PREPARATIONS FOR THE IWASAWA MAIN CONJECTURE

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This is the notes for a talk given at the Princeton number theory working group, intended to set up the necessary background for Wiles' proof of the Iwasawa Main Conjecture for GL(1) using Hida families. It is based very closely on the notes of Chris Skinner's 2009 CMI summer school lecture.

We will use the following notations:

- p is an odd prime.
- $\mathbf{Q}(\mu_{p^\infty}) = \bigcup_{n\geq 0} \mathbf{Q}(\mu_{p^n})$.
- $\mathbf{Q}_\infty \subseteq \mathbf{Q}(\mu_{p^\infty})$ is the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$.
- $\mathbf{Q}_n$ is the unique $\mathbf{Z}/p^n\mathbf{Z}$-extension of $\mathbf{Q}$ in $\mathbf{Q}_\infty$.
- $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ is the p-adic cyclotomic character.
- $\omega : G_{\mathbf{Q}} \to \mu_{p-1} \subseteq \mathbf{Z}_p^\times$ is the Teichmüller character.
- $\Gamma = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} (1 + p\mathbf{Z}_p)$, $\Delta = \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q}) \xrightarrow{\sim} \mu_{p-1}$.
- $\gamma \in \Gamma$ is a fixed topological generator.
- $\Lambda = \mathbf{Z}_p[[\Gamma]] \xrightarrow{\sim} \mathbf{Z}_p[[T]]$, where the isomorphism sends γ to $1 + T$.
- $*$ denotes the Pontrjagin dual.

1. Statement of the IMC

1.1. **Characters.** Let $\mathcal{W}$ be the open disc around 1 of radius 1 defined over $\mathbf{Q}_p$. It is the rigid analytic space associated to Λ. Given an integer k and a p-th power root of unity ζ, let $\phi_{k,\zeta} = \zeta\chi^k(\gamma) \in \mathcal{W}(\mathbf{Q}_p[\zeta])$, also viewed as the homomorphism $\Lambda \to \mathbf{Q}_p[\zeta]$ sending γ to $\zeta\chi^k$. These are the arithmetic points of $\mathcal{W}$.

Let $\Psi : G_{\mathbf{Q}} \to \Lambda^\times$ be the projection to Γ followed by the inclusion. This is a p-adic family of characters over $\mathcal{W}$. For each $\phi \in \mathcal{W}(\bar{\mathbf{Q}}_p)$, write $\Psi_\phi = \phi \circ \Psi$ for its specialization at ϕ. If $\phi = \phi_{k,\zeta}$ is arithmetic, then

$$\Psi_{k,\zeta} := \phi_{k,\zeta} \circ \Psi = \psi_\zeta \omega^{-k}\chi^k$$

where $\psi_\zeta : G_{\mathbf{Q}} \to \mathbf{Z}_p[\zeta]^\times$ is the finite order character defined by projection to Γ followed by sending γ to ζ.

More generally, we want to carry around a Dirichlet character ψ, viewed as a character $G_{\mathbf{Q}} \to \bar{\mathbf{Q}}_p^\times$. Let $\mathcal{O}_\psi = \mathbf{Z}_p[\psi]$, $F_\psi = \mathbf{Q}_p[\psi]$, and $\Lambda_\psi = \Lambda \otimes_{\mathbf{Z}_p} \mathcal{O}_\psi$, then we can view $\psi\Psi : G_{\mathbf{Q}} \to \Lambda_\psi^\times$ as a p-adic family of characters over $\mathcal{W}_{\mathcal{O}_\psi}$. We will denote the conductor of ψ by N_ψ and factorize it as $N_\psi^{(p)} N_{\psi,p}$, where the first term is prime to p, and the second term is a power of p.

1.2. **p-adic L-functions.**

Theorem 1.1. *Suppose ψ is odd. There exists $\mathcal{L}_\psi \in \mathrm{Frac}(\Lambda_\psi)$ such that if $k \geq 0$, then*

$$\phi_{k,\zeta}(\mathcal{L}_\psi) = L^{\{p\}}(0, \psi\Psi_{k,\zeta}) := L(0, \psi\Psi_{k,\zeta})(1 - \psi\psi_\zeta\omega^{-k}(p)p^k)$$

Moreover, $\mathcal{L}_\psi \in \Lambda_\psi$ unless $\psi = \omega^{-1}\psi_\xi$ for some ξ, in which case $(\xi\chi(\gamma)\gamma - 1)\mathcal{L}_\psi \in \Lambda_\psi$.

