NON-VANISHING OF GLOBAL THETA LIFTS AND L-FUNCTIONS

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This is the notes for a talk given at the student-organized theta correspondence seminar. The goal is to state various results on the non-vanishing of global theta lifts, in particular on the relation to special values of L-functions. The local aspects of the theory were covered in a previous talk.

1. General results

We will state the non-vanishing conjectures quite generally, following [GQT14]. Let F be a number field, and let E be an extension of F of degree $d \in \{1, 2\}$. Fix an additive character $\psi : F\backslash\mathbb{A}_F \to \mathbf{C}^\times$. Let $\epsilon_{E/F} : F^\times\backslash\mathbb{A}_F^\times \to \mathbf{C}^\times$ be the character attached to the extension E/F.

If G is an algebraic group, $\mathcal{A}(G) \supseteq \mathcal{A}_{\text{cusp}}(G)$ will denote the space of automorphic (resp. cuspidal automorphic) forms on $[G] := G(F)\backslash G(\mathbb{A}_F)$. If $G = \mathrm{Mp}_n$ is the metaplectic group, then $\mathcal{A}(\mathrm{Mp}_n)$ denotes the space of automorphic forms on $[\mathrm{Mp}_n] := \mathrm{Sp}_n(F)\backslash\mathrm{Mp}(\mathbb{A}_F)$ where the centre acts by the non-trivial character. The cuspidal subspace $\mathcal{A}_{\text{cusp}}(\mathrm{Mp}_n)$ can be defined by the usual condition since the cover splits canonically over the unipotent radical of any parabolic subgroup of Sp_n.

Fix a sign $\epsilon \in \{\pm 1\}$. Let W_n be a $(-\epsilon)$-Hermitian space of dimension n, and let $V_{m,r}$ be an ϵ-Hermitian space of dimension m and Witt index r. In particular, we have a decomposition $V_{m,r} = V_{m_0,0} \oplus V_{2,1}^r$, where $m = m_0 + 2r$. Moreover, in the skew-symmetric case ($E = F$, $\epsilon = -1$), we have $m_0 = 0$, $m = 2r$. The above data determines a dual reductive pair (G, H), which is shown in the following table, where we define $\epsilon_0 = \epsilon$ if $E = F$ and 0 otherwise.

ϵ_0		$G(W_n)$	$H(V_{m,r})$	χ_V
0		U_n	U_m	$\chi_V\vert_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m$
-1	n even	O_n	Sp_{2r}	1
	n odd	O_n	Mp_{2r}	
1	m even	Sp_n	O_m	$\left(\frac{\cdot}{\mathrm{disc}(V)}\right)$
	m odd	Mp_n	O_m	

The big metaplectic group for a pair as above is $\mathrm{Mp}(W_n \otimes_E V_{m,r})$. It has a Weil representation ω_ψ, which can be realized on $\mathcal{S}(X(\mathbb{A}_F))$, where X is a maximal isotropic subspace of $W_n \otimes_E V_{m,r}$.

We make some comments on splitting the metaplectic cover. In the paper [Kud94], given a Hecke character $\chi_V : E^\times\backslash\mathbb{A}_E^\times \to \mathbf{C}^\times$ satisfying certain conditions depending on $V_{m,r}$, Kudla wrote down an explicit splitting $\iota_{\chi_V} : G \hookrightarrow \mathrm{Mp}(W_n \otimes_E V_{m,r})$. Via this, we can define the Weil representation $\omega_{\psi,\chi_V} := \omega_\psi \circ \iota_{\chi_V}$ on G. We use the following choices for χ_V:

$$\chi_V = \begin{cases} \text{Any character such that } \chi_V|_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m & \text{if } \epsilon_0 = 0 \\ \text{The quadratic character attached to } \mathrm{disc}(V) & \text{if } \epsilon_0 = 1 \\ \text{Trivial character} & \text{if } \epsilon_0 = -1 \end{cases}$$

An analogous choice of data for W determines a Weil representation ω_{ψ,χ_W} on H.

