TWO VARIABLE IWASAWA MAIN CONJECTURE

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This is the notes for two consecutive talks given at the number theory working group. The first talk formulates in some generality a two variable Iwasawa main conjecture for Hida families of modular forms. The second talk shows how some Eisenstein cases of it relates to certain cyclicity hypotheses on modules coming from GL(1). The main reference is [FK12], Section 11.

0. Introduction

The purpose of the talks will be to state a two variable main conjecture for modular forms and relate the Eisenstein part of it to the previous talks. We will vaguely sketch the form of the conjecture now for motivation. The details will come later.

The Selmer group we consider will be Greenberg's Selmer group

$$S = \ker\left(H^1(G_{\mathbf{Q},p}, H^*[[\mathbf{Z}_p^\times]]) \to H^1(\mathbf{Q}_p, H^*_{\mathrm{sub}}[[\mathbf{Z}_p^\times]])\right)$$

There will be some way of measuring valuation at a prime $\mathfrak{p} \subseteq \mathfrak{h}[[\mathbf{Z}_p^\times]]$, and the main conjecture is that

$$S^* \sim_{\mathfrak{p}} H[[\mathbf{Z}_p^\times]]/(\mathcal{L}_{\mathrm{MK}})$$

for all primes $\mathfrak{p}$. Subject to technical hypotheses, the residually irreducible case of this conjecture should follow from [SU14, Theorem 3.31]. This still leaves the Eisenstein case, among others.

We will prove the ω^{-1}-part of this conjecture for those $\mathfrak{p}$ such that $\mathfrak{p} \cap \mathfrak{h}$ is contained in an Eisenstein maximal ideal $\mathfrak{m}$, assuming the following two equivalent conditions (assuming a Gorenstein condition)

(1) H^-/IH^- is generated by $(1 - T^*(p))\{0, \infty\}$ as an $\mathfrak{h}/I$-module.
(2) $\{p, 1 - \zeta_{p^r}\}$ generates X as a Λ-module.

where $X = \varprojlim_r \mathrm{Cl}(\mathbf{Q}(\mu_{p^r}))(p)$ is the classical Iwasawa module. Observe that the second condition is a stronger version of the classical cyclicity conjecture for class groups. The method of proof is similar to the proof of the GL(1)-main conjecture assuming Vandiver's conjecture using Stickelberger elements.

1. Homological Algebra

In this section, Λ will be a complete Noetherian local ring with a finite residue field k of characteristic p. Let $Q(\Lambda)$ be its total ring of fractions. Throughout this talk, $(-)^* = \mathrm{Hom}_{\mathrm{cont}}(-, \mathbf{Q}_p/\mathbf{Z}_p)$.

1.1. **Determinant.** Following Fukaya–Kato, we will define a map $[-]$ sending torsion Λ-modules of finite projective dimension to $Q(\Lambda)^\times/\Lambda^\times$ which is multiplicative in short exact sequences. Given a resolution,

$$0 \to L_n \to L_{n-1} \to \cdots \to L_0 \to M \to 0$$

Tensoring by $Q(\Lambda)$ gives an exact sequence of free modules

$$0 \to L_n \otimes_\Lambda Q(\Lambda) \to L_{n-1} \otimes_\Lambda Q(\Lambda) \to \cdots \to L_0 \otimes_\Lambda Q(\Lambda) \to 0$$

This can be split into multiple 3-term exact sequences $0 \to K_i \to L_i \otimes_\Lambda Q(\Lambda) \to K_{i-1} \to 0$ with $K_n = K_0 = 0$. Inductively, each K_i is projective, and we can choose splittings $L_i \otimes_\Lambda Q(\Lambda) \simeq K_i \oplus K_{i-1}$. Therefore, this gives an isomorphism

$$\bigoplus_{i \text{ odd}} L_i \otimes_\Lambda Q(\Lambda) \simeq \bigoplus_i K_i \simeq \bigoplus_{i \text{ even}} L_i \otimes_\Lambda Q(\Lambda)$$

The determinant of this map with respect to a choice of Λ-bases for both sides will be defined to be $[M]$. Since each finite free resolution of M is the direct sum of the minimal resolution and shifts of $\Lambda \xrightarrow{=} \Lambda$, this is independent of the choice of resolution.

Example 1.1. If Λ is a DVR, then $v([M]) = \mathrm{length}(M)$.

Example 1.2. Let $\Lambda = \mathbf{Z}_p[[T]]$ be the classical Iwasawa algebra.

(1) Let $M = \Lambda/(\xi)$, where $\xi \in \Lambda$ is non-zero, then M can be resolved by the complex
$$0 \to \Lambda \xrightarrow{\xi} \Lambda \to M \to 0$$
The determinant of M is therefore $\xi \in \mathrm{Frac}(\Lambda)^\times/\Lambda^\times$.

(2) Let $k = \mathbf{F}_p$ be the residue field. It has the Koszul resolution
$$0 \longrightarrow \Lambda \xrightarrow{\begin{pmatrix} T \\ -p \end{pmatrix}} \Lambda^2 \xrightarrow{(p,T)} \Lambda \longrightarrow k \longrightarrow 0$$
It follows that $[k] = \det\begin{pmatrix} T & p^{-1} \\ -p & 0 \end{pmatrix} = 1$ in $\mathrm{Frac}(\Lambda)^\times/\Lambda^\times$. The finite Λ-modules are all successive extensions of k, so they all have determinant 1.

It follows from the above computations and multiplicativity that if X is a torsion Λ-module, then $([X]) = \mathrm{Ch}(X)$, the characteristic ideal of X. This also agrees with the embedding of the determinant module into $\mathrm{Frac}(\Lambda)$ defined in Section 3 of [PR94].

1.2. **Selmer complex.** We summarize some results on Nekovář's Selmer complex from [Nek06]. The original reason for their introduction was to deal with trivial zeros, but we will use them because they have good finiteness properties. We will use $C^\bullet$ to denote complexes, $\mathbf{R}\Gamma$ to denote the corresponding objects in the derived category, and $H^\bullet$ to denote cohomology.

