma\rangle = 0$.

A point $x \in X$ lies on a wall of type T if it is in the orbit $x_{\check{\lambda}} K^0$, where $\check{\lambda}$ lies on a wall of type T.

Proposition 3.4. *Let $x \in \widetilde{X}$. If the action of K^1 on xK^0 has fewer than $q-1$ orbits, then x lies on a wall of type T.*

Proof. In this proof, we will only work over the generic fibre, so the standard theory of spherical embeddings apply without comment. Further note that we are only working over the open $\mathbf{G}$-orbit $\mathbf{X}^\bullet$, so we will assume that $\mathbf{X}$ is homogeneous.

Over the open orbit $\mathring{\mathbf{X}}$, fix a trivialization of $\widetilde{\mathbf{X}}$. Identify A_X with a subset of $\mathring{X}$ using the orbit map. Let $x \in A_X^+$ be an element such that K^1 splits xK^0 into less than $q-1$ pieces. Then there exists $g \in K$ such that $(x,1)g = (x, \alpha)$, where $\alpha \not\equiv 1 \pmod{\varpi}$. For any $n \geq 1$, we have

$$(x^n, 1)g = (x^{n-1}, 1)(x, 1)g = (x^n, \alpha), \tag{2}$$

where we are using the identification of A_X (as a group) with its orbit through the base point x_0 (a subset of X). It follows from this equation that the orbit $x^n K^0$ also does not split completely.

Let $\check{\lambda}$ be the image of x in Λ_X^+. Suppose for contradiction that $\check{\lambda}$ does not lie on a wall of type T, then the set $\Theta := \{\gamma \in \Delta_X \mid \langle \check{\lambda}, \gamma\rangle = 0\}$ consists only of spherical roots of type G. The idea of the proof is to show that the bundle trivializes at Θ-infinity. This notion is introduced in [SV17, §2.3], and we recall some aspects of their construction.

Let $\mathcal{F}_\Theta$ be the face in $\mathcal{V}$ consisting of vectors orthogonal to all spherical roots in Θ, then it contains $\check{\lambda}$ in its relative interior. Choose a fan decomposition of $\mathcal{F}_\Theta$ so that $\check{\lambda}$ belongs to a face $\mathcal{F}$. Let $\overline{\mathbf{X}}$ be the spherical embedding defined by this fan decomposition, following Luna–Vust theory [Kno91, Theorem 3.3]. The face $\mathcal{F}$ corresponds to a $\mathbf{G}$-orbit $\mathbf{Z}$ satisfying the condition that x^n converges to a point $x^\infty \in \mathbf{Z}$.

The combinatorially trivial $\mathbb{G}_m$-bundle $\widetilde{\mathbf{X}}$ also extends to the boundary using the compactification given by the same fan decomposition. By [SV17, Proposition 2.3.8(3)], the spherical roots of $\mathbf{Z}$ are exactly Θ, so $\mathbf{Z}$ is a spherical variety with only spherical roots of type G. Proposition 2.5 implies that the bundle over $\mathbf{Z}$ must split. On the other hand, taking limit as $n \to \infty$ in equation (2), we see that g translates between the $q-1$ disks above the point x^∞. This is a contradiction, proving that $\check{\lambda}$ must lie on a wall of type T. $\square$

3.3. **Relative Satake isomorphism.** In this subsection, we will introduce the relative Satake isomorphism. In the homogeneous case, the results are unconditional and due to Sakellaridis [Sak13, Sak18]. In a second case, which we call "totally type T", the analogous result holds in the equal-characteristic case [SW22], and we conjecture that it carries over to the mixed characteristic case.

3.3.1. *Hecke algebra and Satake isomorphism.* Fix the Haar measure on G so that K has volume 1, then we can define the Hecke algebra $\mathcal{H}(G) := C_c^\infty(K\backslash G/K, \mathbb{C})$ whose algebra structure is given by convolution

$$(f_1 * f_2)(x) = \int_G f_1(xg^{-1}) f_2(g)\, dg$$

This algebra acts on $C_c^\infty(X, \mathbb{C})$ on the left in the natural way

$$(f \cdot \phi)(x) = \int_G \phi(xg) f(g)\, dg.$$

Note that when $X = G$, this action is different from the convolution.

Recall the classical Satake isomorphism, cf. [Gro98]

$$\mathcal{H}(G) \xrightarrow{\sim} \mathcal{H}(A)^W \simeq \mathbb{C}[\Lambda]^W, \quad f \mapsto \hat{f}(t) := \delta(t)^{\frac{1}{2}} \int_N f(tn) dn \tag{3}$$

where $\Lambda = X_*(\mathbf{A}) = \mathbf{A}(F)/\mathbf{A}(\mathcal{O})$. By restriction, we have a natural ring homomorphism $\mathbb{C}[\Lambda]^W \to \mathbb{C}[\Lambda_X^+]$. Let A^* be the set of unramified characters of A, so $A^* = X^*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ is an algebraic torus. There is a Fourier transform

$$\mathbb{C}[\Lambda] \xrightarrow{\sim} \mathbb{C}[A^*], \quad e^{\check{\lambda}} \mapsto (\chi \mapsto \langle \chi, \check{\lambda} \rangle \in \mathbb{C}^\times) \tag{4}$$

where as before, $e^{\check{\lambda}}$ is a formal symbol representing the monoid element $\check{\lambda}$. We will use the Fourier transform to identify the two spaces.

3.3.2. *Homogeneous case.* In this subsection, suppose in addition that $\mathbf{X}$ is a homogeneous spherical variety. We will also assume Statements 6.3.1 and 7.1.5 from [Sak13]. These are easy to verify for each given $\mathbf{X}$, and it is expected they always hold under the assumptions we have made already.

Let A_X^* (resp. A^*) be the set of unramified characters of A_X (resp. A). Since we are assuming all $\mathbf{B}$ orbits on $\mathbf{X}_{/\bar{F}}$ are defined over F, the tours $\mathbf{A}_X$ is split, and we have the canonical injection $A_X^* \hookrightarrow A^*$. We can alternatively describe A_X^* as the $\mathbb{C}$-points of the dual torus $\check{\mathbf{A}}_X$.