With these choices, we can define

$$\omega_{\psi,\chi_W,\chi_V} = \omega_{\psi,\chi_V} \otimes \omega_{\psi,\chi_W} : G(\mathbb{A}_F) \times H(\mathbb{A}_F) \to \mathrm{GL}(\mathcal{S}(X(\mathbb{A}_F)))$$

There is an intertwining operator $\omega_{\psi,\chi_W,\chi_V} \to \mathcal{A}(G \times H)$ given by the usual theta kernel

$$\theta_{\psi,\chi_W,\chi_V}(\phi; g, h) = \sum_{v \in X(F)} \omega_{\psi,\chi_W,\chi_V}(g, h)\phi(v)$$

Let $\pi \subseteq \mathcal{A}_{\text{cusp}}(G)$ be a cuspidal automorphic representation of G. For any $f \in \pi$, define

$$\theta_{\psi,\chi_W,\chi_V}(\phi, f)(h) = \int_{G(F)\backslash G(\mathbb{A}_F)} \theta_{\psi,\chi_W,\chi_V}(\phi; g, h)\overline{f(g)}dg$$

Since f is rapidly decreasing, this integral converges. The big theta lift of π from G to H is then

$$\Theta_{\psi,\chi_W,\chi_V}(\pi) = \{\theta_{\psi,\chi_W,\chi_V}(\phi, f) \,|\, \phi \in \omega_{\psi,\chi_W,\chi_V}, f \in \pi\}$$

In the sequel, we will often drop the auxiliary data from the notation if they are not essential.

Questions

– When is the the lift non-trivial?
– When is the the lift cuspidal?

The answer to the second question is partially given by the Rallis tower property, which is a global version of the first occurrence properties discussed last time. To set it up, we consider a tower of spaces

$$V_{m_0,0} \subseteq \cdots \subseteq V_{m-2,r-1} \subseteq V_{m,r} \subseteq V_{m+2,r+1} \subseteq V_{m+4,r+2} \subseteq \cdots$$

where each space is the direct sum of the previous space and a split-space of dimension 2. Observe that the character χ_V can be used for all spaces in this tower. We will fix the choices of auxiliary data and write $\Theta_{n,r'}$ for the theta lift to the space with Witt index r'.

Theorem 1.1 (Rallis)**.** *Let $\pi \subseteq \mathcal{A}_{\text{cusp}}(G)$, then*

– $\Theta_{n,n}(\pi) \neq 0$.
– *Suppose r is the first occurrence, i.e.*

$$r = \min\{r' \,|\, \Theta_{n,r'}(\pi) \neq 0\} \leq n$$

then $\Theta_{n,r}(\pi)$ is cuspidal, and $\Theta_{n,r'}(\pi) \neq 0$ for any $r' > r$.

As we will see, the proof follows from a calculation of the constant terms of the theta kernel, entirely analogously to what was done for the Shimura correspondence. This reduces the second problem to the first one, which is resolved by the following much more substantial theorem.

Theorem 1.2 (Rallis, Kudla–Rallis, etc.)**.** *If $\Theta(\pi)$ is cuspidal, then it is non-zero if and only if*

Local obstruction: *The local theta lifts $\Theta_v(\pi_v) \neq 0$ for all v (with compatible auxiliary data).*
Global obstruction: *Non-vanishing of L-value: let $s_0 = \frac{m-n-\epsilon_0}{2}$, then*
– *If $s_0 \geq 0$: $L\big(s_0 + \frac{1}{2}, \pi \times \chi_V\big) \neq 0$.*
– *If $s_0 < 0$: $L(s, \pi \times \chi_V)$ has a pole at $-s_0 + \frac{1}{2}$.*

In the case of the Shimura lift, $G = \mathrm{O}_3$, $H = \mathrm{Mp}_2$, and $\epsilon = -1$. The lifts are all cuspidal except for the case of trivial representation case, which is consistent with the first occurrence theorem. Moreover, the relevant L-value is indeed the central value, as observed before. In face, the proof will also follow the same strategy, but there will be many more analytic and representation-theoretical difficulties.

2. Rallis tower property

The first main theorem was proven by Rallis [Ral84, Theorem I.2.1] in the symplectic-orthogonal case. The unitary case is the same, and we will sketch the computations there. In what follows, both spaces will be Hermitian, and the symplectic form on $W_n \otimes V_{m,r}$ is defined by

$$(w_1 \otimes v_1, w_2 \otimes v_2) := \frac{1}{2}\,\mathrm{Tr}_{E/F}(\delta\langle w_1, w_2\rangle\overline{\langle v_1, v_2\rangle})$$

where $\delta \in E$ is a fixed element such that $\mathrm{Tr}_{E/F}(\delta) = 0$.