Suppose G is profinite group, and M is a topological $\Lambda[G]$-module. We define $C^\bullet_{\mathrm{cont}}(G, M)$ to be the complex of continuous inhomogeneous cochains valued in M. This computes the usual group cohomology $H^i(G, M)$ if M is finite. In general, the category of topological $\Lambda[G]$-modules is not abelian. Nekovář defined an abelian full subcategory of *admissible modules* which is stable under subobjects and quotients. It contains finitely generated Λ-modules and their Pontryagin duals. We can extend $C^\bullet_{\mathrm{cont}}(G, M)$ to a map
$$\mathbf{R}\Gamma_{\mathrm{cont}}(G, -) : D^+({}^{\mathrm{ad}}_{\Lambda[G]}\mathbf{Mod}) \to D^+({}_{\Lambda}\mathbf{Mod})$$

Theorem 1.3 (Proposition 4.2.9 of [Nek06])**.** *Suppose* $\mathrm{cd}_p(G) = d < \infty$ *and* G *satisfies*
$$\text{(F)} \qquad \dim_k H^i(U, k) < \infty \text{ for all open normal subgroup } U \trianglelefteq G \text{ and all } i \geq 0$$
Let $S \subseteq \Lambda$ *be multiplicatively closed. Let* $M \in D^+({}^{\mathrm{ad}}_{\Lambda[G]}\mathbf{Mod})$*. If* $M \otimes_R S^{-1}R \in D^{[a,b]}_{\mathrm{perf}}({}_{S^{-1}R}\mathbf{Mod})$*, then* $S^{-1}\mathbf{R}\Gamma_{\mathrm{cont}}(G, M) \in D^{[a,b+d]}_{\mathrm{perf}}({}_{S^{-1}R}\mathbf{Mod})$*.*

In particular, the conditions are satisfies if G is the absolute Galois group of a local field or the Galois group of the maximal extension of a global field unramified away from a finite number of places (assuming $p > 2$). This is a generalization of the classical finiteness results in Galois cohomology.

Definition 1.4. Let $G_{\mathbf{Q},p}$ be the Galois group of the maximal extension of $\mathbf{Q}$ unramified away from p, G_p be the absolute Galois group of $\mathbf{Q}_p$, and I_p be the inertia group of G_p. Let M be an admissible $\Lambda[G_{\mathbf{Q},p}]$-module with a G_p-stable filtration
$$0 \to M_{\mathrm{sub}} \to M \to M_{\mathrm{quo}} \to 0$$
then its Selmer group is
$$\mathrm{Sel}(M; M_{\mathrm{sub}}) = \ker\left(H^1(G_{\mathbf{Q},p}, M) \to H^1(G_p, M_{\mathrm{quo}})\right)$$
and its complex is
$$\widetilde{C}^\bullet_f(M; M_{\mathrm{sub}}) = \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{\mathbf{Q},p}, M) \to C^\bullet_{\mathrm{cont}}(G_p, M_{\mathrm{quo}})\right)[-1]$$

Our definition of the classical Selmer group differs from the one given by Greenberg [Gre94] in that he took the kernel into $H^1(I_p, M_{\mathrm{quo}})$. The difference injects into $H^1(G_p/I_p, M_{\mathrm{quo}}^{I_p})$, which will be zero in the setting of our main results.

The relation between the Selmer group and the Selmer complex is given by the next lemma.

Lemma 1.5. *There is an exact sequence*
$$0 \to \widetilde{H}^0_f(M; M_{\mathrm{sub}}) \to H^0(G_{\mathbf{Q},p}, M) \to H^0(G_{\mathbf{Q}_p}, M_{\mathrm{quo}}) \to \widetilde{H}^1_f(M; M_{\mathrm{sub}}) \to \mathrm{Sel}(M; M_{\mathrm{sub}}) \to 0$$

Proof. By construction, we have an exact triangle
$$\widetilde{\mathbf{R}\Gamma}_f(G_{\mathbf{Q},p}, M; M_{\mathrm{sub}}) \to \mathbf{R}\Gamma_{\mathrm{cont}}(G_{\mathbf{Q},p}, M) \to \mathbf{R}\Gamma_{\mathrm{cont}}(G_p, M_{\mathrm{quo}})$$
from which the exact sequence follows. ∎

The Selmer complex was introduced to have good formal duality properties. We state a special case of this, which follows from the classical Poitou–Tate duality.

Theorem 1.6 (Proposition 6.7.7 of [Nek06])**.** *Suppose T is a Λ-module of finite type with a continuous $G_{\mathbf{Q},p}$-action. Let T_{sub} be a $G_{\mathbf{Q}_p}$-stable submodule, then*

$$\widetilde{\mathbf{R}\Gamma}_f(T;T_{\mathrm{sub}}) \simeq \mathbf{R}\mathrm{Hom}(\widetilde{\mathbf{R}\Gamma}_f(T^*(1);T^*_{\mathrm{quo}}(1)),\mathbf{Q}_p/\mathbf{Z}_p)[3]$$

as objects of $D({}_{\Lambda}\mathbf{Mod})$.

2. Statements

We use the following notations

- $p>2$ is prime.
- $\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$, viewed as the weight variable Iwasawa algebra.
- $G_\infty = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$. $\chi : G_\infty \xrightarrow{\sim} \mathbf{Z}_p^\times$ is the cyclotomic character.
- $G_{\mathbf{Q},p}$ is the Galois group of the maximal extension of $\mathbf{Q}$ unramified away from p.
- G_p is the absolute Galois group of $\mathbf{Q}_p$.
- $H = \varprojlim_r H^1_{\text{ét}}(X_1(p^r)_{/\bar{\mathbf{Q}}},\mathbf{Z}_p)^{\mathrm{ord}}$, with a natural action of $G_{\mathbf{Q},p}$.
- $\mathfrak{h}$ is the $\mathbf{Z}_p$-algebra of dual Hecke operators on H.
- The diamond operators $\langle a\rangle$ defines a map $\Lambda \to \mathfrak{h}$.
- $I = \langle 1 - T^*(\ell) + \ell\langle\ell\rangle^{-1} | \ell \neq p\rangle$ is the Eisenstein ideal of $\mathfrak{h}$.
- The subscript E denotes taking the Eisenstein component.
- $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$ is the Teichmüller character.

2.1. **Algebraic side.** Let $\mathbb{H} = H(1)\hat{\otimes}_{\mathbf{Z}_p[\{\pm1\}]}\mathbf{Z}_p[[G_\infty]]$, where $\{\pm 1\}$ acts on both modules by complex conjugation. This is a module over $\mathfrak{h}[[G_\infty]]$. Let $G_{\mathbf{Q},p}$ act on $\mathbb{H}$ by its usual action on $H(1)$ and by projection on $\mathbf{Z}_p[[G_\infty]]$ (the dual action is denoted by $\mathbf{Z}_p[[G_\infty]]^\sharp$ in [FK12]). Observe that $[\sigma]\mapsto\chi(\sigma)[\sigma]$ is a Galois-equivariant isomorphism $\mathbf{Z}_p[[G_\infty]] \to \mathbf{Z}_p(1)[[G_\infty]]$, so $\mathbb{H}$ is isomorphic as a $\mathbf{Z}_p[[G_{\mathbf{Q}}]]$-module to its twists.