This is a modified version of the classical Kubota-Leopoldt p-adic L-function. We will let h_ψ denote the denominator of $\mathcal{L}_\psi$, so

$$h_\psi = \begin{cases} \xi\chi(\gamma)\gamma - 1 & \text{if } \psi = \omega^{-1}\psi_\xi \\ 1 & \text{otherwise} \end{cases}$$

and let $g_\psi = h_\psi\mathcal{L}_\psi \in \Lambda_\psi$. If one formally specializes at $k = -1$, then the existence of the denominator recovers the pole of the classical L-function at 1 when the character is trivial.

1.3. **Some results on Λ-modules.** Let $\mathcal{O}$ be a finite extension of $\mathbf{Z}_p$ of residue class degree f, ramification index e, and uniformizer ϖ. Let $\Lambda_{\mathcal{O}} = \Lambda \otimes_{\mathbf{Z}_p} \mathcal{O} \simeq \mathcal{O}[[T]]$. The usual classification theorem of Λ-modules up to pseudo-isomorphisms still hold for $\Lambda_{\mathcal{O}}$, with essentially the same proof:

Proposition 1.2. *Let M be a $\Lambda_{\mathcal{O}}$-module of finite type, then there exists irreducible polynomials $f_1, \cdots, f_n$, possibly 0, such that*

$$0 \to N \to M \to \prod_{i=1}^{n} \Lambda_{\mathcal{O}}/(f_i) \to N' \to 0$$

where N and N' have finite orders. Furthermore, the principal ideal $(\prod_i f_i) \subseteq \Lambda_{\mathcal{O}}$ depends only on M.

The principal ideal is called the characteristic ideal of M, denoted by $\mathrm{Ch}(M)$. If M is $\Lambda_{\mathcal{O}}$-torsion, then $\mathrm{Ch}(M) \neq 0$. Assuming this, let $f \in \Lambda_{\mathcal{O}}$ is any generator of $\mathrm{Ch}(M)$, we can apply the Weierstrass prepration theorem to factor it as $\varpi^{\mu} g(T) u$, where u is a unit, and $g(T)$ is a polynomial of degree λ. The values μ and λ are called the μ- and λ-invariants of M, respectively. They control the sizes of certain quotients of M.

Proposition 1.3. *Let M be a torsion $\Lambda_{\mathcal{O}}$-module. Let $\omega_n = \gamma^{p^n} - 1$, then for a fixed $n_0 \gg 1$ and $n \gg n_0$,*

$$\#\Big(M/\frac{\omega_n}{\omega_{n_0}}M\Big) = p^{\mu f p^n + \lambda e f n + O(1)}$$

Proof. We have an exact sequence

$$0 \to M/N \to \prod_i \Lambda_{\mathcal{O}}/(f_i) \to N' \to 0$$

Choose n_0 sufficiently large so that ω_{n_0} contains all p-power roots of unities roots of all f_i. This is possible since an element of $\Lambda_{\mathcal{O}}$ can have at most finitely many zeroes on $\mathcal{W}$. For any $n \geq n_0$, let $\nu_n = \omega_n/\omega_{n_0}$. Multiply the exact sequence by ν_n and apply the snake lemma. Observe that ν_n is injective on the middle term and N' has finite size. It follows that

$$\#(M/\nu_n M + N) = \prod_i \#\Lambda_{\mathcal{O}}/(\nu_n, f_i)$$

The proposition now follows from an explicit calculation of the terms on the right. □

Remark 1.4. With a more careful analysis, done in Chapter V of [NSW08], one can give explicit bounds on n and n_0, and in fact show that the $O(1)$ error term stabilizes.

Finally, if M is a discrete Λ-module such that M^* is torsion, then we define $\mathrm{Ch}(M) := \mathrm{Ch}(M^*)$.