Let $\delta_{(X)}$ (resp. $\delta_{P(X)}$) be the modulus character for the good Levi subgroup $\mathbf{L}(\mathbf{X})$ (resp. parabolic subgroup $\mathbf{P}(X)$), cf. §2.1. The translate $\delta_{(X)}^{\frac{1}{2}} A_X^*$ defines a subvariety of A^*. Its ring of polynomial functions will be denoted by $\mathbb{C}[\delta_{(X)}^{\frac{1}{2}} A_X^*]$. We make the following canonical identification

$$\mathbb{C}[\Lambda_X] \xrightarrow{\sim} \mathbb{C}[A_X^*] \xrightarrow{\sim} \mathbb{C}[\delta_{(X)}^{\frac{1}{2}} A_X^*] \tag{5}$$

where the first map is the Fourier transform as defined in (4), and the second map is induced by translation.

Theorem 3.5 ([Sak13, Theorem 8.0.2])**.** *There exists an isomorphism*

$$\phi \mapsto \hat{\phi} : C_c^\infty(X, \mathbb{C})^K \simeq \mathbb{C}[\delta_{(X)}^{\frac{1}{2}} A_X^*]^{W_X}$$

such that

(1) The image of the basic function

$$\Phi_0 := \mathbf{1}[\mathbf{X}(\mathcal{O})]$$

is the constant function 1.

(2) Let $C_c^\infty(K\backslash G/K, \mathbb{C}) \simeq \mathbb{C}[A^*]^W$ be the classical Satake isomorphism, then the above isomorphism is equivariant with respect to the natural action on the left hand side and multiplication on the right hand side.

Let $\mathcal{L} \in C_c^\infty(K\backslash G/K, \mathbb{C})$ be a Hecke operator, with Satake transform $\hat{\mathcal{L}} \in \mathbb{C}[A^*]^W$. By restriction and using identification (5), this defines an element

$$\hat{\mathcal{L}}_{(X)} \in \mathbb{C}[\delta_{(X)}^{\frac{1}{2}} A_X^*] \xrightarrow{\sim} \mathbb{C}[\Lambda_X] \tag{6}$$

We would like to compute the function $\mathcal{L} \cdot \Phi_0$ in terms of $\hat{\mathcal{L}}$, which can be done using the inverse Satake isomorphism [Sak18]. To state it, recall that Sakellaridis defined a multiset Θ_X^+ in [Sak13, §7.1]. By our split hypothesis, all the signs $\sigma_{\check{\theta}} = 1$ in the notation of *loc. cit.*, and the remaining two pieces of data can be combined as a representation of $\check{\mathbf{A}} \times \mathbb{G}_m$, cf. [BZSV24, §9.3.5].

Theorem 3.6. *Let Θ_X^+ be the multiset of weights of $\check{\mathbf{A}} \times \mathbb{G}_m$ described in* [Sak13, §7.1]*, then*

$$\delta_{P(X)}^{\frac{1}{2}} \cdot (\mathcal{L} \cdot \Phi_0) := \hat{\mathcal{L}}_{(X)} \cdot \frac{\prod_{\check{\gamma} \in \check{\Phi}_X^+} (1 - e^{\check{\gamma}})}{\prod_{(\check{\theta}, d_{\check{\theta}}) \in \Theta_X^+} (1 - q^{-\frac{1}{2} d_{\check{\theta}}} e^{\check{\theta}})} \Bigg|_{\Lambda_X^+}$$

under the identification (1)*.*

Proof. This is proven in [Sak18] under the additional assumption that $\mathbf{X}$ is wavefront. However, the only time the wavefront hypothesis is used in the derivation is to compute the normalization factor in the Plancherel measure. This exact normalization is not needed for the above result since the expressions do not depend on the choice of the G-invariant measure on X. □

3.3.3. *Strongly tempered case.* We no longer suppose X is homogeneous. Instead, we work in the strongly tempered case, namely $\check{G}_X = \check{G}$. We state a conjectural analogue of Theorem 3.6. In the case when F is an equal-characteristic local field, the conjecture minus the final part is known by the main theorem of [SW22].

In this case, we have $\mathbf{A} = \mathbf{A}_X$, so we can drop the subscript X from all combinatorial data. Moreover, $\delta_{(X)} = 1$ and $\delta_{P(X)} = \delta$. Finally, all elements of Θ_X^+ are in degree 1 by the recipe constructing them. Therefore, the following is the exact analogue of Theorem 3.6, with the final statement playing the role of [Sak13, Statement 7.1.5].

Conjecture 3.7. *Let Θ_X^+ be the multiset of weights of $\check{\mathbf{A}}$ denoted by $\mathfrak{B}^+$ in* [SW22, Theorem 1.1.2]*, then*

$$\delta^{\frac{1}{2}} \cdot (\mathcal{L} \cdot \Phi_0) = \hat{\mathcal{L}} \cdot \frac{\prod_{\check{\gamma} \in \check{\Phi}^+} (1 - e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta_X^+} (1 - q^{-\frac{1}{2}} e^{\check{\theta}})} \Bigg|_{\Lambda^+}$$

Moreover, all weights of Θ_X^+ are minuscule as coweights of $\mathbf{G}$.

Proof in the equal-characteristic case. All references in this proof are to [SW22]. Define the Radon transform $\pi_!$ by the expression

$$\pi_! \phi(t) = \int_N \phi(tn)\, dn.$$

By unwinding definitions, it is easy to check that $\pi_!(\mathcal{L} \cdot \phi) = \hat{\mathcal{L}} * \pi_! \phi$, where the convolution takes place in the space of smooth (not necessarily compactly supported) functions on $\mathbf{A}$. In terms of the ring $\mathbb{C}[[\Lambda]]$, this convolution is just usual multiplication. Since $\pi_! \Phi_0$ is computed in Theorem 1.1.2, the conjecture follows by applying the theory of asymptotics, exactly as in the proof of Corollary 1.2.1. □

3.4. **Integral structure.** Let the assumptions be as in the previous subsection. We give an *ad hoc* definition of an integral structure on $\mathbb{C}[\delta_{(X)}^{\frac{1}{2}} A_X^*]$, whose only purpose is to specify when a half power of q can show up. This is abstractly explained by the analytic shearing construction in [BZSV24, §6.8]. Let $\mathbb{Z}_{(q)} = \mathbb{Z}[q^{-1}]$.