Recall that H has a $G_{\mathbf{Q}_p}$-stable filtration

$$0 \to H_{\mathrm{sub}} \to H \to H_{\mathrm{quo}} \to 0$$

This gives inclusions $\mathbb{H}_{\mathrm{sub}} \subseteq \mathbb{H}$ and $\mathbb{H}^*_{\mathrm{quo}} \subseteq \mathbb{H}^*$.

Definition 2.1. The compact Selmer group is

$$\mathfrak{X} = \mathrm{Sel}(\mathbb{H}^*;\mathbb{H}^*_{\mathrm{quo}})^*$$

The compact Selmer complex is

$$\mathfrak{X}' = \mathbf{R}\mathrm{Hom}(\widetilde{\mathbf{R}\Gamma}_f(\mathbb{H}^*;\mathbb{H}^*_{\mathrm{quo}}),\mathbf{Q}_p/\mathbf{Z}_p)[2]$$

Remark 2.2. Our Selmer group is different from the one defined in [FK12] by a twist, namely $\mathfrak{X} = \mathfrak{X}^{\mathrm{FK}}(1)$. This causes some differences in eigenspaces what follows.

The shift is such that the interesting cohomology of $\mathfrak{X}'$ is in degree 1. The inclusion $\mathbb{H}_{\mathrm{sub}} \subseteq \mathbb{H}$ satisfies the Panchishkin condition, defined in [Gre94]. The Selmer group defined above $\mathfrak{X}$ is a weight-space interpolation of the classical Selmer group of an ordinary modular form. We recall some results of Kato.

Theorem 2.3 (Kato)**.**

(1) $H^2(G_{\mathbf{Q},p},\mathbb{H})$ is a torsion $\mathfrak{h}[[G_\infty]]$-module.
(2) $H^1(G_{\mathbf{Q},p},\mathbb{H})$ is a torsion-free $\mathfrak{h}[[G_\infty]]$-module of rank 1.
(3) $H^1(\mathfrak{X}')$ is a torsion $\mathfrak{h}[[G_\infty]]$-module.
(4) The map $H^1(G_{\mathbf{Q},p},\mathbb{H}) \to H^1(G_p,\mathbb{H}_{\mathrm{quo}})$ is injective.

Proof. The first two statements are interpolated versions of Theorem 12.4 of [Kat04]. They follow axiomatically from the existence of an Euler system with non-torsion base layer [Rub00, Theorem II.3.2], which was construted by Kato.

For the last two statements, Theorem 1.6 and the definition of the Selmer complex gives the exact sequence

$$H^1(G_{\mathbf{Q},p},\mathbb{H}) \to H^1(G_p,\mathbb{H}_{\mathrm{quo}}) \to H^1(\mathfrak{X}')(-1) \to H^2(G_{\mathbf{Q},p},\mathbb{H}) \to 0$$

Classically, this arises from the comparison of Selmer structures: $H^1(\mathfrak{X}') \approx \mathfrak{X}$ has the Greenberg local condition at p, and $H^2(G_{\mathbf{Q},p}, \mathbb{H})$ is isomorphic to the dual of the Selmer group with the strict local condition at p. It follows from local duality that $H^1(G_p, \mathbb{H}_{\text{quo}})$ is free of rank 1 [PR95, Proposition A.2.3]. Therefore, under (1) and (2). statements (3) and (4) are equivalent.

To actually prove them, we introduce the Coleman map

$$\mathrm{Col} : H^1(G_p, \mathbb{H}_{\text{quo}}) \to S_\Lambda[[G_\infty]]$$

In our case, because $H(1)^{\mathrm{Fr}_p=1} = 0$, there are no denominators, and Col is an isomorphism. Kato's Euler system gives an element $z \in H^1(G_{\mathbf{Q},p}, \mathbb{H})$. A key property is that the Coleman map of the restriction of z is a p-adic L-function. The corresponding classical L-values are not all zero by a result of Rohrlich [Roh84], so the image of z is non-trivial. Since both modules are torsion-free, (4) must hold. □

Remark 2.4. (i) Statements (1) and (2) are equivalent statements of the weak Leopoldt conjecture, see [PR95, Proposition 1.3.2] for further formulations.
(ii) The statement that $H^1(G_{\mathbf{Q},p}, \mathbb{H})$ is the "same" as $H^2(G_{\mathbf{Q},p}, \mathbb{H})$ is usually called a "main conjecture without L-functions". Using (3) and (4), this is equivalent to a usual main conjecture. We will see more applications of the ideas.

In the formulation of the main conjecture in [FK06] using determinants, we need to know that $\mathfrak{X}$ has finite projective dimension, which we will do via $\mathfrak{X}'$. They are related by the following lemma.

Lemma 2.5. *Suppose that the following two conditions hold*

(no-pole) $$\left(H^0(G_{\mathbf{Q},p}, \mathbb{H}^*)^*\right)_{\mathfrak{p}} = 0$$

(no-triv-zero) $$\left(H^0(G_p, \mathbb{H}^*_{\text{sub}})^*\right)_{\mathfrak{p}} = 0$$

then $\mathfrak{X}'_{\mathfrak{p}} \simeq \mathfrak{X}_{\mathfrak{p}}$ *in the derived category.*

Proof. It follows from Lemma 1.5 that if (no-pole) and (no-triv-zero) holds, then

$$\widetilde{H}^0_f(\mathbb{H}^*; \mathbb{H}^*_{\text{quo}}) = 0, \quad \widetilde{H}^1_f(\mathbb{H}^*; \mathbb{H}^*_{\text{quo}}) = \mathrm{Sel}(\mathbb{H}^*; \mathbb{H}^*_{\text{quo}})$$

The functor $\mathrm{Hom}(-, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontryagin duality, so it is exact. Therefore, the map $\mathfrak{X}_{\mathfrak{p}} \to \mathfrak{X}'_{\mathfrak{p}}$ is quasi-isomorphic in degrees 0 and 1. It remains to show that the chomology of $\mathfrak{X}'_{\mathfrak{p}}$ is concentrated in degrees 0 and 1. Theorem 1.3 and the definition of $\widetilde{\mathbf{R}\Gamma}$ shows that it is a priori concentrated in degrees $[0, 2]$. By Theorem 1.6, $\mathfrak{X}' \simeq \widetilde{\mathbf{R}\Gamma}_f(\mathbb{H}(1); \mathbb{H}_{\text{sub}}(1))[-1]$, so we just need to show that $\widetilde{H}^1_f(\mathbb{H}(1); \mathbb{H}_{\text{sub}}(1)) = 0$. Using Lemma 1.5 with $M = \mathbb{H}$, this follows from the following two claims

(1) $\mathbb{H}(1)^{G_p}_{\text{quo}} = 0$.
(2) $\ker\left(H^1(G_{\mathbf{Q},p}, \mathbb{H}(1)) \to H^1(G_p, \mathbb{H}_{\text{quo}}(1))\right) = 0$.