1.4. ***p*-adic Selmer groups.** For $k \geq 0$, define

$$\begin{aligned}\mathrm{Sel}(\psi\chi^{-k}) &= H^1_{\mathrm{ur}}(\mathbf{Q}, F_\psi/\mathcal{O}_\psi(\psi\chi^{-k})) \\ &= \ker\Big(H^1(\mathbf{Q}, F_\psi/\mathcal{O}_\psi(\psi\chi^{-k})) \to \prod_\ell H^1(I_\ell, F_\psi/\mathcal{O}_\psi(\psi\chi^{-k}))\Big) \\ \mathrm{Sel}_\infty(\psi) &= H^1_{\mathrm{ur}}(\mathbf{Q}, \Lambda_\psi^*(\psi\Psi^{-1}))\end{aligned}$$

Remark 1.5. For the given module, $H^1_{\mathrm{ur}} = H^1_f$, the Bloch–Kato Selmer group.

Further define

$$X(\psi\chi^{-k}) = \mathrm{Sel}(\psi\chi^{-k})^*,\ X_\infty(\psi) = \mathrm{Sel}(\psi)^*$$

so $X_\infty(\psi)$ is a compact Λ_ψ-module. We have the following control theorem, which explains exactly how $X_\infty(\psi)$ is a p-adic interpolation of the various $X(\psi\chi^{-k})$.

Theorem 1.6. *Suppose ψ is odd and $k \geq 0$. We have a natural map*

$$(X_\infty(\psi) \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi[\zeta])/(\gamma - \zeta\chi^k(\gamma))(X_\infty(\psi) \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi[\zeta]) \to X(\psi\Psi^{-1}_{k,\zeta})$$

If for every $\ell \neq p$, the order of $\psi|_{I_\ell}$ is either 1 or not a power of p, then the map is bijective unless $k = 0$, $\psi|_{G_{\mathbf{Q}_p}} = \psi_\zeta|_{G_{\mathbf{Q}_p}}$, in which case it is surjective with kernel $\mathcal{O}_\psi$.

This theorem is mostly straightforward computations, except for the final part identifying the kernel as exactly $\mathcal{O}_\psi$, which was done in [Gre73]. It is equivalent to the non-vanishing of certain p-adic regulators. Through the IMC, this is also equivalent to the simplicity of certain zeroes of $\mathcal{L}_\psi$.

The Selmer group we have defined is based on a cyclotomic deformation of the character. It is equal to the version defined by taking the limit of finite level Selmer groups as one climbs the cyclotomic tower:

Proposition 1.7. *There are canonical isomorphisms of Λ_ψ-modules*

$$\mathrm{Sel}_\infty(\psi) \simeq \varinjlim_n \mathrm{Sel}(\mathbf{Q}_n, F_\psi/\mathcal{O}_\psi(\psi)) \simeq \mathrm{Sel}(\mathbf{Q}_\infty, F_\psi/\mathcal{O}_\psi(\psi))$$

Proof. This follows from Shapiro's lemma. Some details can be found in sections 3.1 and 3.2 of [SU14]. □

It is also related to the classical definition of Iwasawa using inverse limits of ideal class groups:

Proposition 1.8. *Let L be a field such that ψ factors through $\mathrm{Gal}(L/\mathbf{Q})$. Let $L_\infty = L\mathbf{Q}_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension of L. If p does not divide the degree of the tame part of L over $\mathbf{Q}$, then as Λ_ψ-modules,*

$$X_\infty(\psi)^\iota \simeq \mathrm{Gal}(E_\infty/L_\infty)^\psi \simeq \varprojlim_n \mathrm{Cl}(L_n)^\psi$$

where E_∞ is the maximal unramified abelian extension of L_∞, and ι indicate that $g \in \Gamma$ now acts as g^{-1}.