Definition 3.8. Let ρ (resp. $\rho_{(X)}$, $\rho_{P(X)}$) be the half sum of positive roots for $\mathbf{G}$ (resp. $\mathbf{L}(\mathbf{X})$, $\mathbf{P}(\mathbf{X})$). Define $\mathbb{Z}_{(q)}[[A^*]]^{\not{J}}$ to be the functions in $\mathbb{C}[[A^*]]$ of the form

$$\sum_{\check{\lambda}\in\Lambda_X} a_{\check{\lambda}} q^{\langle\check{\lambda},\rho\rangle} e^{\check{\lambda}}$$

where $a_{\check{\lambda}} \in \mathbb{Z}_{(q)}$. We are again identifying $\mathbb{C}[[A^*]]$ with $\mathbb{C}[[\Lambda]]$ by (1).

The ring $\mathbb{Z}_{(q)}[[\delta_{(X)}^{\frac{1}{2}} A_X^*]]^{\not{J}}$ consists of restrictions of functions in $\mathbb{Z}[[A^*]]^{\not{J}}$.

Proposition 3.9. *Let $\mathcal{L}$ be a Hecke operator such that $\hat{\mathcal{L}}_{(X)} \in \mathbb{Z}_{(q)}[\delta_{(X)}^{\frac{1}{2}} A_X^*]^{\not{J}}$, in the notation of (6). Let $\phi = \mathcal{L}\cdot\Phi_0$, then $\phi(x) \in \mathbb{Z}[q^{-1}]$ for all $x \in X$. Moreover, if x lies on a wall of type T, then $\phi(x) \in (q-1)\mathbb{Z}[q^{-1}]$.*

Proof. Under the various hypotheses discussed in the previous section, the second term in Theorem 3.6 and Conjecture 3.7 lies in the ring $\mathbb{Z}_{(q)}[[\delta_{(X)}^{\frac{1}{2}} A_X^*]]^{\not{J}}$. It follows that

$$\begin{aligned}\phi(x_{\check{\lambda}}) &\in \delta_{P(X)}^{-\frac{1}{2}}(x_{\check{\lambda}})\delta_{(X)}^{\frac{1}{2}}(x_{\check{\lambda}}) q^{\langle\check{\lambda},\rho\rangle}\mathbb{Z}_{(q)}\\ &= q^{\langle\check{\lambda},-\rho_{P(X)}+\rho_{(X)}+\rho\rangle}\mathbb{Z}_{(q)}\end{aligned}$$

Since $\rho = \rho_{P(X)} + \rho_{(X)}$ and $2\rho_{P(X)}$ is integral, this is an integral power of q.

From this computation, we see that $\phi(x_{\check{\lambda}})$ is a polynomial in $\mathbb{Z}[q^{\pm 1}]$, where we now treat q as a formal variable. To prove the required divisibility, we specialize it to $q = 1$, so $\phi(x_{\check{\lambda}})|_{q=1}$ is the coefficient of $e^{\check{\lambda}}$ in the power series expansion of

$$\hat{\mathcal{L}}_{(X)}|_{q=1}\cdot\frac{\mathtt{ad}}{\mathtt{L}},\quad \text{where } \mathtt{ad} = \prod_{\check{\gamma}\in\check{\Phi}_X^+}(1-e^{\check{\gamma}}),\quad \mathtt{L} = \prod_{\check{\theta}\in\Theta^+}(1-e^{\check{\theta}}).$$

Suppose γ is a spherical root of type T, with corresponding coroot $\check{\gamma}$. Let $w_\gamma \in W_X$ be its associated simple reflection. We consider the behaviour of the above expression under w_γ. There are three terms.

(1) The term $\hat{\mathcal{L}}_{(X)}|_{q=1}$ is w_γ-invariant by construction.
(2) Since Φ_X^+ is the positive part of a root system, we see that

$$\frac{w_\gamma\mathtt{ad}}{\mathtt{ad}} = \frac{1-e^{-\check{\gamma}}}{1-e^{\check{\gamma}}} = -e^{-\check{\gamma}}$$

(3) We need to understand the difference between $w_\gamma\Theta^+$ and Θ^+. In the homogeneous case, this is described in [Sak13, Statement 7.1.5]. Since γ has type T, there are two virtual colors D and D' belonging to γ. Their associated valuations v_D and $v_{D'}$ sum to $\check{\gamma}$, so

$$\begin{aligned}\frac{w_\gamma\mathtt{L}}{\mathtt{L}} &= \frac{1-e^{-v_D}}{1-e^{v_D}}\frac{1-e^{-v_{D'}}}{1-e^{v_{D'}}}\\ &= e^{-(v_D+v_{D'})} = e^{-\check{\gamma}}\end{aligned}$$

In the strongly tempered case, the above discussion still holds due to our minuscule hypothesis combined with [SW22, Theorem 7.1.9(iii)].

It follows that

$$w_\gamma\left(\hat{\phi}|_{q=1}\cdot\frac{\mathtt{ad}}{\mathtt{L}}\right) = -\hat{\phi}|_{q=1}\cdot\frac{\mathtt{ad}}{\mathtt{L}} \tag{7}$$

If $w_\gamma\check{\lambda} = \check{\lambda}$, or equivalently if $\check{\lambda}$ lies on the wall defined by γ, then we see that the coefficient of $e^{\check{\lambda}}$ must be 0, as required. □

Remark 3.10. Performing the above computation for a spherical root of type G recovers equation (7), except with a plus sign. Therefore, we gain no information. In particular, all roots are of type G in the group case $\mathbf{H}\backslash\mathbf{H}\times\mathbf{H}$, so we do not see any additional divisibility in the classical Satake isomorphism.