Since $H(1)_{\text{quo}}$ is unramified, taking I_p invariant of $\mathbb{H}(1)_{\text{quo}}$ kills the G_∞ part, so the first statement follows from the purity of H. The second claim was part (4) of Theorem 2.3. Proposition 3.1.4 of [FK12] gives a proof using only the dual exponential map. □

Let $\mathfrak{p}$ be a prime in $\mathfrak{h}[[G_\infty]]$, and suppose the following condition holds

($\mathfrak{p}$-Gor) There exists a maximal ideal $\mathfrak{m} \subseteq \mathfrak{h}$ such that $\mathfrak{h} \cap \mathfrak{p} \subseteq \mathfrak{m}$ and $\mathfrak{h}_{\mathfrak{m}}$ is Gorenstein.

In particular, $H_{\mathfrak{m}}$ is free of rank 2 over $\mathfrak{h}_{\mathfrak{m}}$ by the Λ-adic Eichler–Shimura isomorphism, so $\mathbb{H}_{\mathfrak{p}}$ is free of rank 1 over $\mathfrak{h}[[G_\infty]]_{\mathfrak{p}}$. Applying Proposition 1.3 with $\Lambda = \mathfrak{h}[[G_\infty]]$ gives the following result.

Lemma 2.6. *Suppose* ($\mathfrak{p}$-Gor) *holds, then* $\mathfrak{X}'_{\mathfrak{p}}$ *is a perfect complex in* $D^b({}_{\mathfrak{h}[[G_\infty]]_{\mathfrak{p}}}\mathbf{Mod})$.

In combination with Lemma 2.5, this yields

Corollary 2.7. *Assuming* ($\mathfrak{p}$-Gor), (no-pole), *and* (no-triv-zero), *then* $\mathfrak{X}$ *has finite projective dimension.*

2.2. **Analytic side.** In the past talks, we saw two constructions of essentially the same p-adic L-function $\mathcal{L} \in \mathbb{H}$. The Mazur–Kitagawa p-adic L-function is defined using modular symbols, and the Fukaya–Kato p-adic L-function is defined as the Coleman map applied to Kato's Euler system. Given a classical ordinary cusp form f of weight 2, it determines a cohomology class $\mathrm{per}(f) \in H^1(X_1(p^r)(\mathbf{C}), \mathbf{C})$. For a finite order character ψ, the interpolation relation is roughly

$$(\mathcal{L}, \mathrm{per}(f))(\psi) = G(\psi)L(f, \psi^{-1}, 1)$$

The fact that the two constructions agree is Kato's explicit reciprocity law.

Using the Λ-adic Poincaré pairing introduced in Giada's talk, there is a Hecke-equivariant isomorphism $H \simeq \mathrm{Hom}_{\mathfrak{h}}(H, S_\Lambda)$, we can also view $\mathcal{L}$ as a homomorphism $H \to S_\Lambda[[G_\infty]]$. The Fukaya–Kato construction using the Coleman map gives an extension of this to $\widetilde{\mathcal{L}} : \widetilde{H}_{DM} \to M_{\Lambda,DM}[[G_\infty]] \otimes_\Lambda Q(\Lambda[[G_\infty]])$. Proposition 4.3.6 of [FK12] gives an explicit formula for $\widetilde{\mathcal{L}}(\{0, \infty\})$ in terms of Eisenstein series.

One may expect that the p-adic L-function should live in $\mathfrak{h}[[G_\infty]]$, which can be evaluated on pairs $\mathfrak{h} \to \bar{\mathbf{Q}}_p$ and $G_\infty \to \bar{\mathbf{Q}}_p$, corresponding to f and ψ respectively. If $(\mathrm{Gor})_\mathfrak{p}$ holds, then $\mathbb{H}_\mathfrak{p}$ is free of rank 1 over $\mathfrak{h}[[G_\infty]]_\mathfrak{p}$. In this case, the choice of a basis is essentially the choice of a p-adic period. Without this hypothesis, we can construct a p-adic L-function for each component, but they may not necessarily fit together.

2.3. **Result.** We are ready to state the main conjecture and theorem.

Conjecture 2.8. *Assuming* ($\mathfrak{p}$-Gor), (no-pole), *and* (no-triv-zero),

$$[\mathfrak{X}_\mathfrak{p}] = [(\mathbb{H}/(\mathcal{L})_\mathfrak{p}]$$

Before stating the main theorem, we introduce some eigenspaces. Let $\theta : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$ be a character. Recall that $\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^\times]] \simeq \mathbf{Z}_p[[\mathbf{Z}_p]][(\mathbf{Z}/p\mathbf{Z})^\times] = \prod_{i=0}^{p-1} \mathbf{Z}_p[[\mathbf{Z}_p]]_{(\omega^i)}$. If M is a Λ-module, then $M_\theta = M \otimes_\Lambda \mathbf{Z}_p[[\mathbf{Z}_p]]_\theta$. In particular, all $\mathfrak{h}$ modules have θ-eigenspaces via $\Lambda \to \mathfrak{h}$ by diamond operators. Furthermore, $G_\infty \simeq \mathbf{Z}_p^\times$, so we have a similar decomposition on the cyclotomic variable. We let $\mathfrak{h}[[G_\infty]]_{\theta,\theta'}$ denote the eigenspace where $\Lambda \to \mathfrak{h}$ acts by θ and G_∞ acts by θ'.

Theorem 2.9. *Suppose the following hypotheses hold*

(wt) $$\theta, \theta' \text{ are even}, \quad \theta \neq 1, \omega^2$$

(Cyc-a) $$(H^-/IH^-)_\theta \text{ is generated by } (1 - T^*(p))\{0, \infty\} \text{ as an } (\mathfrak{h}/I)_\theta\text{-module.}$$

For any prime $\mathfrak{p}$ of $\mathfrak{h}[[G_\infty]]_{\theta,\theta'}$ containing the Eisenstein ideal I, Conjecture 2.8 holds.

Proof. We verify here that (no-pole) and (no-triv-zero) are implied by the hypotheses. It follows from Lemma 3.2 that ($\mathfrak{p}$-Gor) holds. The theorem then follows from the much stronger Theorem 2.10.