Proof. The inflation-restriction sequence gives

$$0 \to H^1(\Delta_\psi, F_\psi/\mathcal{O}_\psi(\psi)) \to H^1(\mathbf{Q}_\infty, F_\psi/\mathcal{O}_\psi(\psi)) \to H^1(L_\infty, F_\psi/\mathcal{O}_\psi(\psi))^{\Delta_\psi} \to H^2(\Delta_\psi, F_\psi/\mathcal{O}_\psi(\psi))$$

where $\Delta_\psi = \mathrm{Gal}(L_\infty/\mathbf{Q}_\infty)$, which naturally equals to the part of $\mathrm{Gal}(L/\mathbf{Q})$ not wildly ramified at p. By hypothesis, $p \nmid \Delta_\psi$, so the first and last term vanish, so

$$H^1(\mathbf{Q}_\infty, F_\psi/\mathcal{O}_\psi(\psi)) \xrightarrow{\sim} H^1(L_\infty, F_\psi/\mathcal{O}_\psi(\psi))^{\Delta_\psi}$$

Finally, repeating the same analysis with the local Galois groups shows that

$$\mathrm{Sel}_\infty(\psi) \simeq H^1_{\mathrm{ur}}(L_\infty, F_\psi/\mathcal{O}_\psi(\psi))^{\Delta_\psi} \simeq \varinjlim_n H^1_{\mathrm{ur}}(L_n, F_\psi/\mathcal{O}_\psi(\psi))^{\Delta_\psi}$$

Taking Pontrjagin dual gives the desired conclusion. □

We collect some deeper facts about $X_\infty(\psi)$.

Theorem 1.9 (Iwasawa)**.** *(1) $X_\infty(\psi)$ is a finitely generated torsion Λ-module.*
(2) If $p \nmid N_\psi^{(p)}$, then $X_\infty(\psi)$ has no non-zero finite Λ-submodule.

Proof. The finitely generated part follows very easily from a version of Nakayama's lemma for Λ-modules and the control theorem. The torsion part is also a consequence of the control theorem since it is equivalent to cerrtain specializations being finite. In fact, it will follow from our proof of Theorem 1.11.

The hypothesis of the second part is used when we apply Proposition 1.8. Once we remove the character, the statement (and in fact also the first part of the theorem) becomes equivalent to the weak Leopoldt's conjecture, which is known for cyclotomic $\mathbf{Z}_p$-extensions. The details can be found in Section X.3 of [NSW08]. See also Section 1.3 of [PR95]. □

1.5. Main Conjecture and reduction.

Theorem 1.10 (IMC)**.** *Let ψ be an odd character, then as fractional ideals of Λ_ψ,*

$$\mathrm{Ch}\left(\mathrm{Sel}_\infty(\psi)\right)\mathrm{Ch}\left(H^0(\mathbf{Q}, \Lambda_\psi^*(\psi^{-1}\chi\Psi))\right)^{-1} = (\mathcal{L}_{\psi^{-1}})$$

An easy computation shows that the theorem is equivalent to the statement $\mathrm{Ch}(\mathrm{Sel}_\infty(\psi)) = (g_{\psi^{-1}})$. This will not be proven in this set of notes. Instead, we show that one divisibility is enough:

Theorem 1.11 (Iwasawa)**.** *Let $K = \mathbf{Q}(\mu_N)$. Define Λ-modules*

$$I^-_{\mathrm{alg}}(K) = \prod_{\psi \text{ odd}} \mathrm{Ch}(\mathrm{Sel}_\infty(\psi)), \quad I^-_{\mathrm{an}}(K) = \left(\prod_{\psi \text{ odd}} g_\psi\right)$$

where the products are over odd characters of $\mathrm{Gal}(K/\mathbf{Q})$. Then $I^-_{\mathrm{alg}}(K)$ and $I^-_{\mathrm{an}}(K)$ have the same μ- and λ-invariant.

Proof. Fix $n \gg n_0 \gg 0$, where we will impose conditions on their sizes as we go. Let $\mathcal{O} = \mathcal{O}_\psi[\mu_{p^n}]$. On the analytic side,

$$\begin{aligned}
\#\Lambda_{\mathcal{O}}/(\nu_n, I_{\mathrm{an}}^-(K)) &= \prod_{\zeta^{p^n}=1, \zeta^{p^{n_0}} \neq 1} \#\Lambda_{\mathcal{O}}/(\gamma - \zeta, I_{\mathrm{an}}^-(K)) \\
&= \prod_{\psi \text{ odd}} \prod_{\zeta^{p^n}=1, \zeta^{p^{n_0}} \neq 1} \#\Lambda_{\mathcal{O}}/\phi_{0,\zeta}(g_\psi) \\
&= \#\mathcal{O}/(h_n/h_{n_0})^{N_{\psi,p}}
\end{aligned}$$

where h_n is the minus part of the class group of $K_n = K\mathbf{Q}_n$. The last equality follows from the analytic class number formula.