3.5. **Abstract Euler system I.** To state the abstract Euler system relation, we need to specify the Hecke operator. Our approach depends very little on this choice.

Definition 3.11. A Hecke operator $\mathcal{L} \in C_c^\infty(K\backslash G/K, \mathbb{C})$ is **X**-*integral* if $\hat{\mathcal{L}}_{(X)} \in \mathbb{Z}_{(q)}[\delta_{(X)}^{\frac{1}{2}} A_X^*]^{\mathbb{W}}$.

Example 3.12. Let V be a minuscule representation of $\check{G}$, then the function $t \mapsto \det(1 - q^{-\frac{1}{2}}t|V)$ belongs to $\mathbb{C}[A^*]^W$. The Hecke operator associated to it by the Satake transform lies in $\mathbb{Z}_{(q)}[A^*]$, so it is **X**-integral for all **X**. More generally, replacing t by tX gives a polynomial $\mathcal{H}_V(X)$, all of whose coefficients are **X**-integral. This is the Hecke polynomial considered in [BR94, §6].

Proposition 3.13. *Let $\mathcal{L}$ be an $\widetilde{\mathbf{X}}$-integral Hecke operator, then there exists $\Phi_1 \in C_c^\infty(\widetilde{X}, \mathbb{Z}[q^{-1}])^{K^1}$ such that*

$$\mathrm{Tr}_{K^0}^{K^1} \Phi_1 = \mathcal{L} \cdot \mathbf{1}[\mathbf{X}(\mathcal{O})],$$

where $\mathrm{Tr}_{K^0}^{K^1} := \sum_{g \in K^0/K_1} g$ is the trace operator.

Proof. Let $\phi \in C_c^\infty(\widetilde{X}, \mathbb{C})^{K^1}$. Let $x \in X$. Write $xK^0 = \bigsqcup_{i \in I} x_i K^1$, then

$$(\mathrm{Tr}_{K^0}^{K^1} \phi)(x) = \sum_{i \in I} \frac{[K^0 : K^1]}{[xK^0 : x_i K^1]} \phi(x_i).$$

Since K^1 is normal in K^0, the multiplier in front of each term is the same. In other words, the image of $C_c^\infty(\widetilde{X}, \mathbb{Z}[q^{-1}])^{K^1}$ under the trace map is characterized by the divisibility condition

$$\phi(x) \in \frac{q-1}{[xK^0 : xK^1]} \mathbb{Z}[q^{-1}] \text{ for every } x$$

By Proposition 3.4, if this coefficient is not 1, then x lies on a wall of type T. Proposition 3.9 applied to $\widetilde{\mathbf{X}}$ shows that this divisibility requirement is automatic. $\square$

4. Global picture

Now let F be a global field. The goal of this section is to apply the above theorems and conjectures to construct the tame part of Euler systems over F.

4.1. **Motivic theta series.** Let $\mathbb{A}$ be the ring of adèles of F. Let p be a rational prime. Let S be a finite set of places of F such that

(1) S contains all places above p and ∞.
(2) Away from S, the assumptions made in §3.1 hold [Sak12, Proposition 2.3.5].

In particular, we may extend **X** and **G** to smooth varieties over $\operatorname{Spec} \mathcal{O}_{F,S}$, which allows us to talk about their adèlic points. Again denote these extensions by the same letters.

Let $K^S = \prod_{v \notin S} \mathbf{G}(\mathcal{O}_{F_v})$. Define the *basic vector* to be

$$\delta := \prod_{v \notin S} \mathbf{1}[\mathbf{X}(\mathcal{O}_{F_v})] \in C_c^\infty(\mathbf{X}(\mathbb{A}^S), \mathbb{Z}_p)$$

By construction, it is K^S-invariant.

Definition 4.1. Let $\mathcal{M}$ be a $\mathbb{Z}_p$-module with a smooth $\mathbf{G}(\mathbb{A}^S)$-action. An **X**-*theta series* for $\mathcal{M}$ is a $\mathbf{G}(\mathbb{A}^S)$-equivariant map

$$\Theta : C_c^\infty(\mathbf{X}(\mathbb{A}^S), \mathbb{Z}_p) \to \mathcal{M}$$

The basic element $z \in \mathcal{M}$ is defined to be $\Theta(\delta)$.

Remark 4.2. In general, the discussions of [Sak12, §3] suggests that the correct function space should be a restricted tensor product of smooth functions on $\mathbf{X}^\bullet$ with bounded growth near the boundary $\mathbf{X} - \mathbf{X}^\bullet$. However, this subspace is sufficient for our applications.

If $\mathcal{M}$ is a space of automorphic functions, then the classical theta series is an example of this definition. In the relative Langlands program, such objects are expected to play an important role, cf. [BZSV24, §15.1]. Their motivic analogues should also exist in great generality, though they are harder to construct. We will later specify $\mathcal{M}$ to be certain $\mathbb{Z}_p$-coefficient continuous étale cohomology groups, which we consider as p-adic realizations of motives.

Example 4.3. Consider the case $\mathbf{G} = \mathrm{GL}_2$, $F = \mathbb{Q}$, and $\mathbf{X} = \mathtt{std}$, the 2-dimensional affine space with the standard action of $\mathbf{G}$. The theory of Siegel units (cf. [Kat04, §1.4], [Col04, Théoréme 1.8]) gives a motivic theta series

$$C_c^\infty(\mathbf{X}(\mathbb{A}^S), \mathbb{Z}) \to \varinjlim_{U} \mathcal{O}(Y(U))^\times,$$

where U runs over all open compact subgroups of $\mathbf{G}(\mathbb{A}^\infty)$, and $Y(U)$ is the open modular curve of level U. By taking p-adic étale realization, this produces a motivic theta series valued in $\mathrm{H}^1_{\mathrm{cont}}(Y, \mathbb{Z}_p(1))$. This is exactly the construction of GL_2-Eisenstein classes in weight 2. For higher weights, the coefficient $\mathbb{Z}_p$ is replaced by an integral étale local system, cf. [LSZ22b, Proposition 7.2.4].