The key formula relates the constant term of a theta lift along a maximal parabolic subgroup to the theta lift to a smaller group. In our case, $G = \mathrm{U}(W_n)$ and $H = \mathrm{U}(V_{m,r})$. The maximal parabolic subgroups of H are stabilizers of isotropic subspaces. Fix a decomposition

$$V_{m,r} = X \oplus I \oplus Y$$

where $X \oplus Y$ is a complete polarization of a $(2i)$-dimensional split subspace, and $I = (X \oplus Y)^\perp \simeq V_{m-2i,r-i}$. Let $P = MN$ be the stabilizer of X, where $M = \mathrm{U}(I) \times \mathrm{GL}(X)$ is a Levi subgroup, and $N = N_1 N_2$ is the unipotent radical, with

$$N_1 = \mathrm{Hom}(Y, I),\ N_2 = \{\beta \in \mathrm{Hom}(Y, X) \mid \mathrm{Tr}_{E/F}\langle y, \beta(y)\rangle = 0 \text{ for all } y \in Y\}$$

With respect to dual bases of X and Y, the condition for N_2 is exactly that the matrix of β is skew Hermitian. The embeddings into $\mathrm{U}(V_{m,r})$ can be expressed in matrix forms as

$$m(a,h) = \begin{pmatrix} a & & \\ & h & \\ & & (a^*)^{-1} \end{pmatrix}, \quad n_1(\alpha) = \begin{pmatrix} 1 & -\alpha^* & -\frac{1}{2}\alpha^*\alpha \\ & 1 & \alpha \\ & & 1 \end{pmatrix} \quad n_2(\beta) = \begin{pmatrix} 1 & 0 & \beta \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

where $a \in \mathrm{GL}(X), h \in \mathrm{U}(I), \alpha \in N_1, \beta \in N_2$, and $*$ represents taking adjoint.

We will use the mixed Schrödinger model for the Weil representation. Let $X_0 \oplus Y_0$ be a polarization of $W_n \otimes_E I$, then the Weil representation ω_I on $\mathrm{U}(W_n) \times \mathrm{U}(I)$ can be realized on $\mathcal{S}(X_0(\mathbb{A}_F))$. A polarization of the symplectic space $W_n \otimes_E V_{m,r}$ is $(X_0 \oplus (W_n \otimes_E Y)) \oplus (Y_0 \oplus (W_n \otimes_E X))$, so we can realize the Weil representation for $\mathrm{U}(W_n) \times \mathrm{U}(V_{m,r})$ on

$$\mathcal{S}((W_n \otimes_E Y)(\mathbb{A}_F) \oplus X_0(\mathbb{A}_F)) \simeq \mathcal{S}\big((W_n \otimes_E Y)(\mathbb{A}_F), \mathcal{S}(X_0(\mathbb{A}_F))\big)$$

Elements in the Siegel parabolic fixing $Y_0 \oplus (W_n \otimes_E X)$ have simple actions on this space. In particular, taking into account the choice of splitting, we have

$$\begin{cases} \omega(g,1)\phi(v) = \omega_I(g)(\phi(g^{-1}v)) & g \in \mathrm{U}(W_n) \\ \omega(1,a)\phi(v) = \chi_W(\det(a))\,|\det(a)|^{n/2} \cdot \phi((a^*)^{-1}v) & a \in \mathrm{GL}(X) \subseteq M \\ \omega(1,\beta)\phi(v) = \psi\big(\frac{1}{2}(v, \beta(v))\big)\phi(v) & \beta \in N_2 \\ \omega(1,\alpha)\phi(v) = \mathrm{Heis}_I(\alpha(v))\phi(v) & \alpha \in N_1 \end{cases}$$

where $v \in (W_n \otimes_E Y)(\mathbb{A}_F)$, and Heis_I denotes the Heisenberg representation of $W_n \otimes_E I$. With these formulae, we can compute the constant term of theta lifts.

Proposition 2.1. *Let $f \in \mathcal{A}_{\mathrm{cusp}}(G)$, then for all $h \in \mathrm{U}(I)(\mathbb{A}_F)$,*

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \theta_{n,r}(\phi, f)(nh)dn = \theta_{n,r-i}(\phi(0), f)(h)$$

Proof. We first expand some definitions.

$$\begin{aligned} \int_{[N]} \theta_{n,r}(\phi; g, nh)dn &= \int_{[N_1]} \int_{[N_2]} \sum_{x_0 \in X_0} \sum_{y \in W_n \otimes_E Y} \omega(g, n_2 n_1 h)\phi(y, x_0) dn_2 dn_1 \\ &= \int_{[N_1]} \sum_{x_0 \in X_0} \sum_{y \in W_n \otimes_E Y} \int_{[N_2]} \omega(g, n_2 n_1 h)\phi(y, x_0) dn_2 dn_1 \\ &= \int_{[N_1]} \sum_{x_0 \in X_0} \sum_{y \in W_n \otimes_E Y} \omega(g, n_1 h)\phi(y, x_0) \int_{[N_2]} \psi\Big(\frac{1}{2}(y, n_2(y))\Big) dn_2 dn_1 \end{aligned}$$

where the sums are over the F-points of the vector spaces. For the final integral to be non-zero, we need $(y, n_2(y)) = 0$ for all $n_2 \in N_2$. Let $S \subseteq W_n \otimes_E Y$ denote the subset of all such y, then we can rewrite the final expression as

$$\int_{[N_1]} \sum_{x_0 \in X_0} \sum_{y \in S} \omega(g, n_1 h)\phi(y, x_0) dn_1$$

This used the normalization of the Haar measure giving $[N_2]$ volume 1.

