HIGHER HIDA THEORY, FIRST STAB

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Contents

We attempt to go over the higher Hida theory part of Pilloni's GSp_4 paper, with notations in accordance with [LPSZ]. Mostly it is about carefully analyzing integral Hecke operators on coherent cohomology groups.

There is a very recent Boxer–Pilloni preprint that seems to bypass all these by using Scholze theory. I am not yet sure what's exactly the scope of it.

1. Outline

We want to use the Igusa tower. One would have such thing over the ordinary locus, but it's affine so it has no higher coherent cohomology.

Luckily we have a finer stratification, and over $X^{\geq 1}_{\mathrm{Kl}(p)}$ (etale rank ≥ 1) there is another Igusa tower, just trivializing H. This is one-variable, as opposed to the Igusa tower over the ordinary locus where it has two variables. Using the Igusa tower is not too difficult.

The more subtle part is to relate the ordinary cohomology of X and $X^{\geq 1}$, and this involves careful analysis of Hecke operators.

2. Algebraic preliminaries

2.1. **Ordinary projector on complexes.** Let R be a complete local noetherian ring with finite residue field. Then a typical construction of "ordinary projector" in Hida theory can be summarized by the following

Proposition 2.1. *Let M be a $\mathfrak{m}_R$-adically separated, complete R-module, and $T \in \mathrm{End}_R(M)$. Suppose that T acts locally finitely on M, then there is an idempotent $e \in \mathrm{End}_R(M)$, the ordinary projector, which is defined as*

$$ev = \lim_{N\to\infty} T^{N!}v \quad (\textit{in } \mathfrak{m}_R\textit{-adic topology}),$$

for any $v \in M$.

Definition 2.1. *(1) For $T \in \mathrm{End}_R(M)$, T is locally finite if $T \in \mathrm{End}_{R/\mathfrak{m}_R^n}(M/\mathfrak{m}_R^n)$ is locally finite for all n. By easy induction it is equivalent to requiring this for only $n = 1$.*
(2) Over an artinian local ring (like $R/\mathfrak{m}_R^n$), local finiteness means the module is a union of T-stable finite type submodules.

Quick proof of Proposition 2.1. It's enough to show $T^{N!}v (\mathrm{mod}\, \mathfrak{m}_R^n)$ becomes stationary for $N \gg 0$. Because T is locally finite, one can take a finite type T-stable submodule and work there. Then $\{T^{N!}M\}$ is decreasing chain of submodules of an artinian module so it becomes constant after $N \gg 0$. Then this module that one eventually ends up with is a module of finite order, and T defines a permutation which becomes trivial after possibly enlarging N. □

So what is the corresponding definition/statement for complexes? Note that we want to work with derived categories.

Definition 2.2. *Let $\mathrm{Kom}(R)$ be the category of bounded complexes of $\mathfrak{m}_R$-separated complete flat R-modules. Let $D(R)$ be the category generated by $\mathrm{Kom}(R)$ inside the derived category $D^b(\mathrm{Mod}(R))$.*

- *For $M^\bullet \in \mathrm{Kom}(R)$ and $T \in \mathrm{End}_{\mathrm{Kom}(R)}(M^\bullet)$, T is locally finite if T is locally finite on each M^i.*
- *For $M^\bullet \in D(R)$ and $T \in \mathrm{End}_{D(R)}(M^\bullet)$, T is locally finite if there is a representative on the level of $\mathrm{Kom}(R)$ which is locally finite.*

Now on the ordinary projector on complexes:

- On the level of $\mathrm{Kom}(R)$, one has ordinary projector on each M^i, and this defines a projector $e \in \mathrm{End}_{\mathrm{Kom}(R)}(M^\bullet)$.
- On the level of $D(R)$, one chooses a lift and define a projector on the level of $\mathrm{Kom}(R)$. One can easily show that this does not depend on the choice of lift.

There is a more concrete way of realizing locally finite endomorphisms on $D(R)$.

Lemma 2.1. *For $M^\bullet \in D(R)$ and $T \in \mathrm{End}_{D(R)}(M^\bullet)$, T is locally finite if and only if both of the following hold.*

(1) There exists a representative of $(M^\bullet, T)$ over $\mathrm{Kom}(R)$ (doesn't have to be locally finite).
(2) T acts locally finitely on cohomology groups $H^i(M^\bullet \otimes^{\mathbb{L}}_R R/\mathfrak{m}_R)$ for all i.