We are hiding one subtlety in this definition. In the above references, one need to restrict to functions which vanish at $(0,0)$ to ensure convergence. To get this vanishing automatically, we fix an auxiliary place $v \in S$ and fix the test vector there to vanish at $(0,0)$. This was used in the choices of [LSZ22b, §8.4.4].

Example 4.4. By taking cup product, we obtain the following motivic theta series for $\mathbf{X} = \mathtt{std}^2$ with the diagonal GL_2-action

$$C_c^\infty(\mathbf{X}(\mathbb{A}^S), \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{cont}}(Y, \mathbb{Z}_p(2)).$$

This is Kato's construction of his zeta element, reformulated in [Col04]. As observed by Colmez, this makes verifying their norm relations "almost automatic", since it reduces to some elementary manipulation of indicator functions on $\mathbf{X}$.

4.2. **Abstract Euler system II.** Let $\mathbf{T}$ be a 1-dimensional torus defined over $\mathcal{O}_{F,S}$, the S-integers of F. Let $\widetilde{\mathbf{X}} \to \mathbf{X}$ be a combinatorially trivial $\mathbf{T}$-bundle. Let $\widetilde{\mathbf{G}} = \mathbf{G} \times \mathbf{T}$ be the augmented group. For each square-free ideal $\mathfrak{m}$ not divisible by any place in S, we may form the level structure

$$J[\mathfrak{m}] := \prod_{\substack{v \nmid \mathfrak{m} \\ v \notin S}} \mathbf{T}(\mathcal{O}_{F_v}) \prod_{v \mid \mathfrak{m}} \mathbf{T}(\mathcal{O}_{F_v})^1 \subseteq \mathbf{T}(\mathbb{A}^S), \quad K[\mathfrak{m}] := K^S \times J[\mathfrak{m}] \subseteq \widetilde{\mathbf{G}}(\mathbb{A}^S), \tag{8}$$

where $\mathbf{T}(\mathcal{O}_{F_v})^1$ consists of the elements of $\mathbf{T}(\mathcal{O}_{F_v})$ whose reduction is the identity.

Let $\mathscr{L}$ be the set of places of F away from S where the hypotheses of Section 2 are satisfied for $\widetilde{\mathbf{X}}$. This in particular requires $\mathbf{T}$ to be split at those places. For each $\ell \in \mathscr{L}$, let $\mathcal{H}_\ell$ be an $\widetilde{\mathbf{X}}$-integral Hecke operator in the group $\widetilde{\mathbf{G}}(F_\ell)$ (Definition 3.11). They give rise to the following abstract Euler system.

Theorem 4.5. *Let Θ be an $\widetilde{\mathbf{X}}$-theta element for $\mathcal{M}$. Let $\mathscr{R}$ be the collection of square-free products of places in $\mathscr{L}$. There exists a collection of elements $\{z_\mathfrak{m} \in \mathcal{M}^{K[\mathfrak{m}]} \mid \mathfrak{m} \in \mathscr{R}\}$ such that z_1 is the basic element, and for any $\mathfrak{m} \in \mathscr{R}$ and any $\ell \in \mathscr{L}$ such that $\ell \nmid \mathfrak{m}$, we have the norm relation*

$$\mathrm{Tr}_{K[\mathfrak{m}]}^{K[\mathfrak{m}\ell]} z_{\mathfrak{m}\ell} = \mathcal{H}_\ell \cdot z_\mathfrak{m}$$

Proof. For each $\ell \in \mathscr{L}$, let $\delta^{(\ell)}$ be the element Φ_1 from Proposition 3.13. Define $\delta[\mathfrak{m}]$ to be δ, except the local component at each $\ell | \mathfrak{m}$ are replaced by $\delta^{(\ell)}$. The theorem follows from the $\widetilde{\mathbf{G}}(\mathbb{A}^\infty)$-equivariance of Θ. □

4.3. **Étale classes and Galois action.** We now specialize further and suppose that $\mathbf{G}$ has Shimura varieties defined over a number field E containing F. We can then form Shimura varieties for $\widetilde{\mathbf{G}}$. Let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system defined using an algebraic representation of $\mathbf{G}$, as in for example [HT01, §III.2]. Define

$$\mathcal{M} = \varinjlim_{K^S} \mathrm{H}^d_{\mathrm{cont}}(\mathrm{Sh}_{\widetilde{\mathbf{G}}}(K_S K^S), \mathbb{L}), \tag{9}$$

where K_S is a fixed open compact subgroup of $\widetilde{\mathbf{G}}(\mathbb{A}_S)$, and K^S runs over all open compact subgroups of $\widetilde{\mathbf{G}}(\mathbb{A}^S)$. The cohomology theory used is Jannsen's continuous étale cohomology [Jan88]. Note that all the transition maps are well-defined since we are not changing the level structure at the p-adic places.

Suppose $K \subseteq \mathbf{G}(\mathbb{A}^\infty)$ and $U \subseteq \mathbf{T}(\mathbb{A}^\infty)$ are both open compact, then we have

$$\mathrm{Sh}_{\widetilde{\mathbf{G}}}(K \times U) \simeq \mathrm{Sh}_{\mathbf{G}}(K) \times \mathrm{Sh}_{\mathbf{T}}(U) \tag{10}$$

as algebraic varieties over E. Take $U = J[\mathfrak{m}] \times J_S$ for some appropriate choice of the ramified level structure J_S, then it is often possible to interpret the above product as a base change from E to an extension $E[\mathfrak{m}]$. We single out two cases.

– If $\mathbf{T} = \mathbb{G}_m$ with Shimura cocharacter z^{-1}, then $E[\mathfrak{m}]$ is the cyclotomic extension of conductor $\mathfrak{m}$.

– If $\mathbf{T} = \mathrm{U}(1)$ attached to a quadratic extension E/F, and the Shimura cocharacter is $z \mapsto \bar{z}/z$, then $E[\mathfrak{m}]$ is the anticyclotomic extension of E of conductor $\mathfrak{m}$.

In these cases, the trace from Theorem 4.5 can be identified with a trace map between fields.