The group $R = \mathrm{U}(W_n) \times \mathrm{GL}(Y)$ acts on S. We now study its orbit. For this, it is convenient to choose a basis $f_1, \cdots, f_i$ for Y. Write $y \in W_n \otimes_E Y$ as $w_1 \otimes f_1 + \cdots + w_i \otimes f_i$, then for $\beta \in N_2$,

$$\begin{aligned} (y, \beta(y)) &= \frac{1}{2} \mathrm{Tr}_{E/F} \left(\delta \sum_{1 \le a, b \le i} \langle w_a, w_b \rangle \langle \beta(f_b), f_a \rangle \right) \\ &= \delta \sum_{1 \le a, b \le i} \langle w_a, w_b \rangle \langle \beta(f_b), f_a \rangle \end{aligned}$$

The condition on y is therefore equivalent to requiring $\langle w_a, w_b\rangle = 0$ for all $1 \le a, b \le i$. The orbits of R on S are labelled by the dimension of $\langle w_1, \cdots, w_i\rangle$. More precisely, for each $d \le \min(i, \mathrm{rank}(W_n))$, fix an isotropic subspace $T_d \subseteq W_n$ of dimension d with basis $(w_1, \cdots, w_d)$. The orbit corresponding to d can be represented by $w_1 \otimes f_1 + \cdots + w_d \otimes f_d$. Let $\mathcal{O}$ denote this set of representatives, then we can re-write the sum as

$$\sum_{y\in\mathcal{O}} \sum_{(\gamma_1,\gamma_2)\in R(F)/R_y(F)} \int_{[N_1]} \sum_{x_0\in X_0} \omega(g, n_1 h)\phi((\gamma_1,\gamma_2)y, x_0)dn_1$$

where R_y denotes the stabilizer of y in R. Let $P_d = M_d N_d$ be the parabolic subgroup of $\mathrm{U}(W_n)$ fixing the isotropic subspace T_d, then $R_y = N_d L_d$, where $L_d \subseteq \mathrm{GL}(Y)$ consists of block lower triangular matrices corresponding to the partition $(d, i-d)$.

It's easy to see using the expressions for the Weil representation that $\mathrm{GL}(Y)(F)$ embedded into P as before acts by translation. On the other hand, $\mathrm{U}(W_n)(F)$ acts by translation in the first variable and another Weil representation on the second variable. Since we are summing over $x_0 \in X_0(F)$, the result is invariant in the second variable by the invariance property of the theta kernel. Therefore, the entire group R acts by translation in the above expression, and

$$\int_{[N]} \theta_{n,r}(\phi; g, nh)dn = \sum_{y\in\mathcal{O}} \sum_{\gamma_1\in N_d(F)\backslash G(F)} \sum_{\gamma_2\in L_d\backslash \mathrm{GL}(Y)} \int_{[N_1]} \sum_{x_0\in X_0} \omega(\gamma_1 g, n_1\gamma_2 h)\phi(y, x_0)dn_1$$

where we have further used the commutation relation $\gamma n_1(\beta)\gamma^{-1} = n_1(\beta\circ\gamma^*)$. The action of N_1 is entirely on the space $\mathcal{S}(X_0(\mathbb{A}_F))$ and expressed using the Heisenberg representation. To make it explicit, choose a decomposition $W_n = T_d \oplus T'_d \oplus \Sigma$, where $T_d \oplus T'_d$ is a split subspace, and Σ is its orthogonal complement. Using the Schrödinger model with $X_0 = (\Sigma \otimes_E I)^+ \oplus T_d \otimes_E X$ for some polarization of $\Sigma \otimes_E I$, we can easily compute that

$$\int_{[N_1]} \sum_{x_0\in X_0} \omega(\gamma_1 g, n_1\gamma_2 h)\phi(y, x_0)dn_1 = \sum_{x_0\in X_0} \omega(\gamma_1 g, n_1\gamma_2 h)\phi(y, x_0) \int_{[N_1]} \psi\big(2(x_0|_{T_d\otimes_E X}, n_1(y))\big)dn_1$$