Recall that $\sigma \in I_p$ acts on H_{sub} by $\langle \chi(\sigma) \rangle^{-1}$, so it acts on $\mathbb{H}^*_{\mathrm{sub}}$ by $\omega(\sigma)\chi(\sigma)^{-1}\langle \chi(\sigma) \rangle[\sigma]$. Suppose $a = \chi(\sigma)$ is torsion, then the (θ^{-1}, ω)-eigenspace has σ-action equal to multiplication by $\theta^{-1}\omega(a)$. Since θ is even, this is non-trivial, so $H^0(G_p, (\mathbb{H}^*_{\mathrm{sub}})_{\theta^{-1},\omega}) = 0$. This proves the hypothesis (no-triv-zero).

For (no-pole), consider the short exact sequence of $\mathfrak{h}[[G_{\mathbf{Q},p}]]$-modules from Section 6.3 of [FK12].

$$0 \to \mathcal{P} \to H_\theta/IH_\theta \to \mathcal{Q} \to 0$$

where $\mathcal{P} = H_\theta^-/I_\theta H_\theta^-$. An element $\sigma \in G_p$ acts on $\mathcal{Q}$ as $\langle \chi(\sigma) \rangle^{-1}$, so as before, it does not contribute to the component of H^0 we want. It acts on $\mathcal{P}$ by $\chi(\sigma)^{-1}$, so the eigenspaces match. Instead, Lemma 3.3 shows that (Cyc-a) implies Υ is surjective. But Υ is trivial in the coinvariants of H_θ/IH_θ by construction, so $\mathcal{P}$ also does not contribute. □

Theorem 2.10. *Assuming* (wt) *and* (Cyc-a), *we have an isomorphism* $\mathfrak{X}_{\theta,\theta',E} \simeq (\mathbb{H}/(\mathcal{L}))_{\theta,\theta',E}$.

3. Proofs

3.1. **Baby example.** We will begin by looking at why Vandiver's conjecture, in the weaker form of a cyclicity statement, directly implies the GL(1)-main conjecture. Recall that in classical Iwasawa theory,

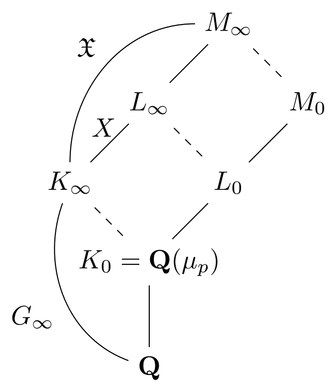
$$X = \varprojlim_r \mathrm{Cl}(\mathbf{Q}(\mu_{p^r}))(p)$$

This is a finitely generated torsion Λ-module. Vandiver's conjecture states that $X^+ = 0$.

Theorem 3.1. *Let $i \not\equiv 1 \pmod{p-1}$ be odd. If Vandiver's conjecture holds, then $X_{\omega^i} \simeq (\Lambda/(\mathcal{L}))_{\omega^i}$, where $\mathcal{L}$ is a version of the Kubota-Leopoldt p-adic L-function.*

Proof. This was first observed by Iwasawa [Iwa69], where he constructed the p-adic L-function using the Stickelberger elements. The conclusion of the theorem is then an easy consequence of cyclicity and the class number formula. Theorem 10.16 of [Was97] contains an exposition of the proof. We use a different proof which reduces it to a "main conjecture without L-functions" and shows that both sides of it are trivial using Vandiver's conjecture. Francesc's talk on Rubin's main conjecture used the same ideas in the imaginary quadratic setting.

Let $K_n = \mathbf{Q}(\mu_{p^{n+1}})$, L_n be the maximal abelian p-extension of K_n unramified everywhere, and M_n be the same except allowing ramification at p. Let $K_\infty = \bigcup_{n\geq 0} K_n$, and similarly define L_∞ and M_∞. Finally, in this proof, let $X = \mathrm{Gal}(L_\infty/K_\infty)$ and $\mathfrak{X} = \mathrm{Gal}(M_\infty/K_\infty)$. The field diagram is



This X is the same as the X before. The $\mathfrak{X}$ is obviously something different, but it can also be interpreted as the dual of a Selmer group, so it plays the same role as the $\mathfrak{X}$ from other sections. We will in fact prove that $\mathfrak{X}_j$ is isomorphic to Λ modulo a p-adic L-function if j is even. A classical reflection principles argument (or global duality) relates $\mathfrak{X}_j$ to X_{1-j}.

Class field theory gives us an exact sequence

$$\varprojlim_n \mathcal{O}^\times_{K_n,p,1} \to \varprojlim_n \mathcal{O}^\times_{K_{n,p},1} \to \mathrm{Gal}(M_\infty/K_\infty) \to \mathrm{Gal}(L_\infty/K_\infty)$$

where the inverse limits are with respect to norms, and the subscripts 1 denote taking elements which are congruent to 1 modulo the unique prime above p in the relevant field. In cohomological languages, this is equivalent to

$$H^1(G_{\mathbf{Q},p}, \mathbf{Z}_p(1)[[G_\infty]]) \to H^1(G_p, \mathbf{Z}_p(1)[[G_\infty]]) \to \mathfrak{X} \to H^2(G_{\mathbf{Q},p}, \mathbf{Z}_p(1)[[G_\infty]])$$

The Galois action on $\mathbf{Z}_p(1)[[G_\infty]]$ is by the canonical projection. The identification of the cohomology groups with the classical objects were done in my previous talk.

In both my talk and Francesc's talk, the cyclotomic units were used. Define

$$z_n = \frac{\zeta^a_{p^{n+1}} - 1}{\zeta_{p^{n+1}} - 1} \in K_n$$

where a is a choice of generator for $(\mathbf{Z}/p^2\mathbf{Z})^\times$. It is classical that $(z_n) \in \varprojlim_n \mathcal{O}^\times_{K_n}$. Therefore, $z = (z_n)$ defines a class in $H^1(G_{\mathbf{Q},p}, \mathbf{Z}_p(1)[[G_\infty]])$. By an abuse of notation, its restriction to the local cohomology group is also denoted by z. We now have an exact sequence

$$H^1(G_{\mathbf{Q},p}, \mathbf{Z}_p(1)[[G_\infty]])/(z) \to H^1(G_p, \mathbf{Z}_p(1)[[G_\infty]])/(z) \to \mathfrak{X} \to H^2(G_{\mathbf{Q},p}, \mathbf{Z}_p(1)[[G_\infty]])$$

One can modify both ends of the terms so that 0 can be put on both ends. The equality of the characteristic ideals of the two modified ends is a "main conjecture without L-functions". We will show that even before modification, both ends are zero.