2.2. **Cohomological correspondences.** To geometrically define Hecke operators, one uses cohomological correspondences. Things get complicated here because we ought to work with integral models of Shimura varieties (as coherent cohomology is what we want to interpolate, so it has to be integral) and they often get complicated. Nevertheless with a slightly more sophisticated theory we can stop worrying.

Very formally a cohomological correspondence would consist of a diagram of the sort

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{p_2} & & \searrow^{p_1} & \\ X & & & & Y \end{array}$$

with coherent sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y so that "some formally cohomological operations" will yield a map $R\Gamma(X, \mathscr{F}) \to R\Gamma(Y, \mathscr{G})$. Those "cohomological operations" should involve pulling back and pushing forward (or taking the trace):

$$R\Gamma(X, \mathscr{F}) \xrightarrow{p_2^*} R\Gamma(C, p_2^*\mathscr{F}) = R\Gamma(Y, R(p_1)_* p_2^*\mathscr{F}) \xrightarrow{???} R\Gamma(Y, \mathscr{G})$$

So it's fairly reasonable to call the diagram together with a morphism $R(p_1)_* p_2^*\mathscr{F} \to \mathscr{G}$ as a cohomological correspondence. In many situations (e.g. p_1 can be "compactified" into a projective morphism), there is the functor

$$p_1^! : D^+(\mathrm{QCoh}(\mathcal{O}_Y)) \to D^+(\mathrm{QCoh}(\mathcal{O}_C))$$

which is right adjoint to $R(p_1)_*$:

$$\mathrm{Hom}_{D(\mathrm{QCoh}(\mathcal{O}_C))}(\mathscr{A}, p_1^!\mathscr{B}) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathrm{QCoh}(\mathcal{O}_Y))}(R(p_1)_*\mathscr{A}, \mathscr{B})$$

for $\mathscr{A} \in D^-(\mathrm{QCoh}(\mathcal{O}_C)), \mathscr{B} \in D^+(\mathrm{QCoh}(\mathcal{O}_Y))$. This is called coherent duality, or less formally relative Serre duality. Thus you can guess that $p_1^!$, somewhat abstractly defined, should have something to do with dualizing complex. With this adjunction, a cohomological correspondence is equivalently the diagram with a morphism $p_2^*\mathscr{F} \to p_1^!\mathscr{G}$.

Proposition 2.2.

(1) $(-)^!$ is compatible with composition of morphisms.

(2) For $\mathscr{A} \in \mathrm{QCoh}(\mathcal{O}_Y)$ and $\mathscr{B}$ a vector bundle on Y,

$$p_1^!\mathscr{A} \otimes_{\mathcal{O}_C} p_1^*\mathscr{B} = p_1^!(\mathscr{A} \otimes_{\mathcal{O}_Y} \mathscr{B})$$

In particular, $p_1^!\mathscr{B} = p_1^!\mathcal{O}_Y \otimes_{\mathcal{O}_C} p_1^*\mathscr{B}$.

(a) For $\mathscr{A} \in D^-(\mathrm{QCoh}(\mathcal{O}_Y))$ and $\mathscr{B} \in D^b(\mathrm{QCoh}(\mathcal{O}_Y))$ ($\mathscr{B}$ has to be quasi-isomorphic to a bounded complex of flat sheaves),

$$p_1^!\mathscr{A} \otimes^{\mathbb{L}}_{\mathcal{O}_C} Lp_1^*\mathscr{B} = p_1^!(\mathscr{A} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathscr{B})$$

(3) Let $f : X \to S$ be a morphism of pure relative dimension n. If f is Cohen-Macaulay (Gorenstein, resp.), then $f^!\mathcal{O}_S = \omega_{X/S}[n]$, where $\omega_{X/S}$ is the dualizing sheaf, which is a coherent sheaf (a line bundle, resp.).

Remark 2.1. Recall Cohen-Macaulay $\supset$ Gorenstein $\supset$ L.c.i. $\supset$ regular.