This integral vanishes unless $x_0 \in (\Sigma \otimes_E I)^+$, so we have

$$\int_{[N]} \theta_{n,r}(\phi; g, nh)dn = \sum_{y\in\mathcal{O}} \sum_{\gamma_1\in N_d(F)\backslash G(F)} \sum_{\gamma_2\in L_d\backslash \mathrm{GL}(Y)} \sum_{x_0\in(\Sigma\otimes_E I)^+} \omega(\gamma_1 g, n_1\gamma_2 h)\phi(y, x_0)$$

Therefore the constant term we are trying to evaluate is equal to

$$\begin{aligned} &\sum_{y\in\mathcal{O}} \int_{[G]} \overline{f(g)} \sum_{\gamma_1\in N_d(F)\backslash G(F)} \sum_{\gamma_2\in L_d\backslash \mathrm{GL}(Y)} \sum_{x_0\in(\Sigma\otimes_E I)^+} \omega(\gamma_1 g, \gamma_2 h)\phi(y, x_0)dg \\ =&\sum_{y\in\mathcal{O}} \sum_{\gamma_2\in L_d\backslash \mathrm{GL}(Y)} \int_{N_d(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \left(\int_{[N_d]} \overline{f(ug)} \sum_{x_0\in(\Sigma\otimes_E I)^+} \omega(ug, \gamma_2 h)\phi(y, x_0)du \right) dg \end{aligned}$$

The term $y = 0$ is exactly equal to the desired right hand side, so we just need to show the terms corresponding to dimensions at least 1 all vanish. Since f is cuspidal, it is enough to show that the sum pairing against $f(ug)$ does not depend on u, which is another computation using the formulae for the Weil representation. □

The second part of the theorem, namely the non-vanishing of theta lifts to larger groups, follows immediately. It remains to prove that $\Theta_{n,n}(\pi) \neq 0$.

Proposition 2.2. *Let $f \in \mathcal{A}_{\mathrm{cusp}}(G)$. If*

$$\int_{G(F)\backslash G(\mathbb{A}_F)} \theta_{n,n}(\phi; g, h)\overline{f(g)}dg = 0$$

for all $\phi \in \omega$, then $f = 0$.

Proof. Let $V = X \oplus Y \oplus I$ be as before, with $\dim X = \dim Y = n$ and $\dim I = m_0$. Let $P = MN$ be the Levi decomposition of the corresponding parabolic subgroup. Choose dual bases for Y and X, then in the decomposition $N = N_1 N_2$ as before, we have

$$N_2 = \mathrm{Skew}_n := \{B \in M_{n,n} \mid \bar{B}^t = -B\}$$

This is self-dual with respect to the trace pairing. We can represent $y \in W_n \otimes_E Y$ as an $n \times n$ matrix whose columns are vectors of W_n with respect to some basis. Let J_W be the matrix of the *skew* Hermitian form $\delta\langle -, -\rangle_{W_n}$. We will compute the Fourier expansion of $\theta_{n,n}$ along the unipotent subgroup N_2 with the character J_W.

$$\begin{aligned}\theta_{n,n}(\phi; g, h)^{N_2, J_W} &= \int_{[\mathrm{Skew}_n]} \theta_{n,n}(\phi; g, n_2(\beta)h)\psi(-\operatorname{Tr}(J_W\beta))d\beta \\ &= \int_{[\mathrm{Skew}_n]} \sum_{x_0 \in X_0} \sum_{y \in W_n \otimes_E Y} \omega(g, n_2(\beta)h)\phi(y, x_0)\psi(-\operatorname{Tr}(J_W\beta))d\beta \\ &= \sum_{x_0 \in X_0} \sum_{y \in W_n \otimes_E Y} \omega(g, h)\phi(y, x_0) \int_{[\mathrm{Skew}_n]} \psi\left(\operatorname{Tr}(y^t J_W y\beta) - \operatorname{Tr}(J_W\beta)\right) d\beta\end{aligned}$$

The integral vanishes unless $y^t J_W y = J_W$, which is the defining relation for $\mathrm{U}(W)$. We view $\mathrm{U}(W)$ as a subset of $W_n \otimes_E Y$ this way, then we have

$$\theta_{n,n}(\phi; g, h)^{N_2, J_W} = \sum_{y \in \mathrm{U}(W)} \sum_{x_0 \in X_0} \omega(g, h)\phi(y, x_0)$$

By choosing ϕ to not depend on x_0, we can view it as a Schwartz function on $(W_n \otimes_E X)(\mathbb{A}_F)$. After unwinding the embedding of $G(\mathbb{A}_F)$ into this space, the above expression is the $G(F)$-average of $\phi|_{G(\mathbb{A}_F)}$.