Let j be even. We will consider the ω^j-eigenspace of this sequence. By Vandiver's conjecture, the last term is zero. The first term is a $\mathbf{Z}_p[[G_\infty]]$-module. By Nakayama's lemma, we just need to prove that z generates it modulo each maximal ideal. This latter module is basically $H^1(G_{K_0,p}, \mu_p)$ up to some error terms which can be controlled. But we have an exact sequence

$$0 \to \mathbf{Z}\Big[\frac{1}{p}, \zeta_p\Big]^\times \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z} \to H^1(G_{K_0,p}, \mu_p) \to \mathrm{Cl}(\mathbf{Q}(\mu_p))[p] \to 0$$

Take its ω^j-component. It is a consequence of Vandiver's conjecture that the last term vanishes. The behaviour of (z) in the first term follows from deciding whether a component of a cyclotomic unit is a p-th power or not, which is in fact also equivalent to Vandiver's conjecture, see Theorem 8.14 of [Was97].

From this discussion, we get an isomorphism

$$H^1(G_p, \mathbf{Z}_p(1)[[G_\infty]]_{\omega^j}) \xrightarrow{\sim} \mathfrak{X}_{\omega^j}$$

The final ingredient is the Coleman map

$$\mathrm{Col} : H^1(G_p, \mathbf{Z}_p(1)[[G_\infty]]) \to \mathbf{Z}_p[[G_\infty]]$$

first defined in [Col79]. It has kernel and cokernel isomorphic to $\mathbf{Z}_p(1)$, so when projected to the ω^j-component, it is an isomorphism. The key formula is

$$\mathrm{Col}(z) = \mathcal{L}$$

where $\mathcal{L}$ is a p-adic L-function. The theorem follows from this. The formula is usually called an explicit reciprocity law, because it is a generalization of the works of Artin, Hasse, Iwasawa, and others on explicit formulae for the local norm residue symbol, see the introduction of [Col79] and [PR94]. In our case, the proof uses the expression of the zeta values in terms of Benoulli numbers and the definition of the Coleman map in terms of Coleman power series. □

3.2. **Cyclicity assumption. From now on, we assume (wt)**, but the $\theta \neq \omega^2$ part will actually be unnecessary until the final part of the talk, where we will need to compare two different normalizations of Kato's Euler system. In any case, if $\theta = \mathbf{1}$ or ω^2, there is no Eisenstein component. The condition on the cyclotomic weight will not enter until the next section.

This secion will prove a few useful equivalent formulations of (Cyc-a), which will be used in the main proof. We will first look more closely into hypothesis (Cyc-a).

Lemma 3.2. *Assuming* (Cyc-a)*, then $\mathfrak{h}_{\theta,E}$ is Gorenstein.*

Proof. We have an isomorphism $H_E^- \simeq H_{\mathrm{quo},E}$ from Gyujin's talks. But $H_{\mathrm{quo}} \simeq S_\Lambda$ by the Eichler–Shimura isomorphism, so it is the dualizing module for $\mathfrak{h}$. The conclusion follows from Nakayama's lemma. □

Recall that Sharifi's conjecture gives explicit isomorphisms $H_{\theta,E}^- \simeq X_{\theta\omega^{-1}}$. It is therefore of interest to study the image of $(1 - T^*(p))\{0, \infty\}$.

Lemma 3.3. *If $a = (1 - T^*(p))\{0, \infty\}$ and $b = \{p, 1 - \zeta_{p^r}\}$, then*

$$\varpi(a) = b, \quad \Upsilon(b) = a$$

Proof. It is straightforward to calculate from the definition that $\varpi(a) = b$. For the other direction, we require some more exact sequences. Gyujin explained that Υ is the connectiong morphism for the restriction map $H^2(G_{\mathbf{Q},p}, -) \to H^2(G_p, -)$ applied to the following short exact sequence from the proof of Theorem 2.9.

$$0 \to \mathcal{P}(2) \to H_\theta/I_\theta H_\theta(2) \to \mathcal{Q}(2) \to 0$$

Suppose x is a lift of $\{p, 1-\zeta_{p^r}\}$ to $H^2(G_{\mathbf{Q},p}, H/IH(2))$. The sequence has a canonical splitting locally, and we can take the image of x_p under this map to obtain an element $z \in H^2(G_p, \mathcal{P}(2)) \simeq H^0(G_p, \mathcal{P}^*(-1))^* \xrightarrow{\sim} \mathcal{P}$, where the first identification is by local duality, and the second map is an isomorphism since $\mathcal{P}(1)$ is a trivial $G_{\mathbf{Q},p}$-module. We now seek to find a good lift x. This requires a study of another extension. We interrupt the proof to state some properties of this sequence. □

There is a short exact sequence $0 \to \mathcal{Q} \to \mathcal{E} \to \mathcal{R} \to 0$, where the modules are

$$\mathcal{Q} = (H_\theta/I_\theta H_\theta)/\mathcal{P} \simeq H_\theta^+/I_\theta H_\theta^+$$

$$\mathcal{E} = \widetilde{H}_{DM,\theta,E}/\ker(H_{\theta,E} \to \mathcal{Q})$$

$$\mathcal{R} = \widetilde{H}_{DM,\theta,E}/H_{\theta,E}$$

The various cohomology groups that appear in the long exact sequence are useful.

Lemma 3.4. *We write $H^i(-)$ for $H^i(G_{\mathbf{Q},p}, -)$.*

(1) *$\mathcal{Q} \simeq \mathfrak{h}_\theta/I_\theta \simeq \Lambda_\theta/(\xi)$ with $\sigma \in G_{\mathbf{Q},p}$ acting by $\theta([\sigma])^{-1}$. There are natural isomorphisms*

$$H^1(\mathcal{Q}(2)) \xrightarrow{\sim} X_{\theta\omega^{-1}}(1) \xrightarrow{\sim} H^2(\mathcal{Q}(2))$$

(2) *$\mathcal{R} \simeq \Lambda_\theta/(\xi)$ with Galois acting by χ^{-1}, and*

$$H^1(\mathcal{R}(1)) \simeq \mathrm{Hom}(\mathbf{Q}_p^\times, \mathbf{Z}_p) \otimes \mathcal{R}(1), \quad H^1(\mathcal{R}(2)) \simeq H^1(\mathbf{Z}_p(1)) \otimes \mathcal{R}(1)$$

(3) *The group $\mathrm{Ext}^1_{\mathbf{Z}_p[[G_{\mathbf{Q},p}]]}(\mathcal{R}, \mathcal{Q})$ may be naturally identified with $H^1(\Lambda_\theta/(\xi)^\sharp(1))$, which is a quotient of $H^1(\Lambda_\theta^\sharp(1))$. The image of $1 - \zeta_{p^\infty}$ under the Kummer map followed by these identifications is the class of the extension of $0 \to \mathcal{Q} \to \mathcal{E} \to \mathcal{R} \to 0$.*