Using these, we see that if C, Y are S-schemes of the same pure relative dimension n where $f : C \to S$ is Cohen-Macaulay and $g : Y \to S$ is Gorenstein, then

$$\omega_{C/S}[n] = f^!\mathcal{O}_S = p_1^!(\omega_{Y/S}[n]) = p_1^!\mathcal{O}_Y \otimes_{\mathcal{O}_C} p_1^*\omega_{Y/S}[n]$$

so $p_1^! \mathcal{O}_Y = \omega_{C/S} \otimes_{\mathcal{O}_C} (p_1^* \omega_{Y/S})^{-1}$.

This fits quite well with the setting of Hecke correspondences because

- X, Y, C are all integral models of Shimura varieties (or something close to them) with some level structure, and most of the cases they are normal and Cohen-Macaulay (maybe local model is always now proven to be so).
- They have codimension ≥ 2 singular locus (generic fiber is smooth, and special fiber is also generically smooth).
- So, if one in addition knows Y is Gorenstein, then there is the fundamental class $\mathcal{O}_C \to p_1^! \mathcal{O}_Y$, i.e. a morphism $p_1^* \omega_{Y/S} \to \omega_{C/S}$.
  (1) Over the smooth locus, this is the same as $p_1^* \Omega^n_{Y/S} \to \Omega^n_{C/S}$, which can be given as the determinant of the natural map $p_1^* \Omega^1_{Y/S} \to \Omega^1_{C/S}$.
  (2) It's normal, so the section over the smooth locus extends over the singular locus, which is of codimension ≥ 2.

  Given this, now we can give a cohomological correspondence by the usual means, namely giving a morphism $p_2^* \mathscr{F} \to p_1^* \mathscr{G}$, since this gives

$$p_2^* \mathscr{F} \to p_1^* \mathscr{G} \xrightarrow{\text{fund}} p_1^! \mathcal{O}_Y \otimes p_1^* \mathscr{G} = p_1^! \mathscr{G}$$

  (usually $\mathscr{F}$ and $\mathscr{G}$ are vector bundles).

3. Defining optimally integral Hecke operators

We now try to actually define integral Hecke operators for certain situations. The covered cases are T_p for modular curve and Siegel threefold of spherical level at p, and U_p for Siegel threefold of Klingen level at p.

3.1. **T_p of modular curves.** Let's try to consider the simplest case, T_p of modular curve. Let $X_1(N)/\mathbb{Q}$ be the modular curve of level $\Gamma_1(N)$, for $(N,p) = 1$. Then what one would do to define T_p is to consider the correspondence diagram

$$\begin{array}{ccc} & X(\Gamma_1(N) \cap \Gamma_0(p)) & \\ {}^{p_2}\swarrow & & \searrow^{p_1} \\ X_1(N) & & X_1(N) \end{array}$$

where $X(\Gamma_1(N) \cap \Gamma_0(p))$ is the (compactified) moduli space of an isogeny $\pi : E_1 \to E_2$ of degree p (plus extra "level N structure"), and p_i only remembers E_i. For $k \geq 0$ and ω^k over $X_1(N)$, the morphism $p_2^* \omega^k \to p_1^* \omega^k$ needed for cohomological correspondence is immediately obtained from the universal isogeny.

Now suppose one would like to do the same thing for the usual integral models over $\mathbb{Z}_p$. The integral model $\mathfrak{X}(\Gamma_1(N) \cap \Gamma_0(p))$ is not smooth over $\mathbb{Z}_p$, but still regular (semistable in fact), and p_1, p_2 are still finite flat. So we can apply the exactly same construction as above to get a Hecke operator

$$T^{naive} : R\Gamma(\mathfrak{X}_1(N), \omega^k) \to R\Gamma(\mathfrak{X}_1(N), \omega^k)$$

But this is not the correct T_p. Namely, the image of this cohomological correspondence T^{naive} is always divisible by $p^{\inf\{1,k\}}$.

Proposition 3.1. *The naive T^{naive} factors through*

$$R\Gamma(\mathfrak{X}_1(N),\omega^k)\to p^{\inf\{1,k\}}R\Gamma(\mathfrak{X}_1(N),\omega^k)\to R\Gamma(\mathfrak{X}_1(N),\omega^k)$$

Proof. By normality of integral models of modular curves, we only need to check at the smooth locus. Since there is nothing to check at the generic fiber, one only needs to check at the generic points of irreducible components of the special fiber. The special fiber of $\mathfrak{X}(\Gamma_1(N)\cap\Gamma_0(p))$ has the usual picture, that there are two irreducible components, each smooth (Igusa curve), intersecting transversally at supersingular points. The natural map

$$\rho:\mathrm{Ig}_1\to\mathfrak{X}_1(N)_0$$

of forgetting trivialization is generically etale.