Let $\langle -, -\rangle$ be the standard inner product on $C_c^\infty(G(F)\backslash G(\mathbb{A}_F))$, then by assumption, $\langle f, \theta_{n,n}(\phi; -, h)\rangle = 0$, so $\langle f, \theta_{n,n}(\phi; g, h)^{N_2, J_W}\rangle = 0$. But by the above computation, this implies $\langle f, -\rangle$ vanishes on a dense subset of $C_c^\infty(G(F)\backslash G(\mathbb{A}_F))$, which implies $f = 0$. $\square$

3. Rallis inner product formula

We now begin proving the second main theorem, so suppose $\Theta_{n,r}(\pi)$ is cuspidal, which always happen by some r by the previous section, and we drop the subscripts in this section. The main tool here is the Rallis inner product formula, which we saw in the talks on Shimura correspondence.

Theorem 3.1 (Rallis, Kudla–Rallis, etc.). *Let $f_1, f_2 \in \pi$ and $\phi_1, \phi_2 \in \omega$. Let $s_0 = \frac{m-n-\epsilon_0}{2}$, then*

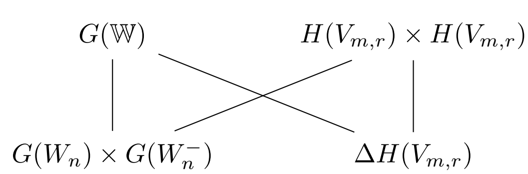
$$\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2)\rangle = c \cdot \begin{cases} \mathrm{Val}_{s=s_0+\frac{1}{2}} L(s, \pi \times \chi_V) \prod_v Z^*(s_0, \phi_{1,v} \otimes \overline{\phi_{2,v}}, f_{1,v}, f_{2,v}) & \text{if } s_0 \geq 0 \\ \mathrm{Res}_{s=-s_0+\frac{1}{2}} L(s, \pi \times \chi_V) \prod_v Z_v^*(-s_0, \phi_{1,v} \otimes \overline{\phi_{2,v}}, f_{1,v}, f_{2,v}) & \text{if } s_0 < 0 \end{cases}$$

where Val *refers to the constant term in the Laurent expansion at the point, c is a constant depending on our set-up, and Z_v^* are the normalized local zeta integrals arising from doubling, more precisely*

$$Z_v^*(s, g, \Phi_v, f_{1,v}, f_{2,v}) = \frac{1}{L_v(s, \pi_v \times \chi_{V,s})} \int_{G(F_v)} \Phi_v(g_v, 1) \cdot \overline{\langle \pi_v(g_v) f_{1,v}, f_{2,v}\rangle} dg_v$$

Remark 3.2. The normalization is not such that the product is finite, but instead, at almost all places, Z^* is a product of local L-factors attached to Hecke characters, so the analytic properties are still well-understood.

The proof uses the doubling method. Let W_n^- be the same space W_n with the form multiplied by -1, so $G(W_n)$ is canonically identified with $G(W_n^-)$. Let $\mathbb{W} = W_n \oplus W_n^-$ be the double of W, with the polarization $\mathbb{W} = W_n^\Delta \oplus W_n^\nabla$, whose terms are the graphs of the identity and the negative map $W_n \to W_n$. We have the following see-saw diagram



In broad strokes, there are three steps in the proof

$$\begin{aligned}\langle\theta(\phi_1,f_1),\theta(\phi_2,f_2)\rangle &\overset{(?1)}{=} \int_{[G(W_n)\times G(W_n^-)]} \theta_{H,G(\mathbb{W})}(\delta(\phi_1\otimes\overline{\phi_2}),1)(g_1,g_2)\overline{f_1(g_1)}f_2(g_2)\chi_V^{-1}(\deg g_2)dg_1dg_2\\ &\overset{(?2)}{=} \int_{[G(W_n)\times G(W_n^-)]} \mathcal{E}\big(s_0,\Phi^{n,r}(\delta(\phi_1\otimes\overline{\phi_2}))\big)\overline{f_1(g_1)}f_2(g_2)\chi_V^{-1}(\deg g_2)dg_1dg_2\\ &\overset{(?3)}{=} L\Big(s_0+\frac{1}{2},\pi\times\chi_V\Big)\cdot\prod_v Z_v^*(s_0,\phi_1,\phi_2,f_1,f_2)\end{aligned}$$

This is obviously wrong, because the end result is different from what was claimed. We now look at them in detail, starting from the end.

3.1. **Doubling integral.** The third step is the doubling method, which was explained in detail last time. It is also the only correct step. The key property it satisfies is that the form

$$Z_v^*: I_{Q(F_v)}^{G(\mathbb{W}_n)(F_v)}(s_0,\chi_{V,v})\otimes\pi_v\otimes(\pi_v^\vee\cdot\chi_{V,v}^{-1})\to\mathbf{C}$$

is non-zero [Yam14, Theorem 5.2]. We saw last time how this implies a close relation with the non-vanishing of the local theta correspondence, so a proof of the global non-vanishing theorem is taking shape.