On the algebraic side, we use Theorem 1.6

$$\begin{aligned}
X_{\mathrm{alg}}^-(K)_{\mathcal{O}}/\nu_n X_{\mathrm{alg}}^-(K)_{\mathcal{O}} &= \prod_{\psi \text{ odd}} X_\infty(\psi)_{\mathcal{O}}/\nu_n X_\infty(\psi)_{\mathcal{O}} \\
&\simeq \prod_{\psi \text{ odd}} \prod_{\zeta^{p^n}=1, \zeta^{p^{n_0}} \neq 1} X_\infty(\psi)_{\mathcal{O}}/(\gamma - \zeta) X_\infty(\psi)_{\mathcal{O}} \\
&\simeq \prod_{\psi \text{ odd}} \prod_{\zeta^{p^n}=1, \zeta^{p^{n_0}} \neq 1} (X(\psi\psi_\zeta^{-1}) \otimes_{\mathcal{O}_\psi} \mathcal{O})
\end{aligned}$$

where we need to make sure n_0 is large enough so that the exceptional case in the control theorem does not occur. A similar inflation-restriction argument to the proof of Proposition 1.8 shows that rationally, $X(\psi\psi_\zeta^{-1})$ is a component of the ideal class group of K_n up to a bounded error term. It follows that

$$\#X_{\mathrm{alg}}^-(K)_{\mathcal{O}}/\nu_n X_{\mathrm{alg}}^-(K)_{\mathcal{O}} = \#\mathcal{O}/(h_n/h_{n_0})^{N_{\psi,p}} \cdot p^{O(1)}$$

Finally, choosing n and n_0 such that Proposition 1.3 applies gives the required result, since the λ- and μ-invariants are coefficients of terms of different growth rate. □

1.6. **Arithmetic consequences.** We state two immediate consequences of the main conjecture.

Corollary 1.12. *If ψ is an odd primitive Dirichlet character of conductor N such that $p \nmid \varphi(N)$, then*

$$\#\mathrm{Cl}(\mathbf{Q}(\mu_N) \otimes \mathcal{O}_\psi)^\psi = \#\,\mathrm{Sel}(\psi) = \#\mathcal{O}_\psi/(w_{\mathbf{Q}(\mu_N)}^\psi L(0, \psi^{-1}))$$

Corollary 1.13. *If $k \geq 2$ is even and $p - 1 \nmid k$, then*

$$\#H_f^1(\mathbf{Q}, \mathbf{Q}_p/\mathbf{Z}_p(1-k)) = \#\mathbf{Z}_p/(\zeta(1-k))$$

Proof. The first corollary is the specialization at $\phi_{0,\psi}$ and requires Proposition 1.8. The second corollary is the specialization at $\phi_{k-1,\omega^{k-1}}$. □

2. Hida Family

We will give a brief introduction to Hida families in this section. The main reference is [Hid93]. Let $M_k(N, \xi, A)$ be the space of classical modular forms of weight k, level $\Gamma_1(N)$, character ξ, and coefficient ring A. Let $S_k(N, \xi, A)$ be the subspace of cusp forms.

2.1. **Ordinary projector.** For a fixed N, define the Hecke operator

$$U_p = \left[\Gamma_1(Np^r)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_1(Np^r)\right] = \sum_{a=0}^{p-1} \Big|_k \left(\begin{smallmatrix}1&a\\0&p\end{smallmatrix}\right)$$

acting on $M_k(\Gamma_1(Np^r), \mathcal{O})$, for some finite extension $\mathcal{O}$ of $\mathbf{Z}_p$. On q-expansion, we have

$$U_p\Big(\sum_{n=0}^\infty a_n q^n\Big) = \sum_{n=0}^\infty a_{np} q^n$$

It preserves p-integrality, so all of its eigenvalues are p-integral. Therefore, the limit

$$e_{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$$

exists and defines an idempotent operator in the Hecke algebra $\mathbb{T}(\Gamma_1(Np^r),\mathcal{O})$. Forms in its image are said to be ordinary. In particular, if f is an eigenform, then it is ordinary if and only if its U_p-eigenvalue has p-valuation 0. Moreover, U_p decreases the level at p, so

$$M_k^{\mathrm{ord}}(\Gamma_1(Np^r),\mathcal{O}) = M_k^{\mathrm{ord}}(\Gamma_1(Np),\mathcal{O})$$