(4) *The connecting map of the exact sequence $0 \to \mathcal{Q}(2) \to \mathcal{E}(2) \to \mathcal{R}(2) \to 0$ sending $H^1(\mathcal{R}(2))$ to $H^2(\mathcal{Q}(2))$ sends the Kummer image of p to $\{p, 1 - \zeta_{p^\infty}\}$.*

Proof. Gyujin did parts (1) and (2) in his talks. Part (4) is a consequence of part (3). Part (3) comes from a detailed study of the geometry of cusps initiated by Ohta and its relation with Siegel units. Its proof is in Section 9.3 of [FK12]. □

Proof of Lemma 3.3 continued. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P} \arrow[r] \arrow[d, shift right] & \mathcal{V} \arrow[r] & \mathcal{R} \arrow[r] & 0 \\
0 \arrow[r] & H_\theta/I_\theta H_\theta \arrow[r] \arrow[u, shift right, "s"'] \arrow[d] & \mathcal{F} \arrow[r] \arrow[u] \arrow[d] & \mathcal{R} \arrow[r] \arrow[u, "="'] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{Q} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{R} \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{F} = \widetilde{H}_{DM,\theta}/I_\theta H_\theta$, s is a G_p-stable splitting, and $\mathcal{V} = \mathcal{F}/\ker s$. The bottom two rows are $G_{\mathbf{Q},p}$-modules, while the top row are G_p-modules.

We have a class $[p] \in H^1(G_{\mathbf{Q},p}, \mathcal{R}(2)) \simeq H^1(G_{\mathbf{Q},p}, \mathbf{Z}_p(1)) \otimes \mathcal{R}(1)$ given by the cup product $\{0, \infty\} \smile p$. After twisting everything by 2, Lemma 3.4 shows that x can be chosen to be the image of $[p]$ under the connecting map of the middle sequence, so $z \in H^2(G_p, \mathcal{P}(2))$ is the image of $[p]$ under the connecting map of the top sequence. Since $[p]$ is defined as a cup product, we have $z = z' \smile p$, where z' is the image of

$$\{0, \infty\} \in H^0(G_p, \mathcal{R}(1)) \to H^1(G_p, \mathcal{P}(1))$$

This can be done using the description of the Galois action on $\mathcal{F}$. The result is in fact the cup product of $(1 - T^*(p))\{0, \infty\} \in H^0(G_p, \mathcal{P}(1))$ with the unramified class $\nu \in H^1(G_p, \mathbf{Z}_p)$. The image of $\nu \smile p$ under $H^2(G_p, \mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Z}_p$ is 1, so we are done. □

We can now state two equivalent conditions to (Cyc-a).

Lemma 3.5. *Consider the hypotheses*

(Cyc-b) $\{p, 1 - \zeta_{p^r}\}$ generates $X_{\theta\omega^{-1}}$ as a $\Lambda_{\theta\omega^{-1}}$-module.

(Cyc-c) $H^2(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta,E}(2)) = 0.$

then (Cyc-a) $\implies$ (Cyc-b), (Cyc-c). *If $\mathfrak{h}_{\theta,E}$ is Gorenstein, then* (Cyc-b) $\implies$ (Cyc-a).

Proof. It was observed in Preston's talk that if H^-/IH^- is free of rank 1 over $\mathfrak{h}/I$, which is implied by (Cyc-a), then Fukaya–Kato's result implies Sharifi's conjecture rationally. In particular, ϖ is surjective since X has μ-invariant 0 [FW79]. It follows from Lemma 3.3 that (Cyc-a) $\implies$ (Cyc-b).

To prove (Cyc-c), first observe that

$$\widetilde{H}_{DM,\theta}/I_\theta\widetilde{H}_{DM,\theta} \simeq \mathcal{E}$$

under (Cyc-a), so we have an isomorphism

$$H^2(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta,E}(2)) \otimes_{\mathfrak{h}_\theta} \mathfrak{h}_\theta/I_\theta \simeq H^2(G_{\mathbf{Q},p}, \mathcal{E}(2))$$

Lemma 3.4 gives the exact sequence

$$H^1(G_{\mathbf{Q},p}, \mathcal{R}(2)) \to H^2(G_{\mathbf{Q},p}, \mathcal{Q}(2)) \to H^2(G_{\mathbf{Q},p}, \mathcal{E}(2)) \to 0$$

where the boundary map is surjective by (Cyc-b), which we know follows from (Cyc-a), so (Cyc-c) follows.

Finally, assuming $\mathfrak{h}_{\theta,E}$ is Gorenstein, then $H^-_\theta/I_\theta H^-_\theta$ is free of rank 1 over $\mathfrak{h}_\theta/I_\theta$. Preston's argument in his talk showed that Υ is surjective in this case. Therefore, by Lemma 3.3, (Cyc-b) implies (Cyc-a). ☐

Remark 3.6. It was shown by Wake–Wang-Erickson [WWE18] that Greenberg's conjecture (X^+ is finite) implies that $\mathfrak{h}_{\theta,E}$ is Gorenstein.

3.3. **Sketch of proof.** First observe that taking eigenspaces and Eistenstein component are all exact functors, so they commute with cohomology. This will be used without comment.

We can do everything so far with H replaced by $\widetilde{H}_{DM}$. Let $\widetilde{\mathfrak{X}}$ denote the corresponding $\mathfrak{X}$. There is a short exact sequence

$$0 \to H \to \widetilde{H}_{DM} \to \Lambda/(\xi) \to 0$$

where the Galois action on the quotient is by χ^{-1}, so the difference between $\mathbb{H}$ and $\widetilde{\mathbb{H}}$ is the trivial Galois-module $\Lambda^*[\xi]$ tensered with $(\mathbf{Z}_p[[G_\infty]])^{*,\sharp}$. The error terms in the long exact sequence in global cohomology are therefore related to the Iwasawa cohomology of $\mathbf{Q}_p/\mathbf{Z}_p$. It is a classical result of Iwasawa [Iwa73] that the weak Leopoldt's conjecture holds for $\mathbf{Z}_p(1)$, see also [NSW08, Theorem 10.3.25] and [PR95, Proposition1.3.2] for a cohomological perspective. Therefore,

$$H^2(G_{\mathbf{Q}(\mu_{p^\infty})}, \mathbf{Q}_p/\mathbf{Z}_p) = 0, \quad H^1(G_{\mathbf{Q}(\mu_{p^\infty})}, \mathbf{Q}_p/\mathbf{Z}_p)_+ = 0$$

By (wt), we have $\mathfrak{X}_{\theta'} \simeq \widetilde{\mathfrak{X}}_{\theta'}$.