Locally at $\ell \in \mathscr{L}$, we have the following description of the Hecke algebra for $\widetilde{\mathbf{G}}$ as a ring of Laurent polynomials

$$\mathcal{H}(\widetilde{\mathbf{G}}(F_\ell), \mathbb{C}) \simeq \mathcal{H}(\mathbf{G}(F_\ell), \mathbb{C})[\mathtt{T}^{\pm 1}].$$

Here, $\mathtt{T}$ is the indicator function of $\varpi\mathcal{O}_{F_\ell}^\times \subseteq \mathbf{T}(F_\ell)$. Moreover, a Hecke operator is $\widetilde{\mathbf{X}}$-integral if and only if all of its coefficients on the right hand side are $\mathbf{X}$-integral. In particular, if V is a minuscule representation of $\check{G}$, then we may use the Hecke polynomial $\mathcal{H}_V(\mathtt{T})$ defined in Example 3.12.

Under the identification (10), the action of $\mathtt{T}$ matches with the action of the geometric Frobenius element Frob_ℓ^{-1}. In the $\mathbf{T} = \mathrm{U}(1)$ case, this requires further comments: the identification of $\mathbf{T}_{/F_\ell}$ with $\mathbb{G}_m$ depends on the choice of a place λ of E above ℓ, and Frob_ℓ really means the arithmetic Frobenius at λ. Using this, Theorem 4.5 leads to the following Corollary.

Corollary 4.6. *Let the setting be as in Theorem 4.5, with $\mathcal{M}$ given by equation (9). Let V be a minuscule representation of $\check{G}$. Then there exists a collection of elements*

$$\left\{ z_\mathfrak{m} \in \mathrm{H}^d_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}/E[\mathfrak{m}]}, \mathbb{L}) \,\middle|\, \mathfrak{m} \in \mathscr{R} \right\}$$

satisfying the norm relations

$$\mathrm{Tr}_{E[\mathfrak{m}]}^{E[\mathfrak{m}\ell]} z_{\mathfrak{m}\ell} = \mathcal{H}_V(\mathrm{Frob}_\ell^{-1}) \cdot z_\mathfrak{m}$$

whenever $\mathfrak{m}, \mathfrak{m}\ell \in \mathscr{R}$.

Remark 4.7. It is crucial that the Shimura cocharacter on $\mathbf{T}$ is non-trivial, otherwise we do not recover a Galois action. In particular, choosing the trivial equivariant bundle $\widetilde{\mathbf{X}} = \mathbf{X} \times \mathbf{T}$ does not lead to interesting arithmetic in any of the examples we consider, as expected.

This result of Corollary 4.6 may be called a "motivic Euler system", cf. [LSZ22a, §9.4]. We now explain the steps required to convert this to an Euler system in Galois cohomology, proving Corollary 1.2 from the introduction. These steps are all standard in the literature.

(1) Null-homologous modification: the main result of [MS19] constructs an element $\mathtt{t}^{\pm}$ in the Hecke algebra of $\mathbf{G}$ whose action on the cohomology of $\mathrm{Sh}_\mathbf{G}$ are the two sign projectors. In particular, $\mathtt{t} := \mathtt{t}^{(-1)^{d+1}}$ annihilates the cohomology group $\mathrm{H}^d(\mathrm{Sh}_{\mathbf{G}/\bar{E}}, \mathbb{L})$. Therefore, for each $\mathfrak{m}$ as above, $\mathtt{t} \cdot z_\mathfrak{m}$ is cohomologically trivial, and the spectral sequence in continuous étale cohomology [Jan88, Remark 3.5(b)] gives an element

$$\tilde{c}_\mathfrak{m} \in \mathrm{H}^1(E[\mathfrak{m}], \mathrm{H}^{d-1}(\mathrm{Sh}_{\mathbf{G}/\bar{E}}, \mathbb{L}))$$

A simpler argument is given in [LL21, Proposition 6.9(1)] which also suffices for our purpose.

(2) Projection to Galois representations: Kottwitz's conjecture gives a concrete description of the cohomology group $\mathrm{H}^{d-1}(\mathrm{Sh}_{\mathbf{G}/\bar{E}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. In the case considered in Corollary 1.2, it gives a projection from this group to the Galois representation ρ_π. Moreover, the Hecke polynomial specialized to the Satake parameters of π agrees with the characteristic polynomial of ρ_π.

The integral structure given by the lattice $\mathbb{L}$ is mapped to a lattice T_π in ρ_π. The image of $\tilde{c}_\mathfrak{m}$ under this projection is the class $c_\mathfrak{m} \in \mathrm{H}^1(E[\mathfrak{m}], T_\pi)$.

In the following two subsections, we will explain how pushforwards of cycles and Eisenstein classes both give instances of such a motivic theta series. In these settings, the above corollary produces many of the known motivic Euler systems in the literature.

4.4. **Special cycles.** Suppose that $\mathbf{X}$ is homogeneous, then $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$, where we recall that $\mathbf{H}$ is the stabilizer of a point in the open orbit. In this case, $\mathbf{X}$ is automatically smooth, and it is affine if and only if $\mathbf{H}$ is reductive. Assume the following conditions.

(1) Both $\mathbf{G}$ and $\mathbf{H}$ have Shimura varieties, and $\dim \mathrm{Sh}_\mathbf{G} = 2 \dim \mathrm{Sh}_\mathbf{H} + 1$.
(2) ξ is an algebraic representation of $\mathbf{G}_\infty$ whose restriction to $\mathbf{H}_\infty$ contains the trivial representation.

Under the above assumptions, the discussions of [LS24, §2.4] carries over verbatim, and Proposition 2.7 of *op. cit.* gives a motivic theta series.