A fundamental property of ordinary forms is

Theorem 2.1 (Hida). *Let ψ be a character of $(\mathbf{Z}/Np^\alpha\mathbf{Z})^\times$. Suppose $\mathcal{O}$ contains the image of ψ. The value*

$$\mathrm{rank}_{\mathcal{O}}\, M_k^{\mathrm{ord}}(Np^\alpha, \psi\omega^{-k}, \mathcal{O})$$

is independent of k if $k \geq 2$. The same statement holds with M_k^{ord} replaced by S_k^{ord}.

Proof. This is Theorem 1 of 7.2 in [Hid93]. Its proof uses the control theorem for Hida families. However, we need to first establish that the ranks are bounded, which is proven using cohomological methods. □

2.2. **Λ-adic forms.** Let ψ be an odd Dirichlet character as before, with conductor $N_\psi = Np^r$, where $p \nmid N$. Let $\mathbb{I}$ be a complete local finite integral Λ_ψ-algebra, corresponding to a rigid analytic space $\mathcal{V}$, which is a finite cover of $\mathcal{W}$.

Definition 2.2. An $\mathbb{I}$-adic modular form of character ψ is a formal q-expansion

$$\mathbf{f} = \sum_{n=0}^{\infty} a_n q^n, \quad a_n \in \mathbb{I}$$

such that for each point $\phi \in \mathcal{V}$ lying above an arithmetic point $\phi_{k,\zeta} \in \mathcal{W}$ with $k \gg 0$ and $|\zeta - 1|_p \ll 1$,

$$\mathbf{f}_\phi = \sum_{n=0}^{\infty} \phi(a_n) q^n \in M_{k+1}(Np^{\max(r,t+1)}, \psi\psi_\zeta\omega^{-k}, \phi(\mathbb{I}))$$

We say the form $\mathbf{f}$ is ordinary or a Hida family if all of its specializations are ordinary. It is cuspidal if all of its specializations are cuspidal.

An example which will be used later is the Eisenstein family

$$\mathbf{E}_\psi = \frac{1}{2}g_\psi + h_\psi \sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ (d,N_\psi p)=1}} \prod_{\ell^e || d} \psi\Psi(\mathrm{Fr}_\ell^e)\Big)q^n$$

where Fr_ℓ is the arithmetic Frobenius at ℓ. Its specializations are classical Eisenstein series

$$\phi_{k,\zeta}(\mathbf{E}_\psi) = \phi_{k,\zeta}(h_\psi)\Big(\frac{1}{2}L(-k,\psi\psi_\zeta\omega^{-k}) + \sum_{n=1}^{\infty} \sigma_k^{\mathbf{1},\psi\psi_\zeta\omega^{-k}}(n)q^n\Big)$$

where for a character φ,

$$\sigma_k^{\mathbf{1},\varphi}(n) = \sum_{\substack{d|n \\ (d,N_\psi p)=1}} \varphi(d)d^k$$

Let $\mathbf{M}(\psi,\mathbb{I})$ be the space of all $\mathbb{I}$-adic forms. Similarly define $\mathbf{M}^{\mathrm{ord}}(\psi,\mathbb{I})$, $\mathbf{S}(\psi,\mathbb{I})$, and $\mathbf{S}^{\mathrm{ord}}(\psi,\mathbb{I})$.