We have an exact sequence, analogous to the one used in Theorem 2.3

$$H^1(G_{\mathbf{Q},p}, \widetilde{\mathbb{H}}(1)) \to H^1(G_p, \widetilde{\mathbb{H}}_{\mathrm{quo}}(1)) \to \widetilde{\mathfrak{X}} \to H^2(G_{\mathbf{Q},p}, \widetilde{\mathbb{H}}(1))$$

Condition (Cyc-c) shows that the final term is trivial when localized to the base level, so by Nakayama's lemma, it is trivial.

Fukaya and Kato defined a map

$$\gamma \mapsto z_\gamma : \widetilde{H} \to H^1(G_{\mathbf{Q},p}, \widetilde{\mathbb{H}}(1))$$

which interpolates Kato's Euler system in an ordinary family. Quotienting by $z_{\{0,\infty\}}$ and taking the appropriate components give an exact sequence

$$H^1(G_{\mathbf{Q},p}, \widetilde{\mathbb{H}}_{\theta,E}(1))/(z_{\{0,\infty\}}) \to H^1(G_p, \widetilde{\mathbb{H}}_{\mathrm{quo},\theta,E}(1))/(z_{\{0,\infty\}}) \to \widetilde{\mathfrak{X}}_{\theta,E} \to 0$$

where by an abuse of notation, we denote the restriction of $z_{\{0,\infty\}}$ by the same symbol. We now show that the first term is zero. By Nakayama's lemma, this is equivalent to the base layer of $z_{\{0,\infty\}}$ generates $H^1(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta,E}(2))$. We want to further quotient out by the Eisenstein ideal. There is a spectral sequence

$$E_2^{ij} = \mathrm{Tor}_{-i}^{\mathfrak{h}_\theta}(H^j(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta,E}(2)), \mathfrak{h}_\theta/I_\theta) \Rightarrow H^{i+j}(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta,E}/I_\theta\widetilde{H}_{DM,\theta,E}(2))$$

But by (Cyc-c), $H^2(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta,E}(2)) = 0$, so the low degree terms give

$$H^1(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta,E}(2)) \otimes_{\mathfrak{h}_\theta} \mathfrak{h}_\theta/I_\theta \simeq H^1(G_{\mathbf{Q},p}, \widetilde{H}_{DM,\theta}/I_\theta\widetilde{H}_{DM,\theta}(2))$$

Recall that in the proof of Lemma 3.5, we observed that (Cyc-a) implies $\widetilde{H}_{DM,\theta}/I_\theta\widetilde{H}_{DM,\theta} = \mathcal{E}$. We just need to show that the image of $z_{\{0,\infty\}}$ generate $H^1(G_{\mathbf{Q},p}, \mathcal{E}(2))$. From Lemma 3.4, there is an exact sequence

$$0 \to H^1(G_{\mathbf{Q},p}, \mathcal{Q}(2)) \to H^1(G_{\mathbf{Q},p}, \mathcal{E}(2)) \to H^1(G_{\mathbf{Q},p}, \mathcal{R}(2)) \to H^2(G_{\mathbf{Q},p}, \mathcal{Q}(2))$$

where the final map is given by

$$\Lambda_\theta/(\xi) \to X_{\theta\omega^{-1}}(1), \ 1 \mapsto \{p, 1-\zeta_{p^\infty}\}$$

so it is an isomorphism by (Cyc-b). Therefore, $H^1(G_{\mathbf{Q},p}, \mathcal{E}(2)) \simeq H^1(G_{\mathbf{Q},p}, \mathcal{Q}(2)) \simeq X_{\theta\omega^{-1}}(1)$. The image of $z_{\{0,\infty\}}$ under this composite is $\{p, 1-\zeta_{p^\infty}\}$. To see this, consider the following approximate commutative triangle from [FK12, Theorem 3.3.9]

$$\begin{tikzcd} H \arrow[r, "``z\text{''}"] \arrow[dr, "\varpi"'] & H^1(G_{\mathbf{Q},p}, H(2)) \arrow[d, "\text{ev. at } \infty"] \\ & X \end{tikzcd}$$

We did not give a definition of z, and the "z" in the above diagram has a different normalization. In fact, $(1-T^*(p))\{0,\infty\}$ is sent to $z_{\{0,\infty\}}$. From Gyujin's talks, evaluation at ∞ is closely related to the boundary map identifying $H^1(G_{\mathbf{Q},p},\mathcal{Q}(2))$ with $X(1)$, so the claim holds. Now, by (Cyc-b), $\{p,1-\zeta_{p^\infty}\}$ is a generator, so we have shown that $z_{\{0,\infty\}}$ is a generator of $H^1(G_{\mathbf{Q},p},\widetilde{\mathbb{H}}_{\theta,E}(1))$.

We now have an isomorphism

$$\widetilde{\mathfrak{X}}_{\theta,E} \xrightarrow{\sim} H^1(G_p,\widetilde{\mathbb{H}}_{\mathrm{quo},\theta,E}(1))/(z_{\{0,\infty\}})$$

As in the proofs of Theorem 2.3 and Theorem 3.1, there is a Coleman map

$$\mathrm{Col}: H^1(G_p,\widetilde{\mathbb{H}}_{\mathrm{quo}}) \xrightarrow{\sim} M_{\Lambda,DM}[[G_\infty]]$$

Recall that Fukaya–Kato defined $\mathrm{Col}(z_\gamma)=\widetilde{\mathcal{L}}(\gamma)$, and Kato's explicit reciprocity law shows that this is a p-adic L-function interpolating classical L-values. This gives an isomorphism.

$$\left(H^1(G_p,\widetilde{\mathbb{H}}_{\mathrm{quo}}(1))/(z_{\{0,\infty\}})\right)_{\theta,\theta',E} \simeq \left(M_{\Lambda,DM}(1)[[G_\infty]]/(\widetilde{\mathcal{L}}(\{0,\infty\}))\right)_{\theta,\theta',E}$$

It remains to compare this to $\mathbb{H}/(\mathcal{L})$. We have a map $H_E^- \xrightarrow{\sim} S_{\Lambda,E}$ given by pairing with $(1-T^*(p))\{0,\infty\}$. By the comparison between $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, this sends $\mathcal{L}$ to $(1-T^*(p))\widetilde{\mathcal{L}}(\{0,\infty\})$, so

$$(\mathbb{H}/(\mathcal{L}))_{+,E} \xrightarrow{\sim} S_{\Lambda,E}(1)[[G_\infty]]/((1-T^*(p))\widetilde{\mathcal{L}}(\{0,\infty\})) \simeq M_{\Lambda,DM,E}(1)[[G_\infty]]/(\widetilde{\mathcal{L}}(\{0,\infty\}))$$

where the inverse of the last isomorphism is acting by $1-T^*(p)$, which is injective on S_Λ by the purity of H. Since (wt) assumes θ is even, we are done. $\square$

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