Proposition 4.8. *Let $d = \dim \mathrm{Sh}_{\mathbf{H}}$. Let $\mathbb{L}$ be the $\mathbb{Z}_p$-local system on $\mathrm{Sh}_{\mathbf{G}}$ attached to ξ. The diagonal cycle construction defines an $\mathbf{X}$-theta series*

$$\Theta : C_c^\infty(\mathbf{X}(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \to \mathrm{H}^{2d}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}, \mathbb{L}(d))$$

Remark 4.9. The algebraic representation ξ determines the archimedean part of the representations distinguished by Θ. The above construction depends on the choice of an invariant vector in $\xi|_{\mathbf{H}}$. It is likely that including the archimedean place in the space $\mathbf{X}(\mathbb{A}^{p\infty})$ would give a more canonical construction.

Since we are constructing motivic classes using cycles, we should limit our deformations to the self-dual setting. Therefore, it seems necessary to take $\mathbf{T} = \mathrm{U}(1)$, defined with respect to a CM extension E/F. To satisfy the conditions in §2, we must restrict our attention to split primes. The result is a split anticyclotomic Euler system, in the sense of [JNS24].

We now explain the content of Corollary 4.6 in a few cases. In each case, we need to specify the data of an embedding of reductive groups $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$ and a character $\nu : \mathbf{H} \to \mathrm{U}(1)$. The latter is used to specify the $\mathrm{U}(1)$-bundle over $\mathbf{X}$. In all cases, the local spherical variety appears in the table [Sak13, Appendix A], so they have been verified to satisfy the combinatorial conditions alluded to in §3.3.2.

4.4.1. *Gan–Gross–Prasad* [LS24]. Let $\mathtt{V}_n \subseteq \mathtt{V}_{n+1}$ be Hermitian spaces of dimensions n, $n+1$ respectively which are nearly definite. Consider the setting

$$\mathbf{H} = \mathrm{U}(\mathtt{V}_n), \quad \mathbf{G} = \mathrm{U}(\mathtt{V}_n) \times \mathrm{U}(\mathtt{V}_{n+1})$$
$$\iota : h \mapsto (h, \mathrm{diag}(h, 1)), \quad \nu : h \mapsto \det h$$

The intersection of $\mathbf{H}$ with a good Borel $\mathbf{B}$ is trivial, so the bundle is automatically combinatorially trivial. Take V to be the standard tensor product representation of degree $n(n+1)$, then Corollary 4.6 recovers [LS24, Proposition 5.4], bypassing the explicit matrix calculations in §4 of *op. cit.*

4.4.2. *Friedberg–Jacquet* [GS23]. Let $\mathtt{V}_{2n}$ be a Hermitian space with signature $(1, 2n-1)$ at one archimedean place and $(0, 2n)$ at the other archimedean places. Let $\mathtt{W}$ be a totally definite subspace of dimension n, and let $\mathtt{W}^\perp$ be its dual. Consider the setting

$$\mathbf{H} = \mathrm{U}(\mathtt{W}^\perp) \times \mathrm{U}(\mathtt{W}), \quad \mathbf{G} = \mathrm{U}(\mathtt{V})$$
$$\iota : (a, b) \mapsto \mathrm{diag}(a, b), \quad \nu : (a, b) \mapsto \frac{\det a}{\det b}$$

At a split place, the local picture is $\mathrm{GL}_n \times \mathrm{GL}_n \backslash \mathrm{GL}_{2n}$. For an appropriate choice of the Borel subgroup $\mathbf{B}$, the intersection $\mathbf{A} \cap \mathbf{H}$ has the form

$$(\mathrm{diag}(z_1, \cdots, z_n), \mathrm{diag}(z_n, \cdots, z_1)),$$

so the bundle defined by ν is combinatorially trivial (which explains why $(a, b) \mapsto \det a \det b$ is not the correct choice for ν). In this case, the equivariant bundle is non-trivial even as a line bundle, which explains why it is essential to include the additional $\mathrm{U}(1)$-factor.

Take V to be the standard representation of degree $2n$, then Corollary 4.6 recovers the tame part of the split anticyclotomic Euler system for symplectic representations constructed by Graham and Shah [GS23, Proposition 9.27] without assumptions on the prime p.

4.4.3. *Cornut's Euler system* [Cor18]. Let F be a totally real field. Let $\mathtt{V}$ be a quadratic space over F of signature $(2, 2n-1)$ at one archimedean place and istotropic at the other archimedean places. Suppose $\mathtt{V}$ splits over E, then there is an E-Hermitian F-hyperplane $\mathtt{W} \subseteq \mathtt{V}$, unique up to $\mathbf{G}$-translation. Let

$$\mathbf{H} = \mathrm{U}(\mathtt{W}), \quad \mathbf{G} = \mathrm{SO}(\mathtt{V})$$

The definition gives an inclusion $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$. The character ν is again the determinant.

At a split place ℓ, the local picture is $\mathrm{GL}_n \backslash \mathrm{SO}_{2n+1}$. The combinatorially trivial condition is automatic. Take V to be the standard representation of degree $2n$. Corollary 4.6 then recovers a split anticyclotomic Euler system for representations of $\mathrm{SO}(\mathtt{V})$, where the "Euler factor" is the one corresponding to the standard L-function. This is the split analogue of the one constructed by Cornut [Cor18].

Remark 4.10. In this case, $\mathbf{A} = \mathbf{A}_{\mathbf{X}}$, but the little Weyl group W_X is smaller, so we do not have multiplicity one [Sak08, Theorem 1.2.1]. This observation also shows that $\mathbf{X}$ is not wavefront, which suggests that a representation π which is distinguished by $\mathbf{X}$ is necessarily not stable. If this is the case, then Kottwitz's conjecture predicts that the Galois representation appearing in the orthogonal Shimura variety is not the standard one. We hope further automorphic studies of the setting will shed light on this question.

4.5. **Eisenstein classes.** We now move to a family of non-homogeneous spherical varieties. Suppose V is an affine space with a spherical action of a group $\mathbf{H}$. Let

$$\mathbf{X} = \mathbf{G} \times^{\mathbf{H}} V$$

In a change of notation from §2, the stabilizer of a generic point is no longer $\mathbf{H}$. In the Eisenstein class applications below, the stabilizer is a mirabolic subgroup of $\mathbf{H}$, which gives an explanation for their prominence in [Loe21].