3.2. **Siegel–Weil formula.** The second step is the Siegel–Weil formula describing the theta lift of the trivial representation of $H(V_{m,r})$ as an Eisenstein series. More specifically, $\mathcal{E}$ is attached to the Siegel parabolic subgroup Q fixing W_n^Δ. To add some explanation, recall that we saw an intertwining operator last time

$$\Phi^{n,r}:\omega_{\psi,\mathbb{W},\chi_V}=\mathcal{S}\big((W_n^\nabla\otimes_E V_{m,r})(\mathbb{A}_F)\big)\to I_Q^{G(\mathbb{W}_n)}(s_0,\chi_V),\ \varphi\mapsto(g\mapsto\omega(g)\varphi(0))$$

which just comes from an explicit description of the Schrödinger model. This makes it believable that the theta lift of the trivial representation locally generates the same representations as the Siegel Eisenstein series. Of course, a lot of work is needed to understand the decomposition of the principal series at the special point.

Globally, a more serious issue appears, namely the integral defining the theta lift need not converge at all. A related issue is that the Eisenstein series may have a pole at the point of evaluation. The regularization process used to circumvent them has been resolved through the works of many people. The introduction of [GQT14] has a good survey. In fact, this is where the point of evaluation switches from $s_0+\frac{1}{2}$ to $-s_0+\frac{1}{2}$.

3.3. **See-saw.** If everything converges, then the first step is entirely formal, but due to the analytic difficulties just discussed, some care is necessary. First we should be more precise about Weil representations.

Let $W_n\otimes_E V_{m,r}=X\oplus Y$ be a polarization. On the space $\mathbb{W}\otimes_E V_{m,r}$, both $X\oplus X$ and $W_n^\nabla\otimes_E V_{m,r}$ are maximal isotropic subspaces, so the Weil representation on $\mathrm{Mp}(\mathbb{W}\otimes_E V_{m,r})$ can be realized on both $\mathcal{S}((W_n^\nabla\otimes_E V_{m,r})(\mathbb{A}_F))$ and $\mathcal{S}((X\oplus X)(\mathbb{A}_F))\simeq\mathcal{S}(X(\mathbb{A}_F))\otimes\mathcal{S}(X(\mathbb{A}_F))$. The intertwining operator is given by the partial Fourier transform

$$\delta:\phi_1\otimes\phi_2\mapsto\Big((x,y)\mapsto\int_{X(\mathbb{A}_F)}\phi_1(u+x)\phi_2(u-x)\psi(2\langle u,y\rangle)du\Big)\in\mathcal{S}((W_n\otimes_E V_{m,r})(\mathbb{A}_F))$$

Observe that the Weil representation for $G(W_n)\times H(V_{m,r})$ is defined on $\mathcal{S}(X(\mathbb{A}_F))$. One can check that, as representations of $G(\mathbb{A}_F)\times G(\mathbb{A}_F)$, δ intertwines

$$\delta:\omega_{\psi,W_n,\chi_V}\boxtimes\big(\omega^\vee_{\psi,W_n,\chi_V}\cdot(\chi_V\circ\det)\big)\to\omega_{\psi,\mathbb{W},\chi_V}|_{G\times G}$$

where we added subscripts to differentiating between W_n and $\mathbb{W}$. This was done by Kudla [Kud94] in the symplectic-orthogonal case and Harris–Kudla–Sweet [HKS96] in the unitary case.

Let $f_1,f_2\in\pi$, then by assumption, $\theta(\phi,f_i)$ is cuspidal for $i=1,2$. A formal change of order of integration gives the formula

$$\langle\theta_{G,H}(\phi_1,f_1),\theta_{G,H}(\phi_2,f_2)\rangle=\int_{[G(W_n)\times G(W_n^-)]}\theta_{H,G(\mathbb{W})}(\delta(\phi_1\otimes\overline{\phi_2}),1)(g_1,g_2)\overline{f_1(g_1)}f_2(g_2)dg_1dg_2$$

where $\theta_{H,G(\mathbb{W})}(-,1)$ is the theta lift of the trivial representation of H to $G(\mathbb{W})$, and δ is applied to be able to work in the more convenient Schrödinger model for this pair. Of course, this manipulation does not make sense, and the regularization process used to prove the regularized Siegel–Weil formula also gives a correct manipulation here.

4. Non-vanishing theorems

We are now ready to discuss the second main theorem. From the Rallis inner product formula, we clearly need the following local ingredient.