In terms of moduli problems, the two components correspond to the kernel of the isogeny $E_1\to E_2$ being multiplicative and etale, respectively.

- Over the "étale" component, namely if ξ is the generic point of the étale component, $(p_2^*\omega^k)_\xi\to(p_1^*\omega^k)_\xi$ is an isomorphism. On the other hand, after identifying the component with Ig_1, it maps to $\mathfrak{X}_1(N)_0$ via $F\circ\rho$, where F is the absolute Frobenius of Ig_1. So, the trace $(p_1^*\mathcal{O})_\xi\to(p_1^!\mathcal{O})_\xi$ factors through $(p_1^*\mathcal{O})_\xi\to p(p_1^!\mathcal{O})_\xi\hookrightarrow(p_1^!\mathcal{O})_\xi$. Thus, the morphism $p_2^*\omega^k\to p_1^!\omega^k$ over ξ factors through $(p_2^*\omega^k)_\xi\to p(p_1^!\omega^k)_\xi\to(p_1^!\omega^k)_\xi$.
- Over the "multiplicative" component, if ξ' is the generic point of the multiplicative component, $(p_2^*\omega)_{\xi'}\to(p_1^*\omega)_{\xi'}$ factors through $(p_2^*\omega)_{\xi'}\to p(p_1^*\omega)_{\xi'}\to(p_1^*\omega)_{\xi'}$, so $(p_2^*\omega^k)_{\xi'}\to(p_1^*\omega^k)_{\xi'}$ factors through $(p_2^*\omega^k)_{\xi'}\to p^k(p_1^*\omega^k)_{\xi'}\to(p_1^*\omega^k)_{\xi'}$. On the other hand, after identifying the component with Ig_1, it maps to $\mathfrak{X}_1(N)_0$ via ρ, so the trace map $(p_1^*\mathcal{O})_\xi\to(p_1^!\mathcal{O})_\xi$ is an isomorphism. Thus, the morphism $p_2^*\omega^k\to p_1^!\omega^k$ over ξ factors through $(p_2^*\omega^k)_\xi\to p^k(p_1^!\omega^k)_\xi\to(p_1^!\omega^k)_\xi$.

Thus optimally it factors through $p_2^*\omega^k\to p^{\inf\{1,k\}}p_1^!\omega^k\to p_1^!\omega^k$. □

In fact this is a quite typical situation, and one can even predict what could be the optimal p-power should be. Indeed, T^{naive} is the same as convoluting $\mathbf{1}_{\mathrm{GL}_2(\mathbb{Z}_p)\left(\begin{smallmatrix}1&0\\0&p^{-1}\end{smallmatrix}\right)\mathrm{GL}_2(\mathbb{Z}_p)}$ in the spherical Hecke algebra. Thus, if $\pi=\pi_f\otimes\pi_\infty$ appears in $R\Gamma(X_1(N),\omega^k)$, then the local-global compatibility (really Satake isomorphism) says that

$$T^{naive}|\pi_p=p\,\mathrm{Tr}(\mathrm{Frob}_p^{-1}|\rho_\pi),$$

where ρ_π is the Galois representation attached to π. Since ρ_π is crystalline and has Hodge-Tate weights 0 and $k-1$, as the Newton polygon should lie over the Hodge polygon,

$$v(T^{naive}|\pi_p)\geq 1+\inf\{0,k-1\}=\inf\{1,k\}.$$

In fact by this way we see that the bound is optimal as the equality is achieved at the "ordinary representations" (in the sense of Katz–Mazur) which really do exist.

3.2. T_p **of Siegel threefold with hyperspecial level.** Now we construct T_p and U_p for Siegel threefolds. We use the notation of Loeffler–Pilloni–Skinner–Zerbes.