We expect there to be a general pushforward construction sending motivic theta series for V to ones for $\mathbf{X}$. If both $\mathbf{H}$ and $\mathbf{G}$ have Shimura varieties, then this should just be the usual pushforward construction, as described for example in [Loe21]. The following proposition describes that construction in our framework.

Proposition 4.11. *Suppose we have a V-theta series*

$$\Theta_{\mathbf{H}} : C_c^\infty(V(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \to \mathrm{H}^i_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{H}}, \mathbb{Z}_p(j)),$$

then its pushforward defines an $\mathbf{X}$-theta series

$$\Theta_{\mathbf{G}} : C_c^\infty(\mathbf{X}(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \to \mathrm{H}^{i+2d}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}, \mathbb{Z}_p(j+d))/\mathrm{tors},$$

where $d = \dim \mathrm{Sh}_{\mathbf{G}} - \dim \mathrm{Sh}_{\mathbf{H}}$ is the codimension.

Proof. Exactly as in [LSZ22b, §8.2], one may define a map

$$\Theta'_{\mathbf{G}} : C_c^\infty((\mathbf{G} \times V)(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \to \mathrm{H}^{i+2d}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}, \mathbb{Q}_p(j+d))$$

which is left $\mathbf{G}$-equivariant and right $\mathbf{H}$-invariant. Therefore, we have a commutative diagram

$$\begin{tikzcd} C_c^\infty((\mathbf{G} \times V)(\mathbb{A}^{p\infty}), \mathbb{Z}_p) \arrow[d] \arrow[dr, "\Theta_{\mathbf{G}}"] & \\ C_c^\infty(\mathbf{X}(\mathbb{A}^{p\infty}), \mathbb{Q}_p) \arrow[r, "\Theta'_{\mathbf{G}}"] & \mathrm{H}^{i+2d}_{\mathrm{cont}}(\mathrm{Sh}_{\mathbf{G}}, \mathbb{Q}_p(j+d)) \end{tikzcd}$$

where the downward arrow is the $\mathbf{H}$-coinvariant map. One simply observes that the coinvariant map and $\Theta'_{\mathbf{G}}$ both introduce the same volume factors on a basis of functions, so $\Theta_{\mathbf{G}}$ is integral. □

Remark 4.12. The proof of [LS24, Proposition 2.7] should carry over to this setting, thereby removing the "mod torsion" part of the statement. One can likely axiomatize it using Loeffler's definition of Cartesian cohomology functors [Loe21, GS23].

If $\mathbf{H} = \mathrm{GL}_2$ and V is its standard two-dimensional representation, then Example 4.3 gives a V-theta series as above, with $i = j = 1$. This gives rise to the tame parts of the following Euler systems, with the caveat that we need the function-level statements of [SW22] in mixed characteristics.

4.5.1. *Rankin–Selberg* [LLZ14]. Let $\mathbf{G} = \mathrm{GL}_2 \times \mathrm{GL}_2$ and $\mathbf{H} = \mathrm{GL}_2$, embedded diagonally in $\mathbf{G}$. Let

$$\mathbf{X} = \mathbf{G} \times^{\mathbf{H}} \mathtt{std}$$

The pushforward construction gives rise to Beilinson–Flach elements considered in [LLZ14]. To obtain an Euler system, we endow the trivial line bundle over $\mathbf{X}$ with the $\mathbf{G}$-action through the character $(g_1, g_2) \mapsto \det g_1$. Our main theorem applied to the associated $\mathbb{G}_m$-bundle recovers the tame norm relations of Beilinson–Flach elements [LLZ14, Theorem 3.4.1].

4.5.2. *Asai representation* [Gro20]. Let F be real quadratic field. In the previous subsection, take instead

$$\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2/F}, \quad \mathbf{H} = \mathrm{GL}_2$$

This gives rise to the Asai–Flach classes considered in [Gro20]. The $\mathbb{G}_m$-bundle defined above recovers the tame norm relation at primes ℓ which split in F.

4.5.3. $\mathrm{GSp}_4 \times \mathrm{GL}_2$ [HJR20]. Let $\mathbf{G} = \mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2$ and $\mathbf{H} = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$. By $\mathtt{std}$ we now mean the 2-dimensional affine space where $\mathbf{H}$ acts through its first component. The variety $\mathbf{X} = \mathbf{G} \times^{\mathbf{H}} \mathtt{std}$ is spherical. The trivial line bundle on $\mathbf{X}$ with the obvious character $\mathbf{G} \to \mathbb{G}_m$ gives rise to the tame norm relation [HJR20, Proposition 8.17].

4.5.4. $\mathrm{GU}(2,1)$ [LSZ22a]. Let E be an imaginary quadratic field. Consider the pair of algebraic groups defined over $\mathbb{Q}$:

$$(\mathbf{G}, \mathbf{H}) = (\mathrm{GU}(2,1), \mathrm{GL}_2 \times \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m).$$

The variety $\mathbf{X} = \mathbf{G} \times^{\mathbf{H}} \mathtt{std}$ is spherical by [LSZ22a, Lemma 2.5.1]. The authors also defined a character $\mu : \mathbf{G} \to \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ in §2.2 of *op. cit.* and use it to vary the class over field extensions. In our set-up, this corresponds to the trivial $\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$-bundle over $\mathbf{X}$ with the action of $\mathbf{G}$ given by μ.

An interesting additional feature is that we have a 2-variable $\mathbb{Z}_p$-deformation, since this is a 2-dimensional torus bundle. Let $\ell = w\bar{w}$ be a split prime, then Theorem A of *op. cit.* includes the hypothesis that at most one of w and $\bar{w}$ divides the ideal $\mathfrak{m}$. Our result also implies the tame norm relation at ℓ under this hypothesis: allowing $w\bar{w}|\mathfrak{m}$ would require a divisibility by $(\ell-1)^2$, which appears to be false in this case. On the other hand, at an inert place, the divisibility requirement is by $\ell^2 - 1 = (\ell-1)(\ell+1)$, and this can be detected by specializing at $\ell = 1$ and $\ell = -1$ separately. We hope to examine the inert case in a future paper.

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