Theorem 4.1. *The normalized local zeta integral*

$$Z_v^*(s_0;\phi_v,f_{1,v},f_{2,v}): I_{Q(F_v)}^{G(\mathbb{W}_n)(F_v)}(s_0,\chi_{V,v}) \to \pi_v^\vee \otimes (\pi_v \cdot \chi_{V,v})$$

does not vanish on the image of $\mathcal{S}((W_n^\nabla \otimes_E V_{m,r})(\mathbb{A}_F))$ under $\Phi^{n,r}$ if and only if the local theta lift $\Theta_v(\pi_v)$ is non-trivial.

Remark 4.2. This is not entirely known if v is a real place, due to the more complicated structures of the principal series. Proposition 11.6 of [GQT14] contains more details and references.

We also need to know the analytic behaviour of classical L-functions. Through the doubling method, this can be deduced from the analytic behaviour of certain Siegel Eisenstein series, which was also important in the statements and proofs of the regularized Siegel–Weil formulas. We recall two fundamental facts, whose proofs can be found in [GPSR87].

Proposition 4.3 (Piatetski-Shapiro–Rallis)**.** *Let π be a cuspidal automorphic form on $G(W_n)$, and let χ be a finite order character of $E^\times\backslash\mathbb{A}_E^\times$.*

– *$L(s,\pi\chi)$ converges absolutely if* $\mathrm{Re}(s) > \frac{n+\epsilon_0+1}{2}$.
– *If χ is not conjugate self-dual, then $L(s,\pi\times\chi)$ is entire. Otherwise, $L(s,\pi)$ can have poles only at*

$$\frac{1}{2} + \{1 \le j \le 2(n+\epsilon_0) \,|\, j \equiv \delta \pmod 2\}$$

where $\delta = n$ if $E = F$ and $\epsilon_{E/F}^\delta = \chi|_{\mathbb{A}_F^\times}$ if $E \neq F$.

In particular, note that the possible locations of poles are exactly the special points showing up in the main non-vanishing theorems. In fact, consider the theta lift from $G(W_n)$ to a Witt tower $V_{m_0,0} \subseteq V_{m_0+2,1} \subseteq \cdots$. We have the following regions:

$$\underbrace{V_{m_0,0} \subseteq \cdots \subseteq V^?_{n+\epsilon_0-1,-}}_{\text{First term range}} \subseteq \underbrace{V^?_{n+\epsilon_0,-}}_{\text{Centre/Boundary}} \subseteq \underbrace{V^?_{n+\epsilon_0+1,-} \subseteq \cdots \subseteq V^?_{2(n+\epsilon_0),-}}_{\text{Second term range/Converget range}} \subseteq \underbrace{V^?_{2(n+\epsilon_0)+1,-} \subseteq \cdots}_{\text{Convergent range}}$$

where we have used $-$ to replace either the dimension or the Witt index, and ? denotes a space which may not exist for parity reasons. There are qualitative differences in the proof of the Siegel–Weil formula between the regions, with the convergent range corresponding to when the un-regularized theta integral converges absolutely. Another observation is that the possible poles control both the first term range and the second term/convergent range.

Proof of Theorem 1.2. If the theta lift is non-zero, then the theorem follows immediately from the Rallis inner product formula. In the other direction, first observe that the range of absolute convergence exactly coincides with the range where we know the theta lift is automatically non-vanishing by the Rallis tower property, so we may suppose that $m \le 2(n+\epsilon_0)$.

The local hypothesis implies (up to the difficulty at real places) that the local zeta integrals are non-zero, so the inner product of the theta lift is entirely controlled by the analytic behaviour of $L(s,\pi\times\chi_V)$. If we are in the first term range, then since the L-function can have at most a simple pole, it being non-holomorphic at $-s_0+\frac{1}{2}$ is exactly equivalent to its residue at $-s_0+\frac{1}{2}$ is non-zero, so the theta lift is non-trivial.

Otherwise, we are in the second term or convergent range. We need to show that the L-function is holomorphic at $s_0+\frac{1}{2}$. By the cuspidality assumption, all lower theta lifts are trivial. In particular, the theta lifts to the first term range are trivial. This does not imply holomorphicity immediately since it could be trivial due to local obstructions. Instead, suppose the L-function has a pole there, then there exists a space $V'_{2(n+\epsilon_0)-m,-}$ in the first term range such that the theta lift of π from G to H' is non-trivial. This space cannot be in the Witt tower for V, so there exists a place where they belong to different local Witt towers. But now we have non-trivial local theta lifts to two different Witt towers, occurring at dimensions m and $2(n+\epsilon_0)-m$. This contradicts the lower bound of the conservation relation [SZ15]. ∎

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