- $G=\mathrm{GSp}_4$ associated to $J=\begin{pmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{pmatrix}$, and the Siegel (Klingen, resp.) parabolic is

$$P_S=\begin{pmatrix}*&*&*&*\\ *&*&*&*\\ &&*&*\\ &&*&*\end{pmatrix}\quad\left(P_{\mathrm{Kl}}=\begin{pmatrix}*&*&*&*\\ &*&*&*\\ &*&*&*\\ &&&*\end{pmatrix},\ \text{resp.}\right)$$

- Choose a tame level K^p, and let X (X_{par}, $X_{\mathrm{Kl}(p)}$, resp.) be some toroidal compactification of integral model over $\mathbb{Z}_p$ of Siegel threefold of level K^pK_p with $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$ ($K_p = K_{par} = \left\{ \begin{pmatrix} * & * & * & */p \\ * & * & * & */p \\ * & * & * & */p \\ p* & p* & p* & * \end{pmatrix} \right\}$, $K_p = \mathrm{Kl}(p) = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} (\mathrm{mod}\, p) \right\}$, resp.).
  - The open Shimura variety of hyperspecial level Y parametrizes (G, λ, ψ), where G is an abelian surface, λ is a ($\mathbb{Z}_p^\times$-multiple of) principal polarization, and ψ is a prime-to-p level structure.
  - The open Shimura variety of paramodular level Y_{par} parametrizes (G, λ, ψ) where G is an abelian surface, λ is a ($\mathbb{Z}_p^\times$-multiple of) degree p^2 polarization, ψ is a prime-to-p level structure.
  - The open Shimura variety of Klingen level $Y_{\mathrm{Kl}(p)}$ parametrizes (G, H, λ, ψ), where G is an abelian surface, λ is a ($\mathbb{Z}_p^\times$-multiple of) principal polarization, $H \subset G[p]$ is a rank p ffgs and ψ is a prime-to-p level structure.
  - There is a natural map $Y_{\mathrm{Kl}(p)} \to Y$ by forgetting H.
  - There is a natural map $Y_{\mathrm{Kl}(p)} \to Y_{par}$ (which is visible from group description) by sending $(G, H, \lambda, \psi) \mapsto (G/H^\perp, \lambda', \psi)$. Here $H^\perp$ (rank p^3) is obtained from H via Weil pairing on $G[p]$, so that there is a degree p isogeny $G/H^\perp \to G/G[p] = G$; then λ' is a degree p^2 isogeny induced from λ and the degree p isogeny.
  - By analyzing local models, one knows that X is smooth and $X_{par}, X_{\mathrm{Kl}(p)}$ are l.c.i.
- T is the diagonal, and

  $$X^*(T) = \{(r_1, r_2; c) \in \mathbb{Z}^3 \mid c \equiv r_1 + r_2 (\mathrm{mod}\, 2)\}$$

  identified via

  $$(r_1, r_2; c) \leftrightarrow \begin{pmatrix} st_1 & & & \\ & st_2 & & \\ & & st_2^{-1} & \\ & & & st_1^{-1} \end{pmatrix} \mapsto t_1^{r_1} t_2^{r_2} s^c$$
- The weight $\lambda(r_1, r_2; c)$ is P_S-dominant iff $r_1 \geq r_2$. Let $\omega(r_1, r_2; c)$ be the corresponding automorphic vector bundle. Since the central twists can be matched easily we usually ignore c.

One would like to construct a cohomological correspondence on X related to the classical Hecke operator $T^{class} = [\mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p)]$. The first natural attempt is to define integral correspondence

$$X\left(\mathrm{GSp}_4(\mathbb{Z}_p) \cap \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix}^{-1}\right)$$

$$\swarrow \qquad\qquad \searrow$$

$$X \qquad\qquad\qquad\qquad X$$

However, $\mathrm{GSp}_4(\mathbb{Z}_p) \cap \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix}^{-1}$ is not a parahoric subgroup (this is a parahoric subgroup only when the diagonal matrix is associated with a miniscule coweight, whereas $\begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix}$ is associated with $t \mapsto \mathrm{diag}(t^2, t, t, 1)$, which is not miniscule), and we don't know how to define a good integral canonical model of Shimura varieties of non-parahoric level. There is a roundabout way though: there is a correspondence diagram

$$\begin{array}{ccc} & X_{\mathrm{Kl}(p)} & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X & & X_{par} \end{array}$$

and one can think of cohomological correspondences going both directions.