Theorem 2.3 (Wiles). *The spaces $\mathbf{M}^{\mathrm{ord}}(\psi,\mathbb{I})$ and $\mathbf{S}^{\mathrm{ord}}(\psi,\mathbb{I})$ are free of finite rank over $\mathbb{I}$.*

Proof. This is a consequence of the boundedness part of Theorem 2.1. □

For $\ell \nmid Np$, we have Hecke operators T_ℓ acting on the $\mathbb{I}$-adic forms, defined by interpolating the usual T_ℓ action on q-expansions

$$a_n(T_\ell \mathbf{f}) = \sum_{d|(\ell,n)} [\omega^{-1}(d)d]d^{-1}\psi(d)a_{\ell n/d^2}(\mathbf{f})$$

where $[z]$ denote the multiplicative element in Λ which equals to z. Similarly, we have U_ℓ for $\ell|Np$. In particular, the ordinary projector e_{ord} is still defined, which projects $\mathbf{M}(\psi,\mathbb{I})$ onto $\mathbf{M}^{\mathrm{ord}}(\psi,\mathbb{I})$. We define the $\mathbb{I}$-adic Hecke algebra $\mathbf{T}^{\mathrm{ord}}(\psi,\mathbb{I})$ to be the $\mathbb{I}$-subalgebra of $\mathrm{End}_{\mathbb{I}}(\mathbf{S}^{\mathrm{ord}}(\psi,\mathbb{I}))$ generated by the Hecke operators T_ℓ and U_ℓ. By the previous theorem, $\mathbf{T}^{\mathrm{ord}}(\psi,\mathbb{I})$ is a finite $\mathbb{I}$-algebra. It can also be constructed as an inverse limit of the usual Hecke algebras acting on modular forms of bounded weight.

The importance of Hida family lies in the following control theorem.

Theorem 2.4 (Hida)**.** *(1) Any classical modular form lies in a Λ-adic family.*

(2) Any normalized classical eigenform of level divisible by p is the specialization of a normalized $\mathbb{I}$-adic eigenform, for a suitable $\mathbb{I}$.

(3) If $k \geq 2$ and ζ is a primitive p^t-th root of unity, then for any point ϕ of $\mathcal{V}$ lying above $\phi_{k,\zeta}$,

$$\mathbf{M}^{\text{ord}}(\psi, \mathbb{I}) \otimes_{\mathbb{I},\phi} \bar{\mathbf{Q}}_p \simeq M_k^{\text{ord}}(Np^{t+1}, \psi\psi_\zeta\omega^{-k}, \bar{\mathbf{Q}}_p)$$

The same statement holds for cusp forms.

This implies the exact statement of Theorem 2.1.

2.3. Λ-adic Galois representations.

Theorem 2.5 (Hida, Wiles)**.** *Let $\mathbf{f}$ be a normalized cuspidal $\mathbb{I}$-adic eigenform with character ψ, then there exists a Galois representation $\rho_{\mathbf{f}} : G_{\mathbf{Q}} \to GL_2(\text{Frac}(\mathbb{I}))$ such that*

(1) $\rho_{\mathbf{f}}$ is continuous with respect to the usual topology of $\mathbb{I}$.

(2) $\rho_{\mathbf{f}}$ is irreducible.

(3) $\det \rho_{\mathbf{f}} = \psi\Psi$.

(4) $\rho_{\mathbf{f}}$ is unramified away from Np and $\operatorname{Tr} \rho_{\mathbf{f}}(\text{Fr}_\ell) = a_\ell(\mathbf{f})$ for all $\ell \nmid Np$.

(5) If $\mathbf{f}$ is ordinary, then

$$\rho_{\mathbf{f}}|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \alpha_p^{-1}\psi\Psi & * \\ 0 & \alpha_p \end{pmatrix}$$

where α_p is unramified and $\alpha_p(\text{Fr}_p) = a_p(\mathbf{f})$.

The construction and first 4 properties were proven by Hida in [Hid86], where he also showed that $\phi(\rho_{\mathbf{f}}) = \rho_{\mathbf{f}_\phi}$ if the weight of ϕ is at least 2. The final property was shown by Wiles in [Wil88], where the construction was also generalized to arbitrary totally real fields.

References

[Gre73] Ralph Greenberg. On a certain ℓ-adic representation. *Invent. Math.*, 21:117–124, 1973.

[Hid86] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.

[Hid93] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number Fields*, volume 323 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 2nd edition, 2008.

[PR95] Bernadette Perrin-Riou. Fonctions L p-adiques des représentations p-adiques. *Astérisque*, (229):198, 1995.

[SU14] Christopher Skinner and Eric Urban. The Iwasawa main conjectures for GL_2. *Invent. Math.*, 195(1):1–277, 2014.

[Wil88] A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.