- Over $Y_{\mathrm{Kl}(p)}$, there is a universal isogeny $G \to G/H^{\perp}$ (degree p^3), which gives rise to $T_1^{naive} : p_2^*\omega(k_1,k_2)[1/p] \to p_1^*\omega(k_1,k_2)[1/p]$. After scaling with some (preferrably optimal) power of p, this restricts to a morphism of lattices. Composed with fundamental class one gets the direction $T_1 : R\Gamma(X_{par}, \omega(k_1,k_2)) \to R\Gamma(X, \omega(k_1,k_2))$. The corresponding classical Hecke operator is associated with the double coset $[\mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} K_{par}]$.
- Over $Y_{\mathrm{Kl}(p)}$, there is again a universal isogeny, this times $G/H^{\perp} \to G$ (degree p), which gives rise to $T_2^{naive} : p_1^*\omega(k_1,k_2)[1/p] \to p_2^*\omega(k_1,k_2)[1/p]$. After scaling with some (preferrably optimal) power of p, this restricts to a morphism of lattices. Composed with fundamental class one gets the direction $T_2 : R\Gamma(X, \omega(k_1,k_2)) \to R\Gamma(X_{par}, \omega(k_1,k_2))$. The corresponding classical Hecke operator is associated with the double coset $[K_{par} \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p)]$.

Thus, the classical Hecke operator corresponding to $T := T_1 \circ T_2$ is contained in the double coset

$$[\mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} K_{par}] \star [K_{par} \begin{pmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p)]$$

$$= [\mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p)] + (1+p+p^2+p^3)[\mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p)],$$

which is sort of expected. Even though there is an unwanted "junk", it is simple, and it can be "ruled out" as it has higher p-divisibility as we will see below.

Now it is about finding the optimally integral T_1 and T_2. We know all integral models are normal, we can use the same trick and worry only about what happens at the ordinary locus of $Y_{\mathrm{Kl}(p)}$. It has two components depending on the étale rank of $H^{\perp}$. Since $H^{\perp} \subset G[p]$, and G is ordinary, $H^{\perp}$, which is of rank p^3, can have étale rank 1 or 2 (and correspondingly multiplicative rank 2 or 1). Using that $\omega(k_1,k_2) = \mathrm{Sym}^{k_1-k_2}(\omega_G) \otimes \det(\omega_G)^{k_2}$, it is about p-divisibility of $\omega_G \to \omega_{G/H^{\perp}}$ and $\omega_{G/H^{\perp}} \to \omega_{G/H^{\perp}/H'}$ (easy) and about p-divisibility of trace map from the two components of $(Y_{\mathrm{Kl}(p)})_1^{\mathrm{ord}}$ to $(Y_{par})_1$ and Y_1 (subscript 1 means singular fiber). The latter problem is analyzed again by local models. One calculates the following.

- Over the locus where $H^{\perp}$ has etale rank 2, multiplicative rank 1 = H is multiplicative,
  - T_1^{naive} has divisibility of p^{k_2} from isogeny and p^2 from trace map, so in total p^{k_2+2},
  - T_2^{naive} has divisibility of p^{k_2} from isogeny and none from trace map, so in total p^{k_2}.
- Over the locus where $H^{\perp}$ has etale rank 1, multiplicative rank 2 = H is etale,
  - T_1^{naive} has divisibility of $p^{k_1+k_2}$ from isogeny and none from trace map, so in total $p^{k_1+k_2}$,
  - T_2^{naive} has no extra divisibility from isogeny and p from trace map, so in total p.

Thus, we see that the optimally integral T_1 and T_2 are

$$T_1 = \frac{1}{p^{\inf\{k_2+2, k_1+k_2\}}} T_1^{naive},$$

$$T_2 = \frac{1}{p^{\inf\{k_2,1\}}} T_2^{naive}.$$

So, if $k_2 \geq 2$, then the optimal p-divisibility is obtained at the component $* \xrightarrow{\text{et rk 2, mul rk 1}} * \xrightarrow{\text{et}} *$.

Remark 3.1. We realized our promise that we can "rule out" the unwanted piece

$$[\mathrm{GSp}_4(\mathbb{Z}_p) \begin{pmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \mathrm{GSp}_4(\mathbb{Z}_p)],$$

because it corresponds to $H' = G[p]/H^{\perp}$, so the unwanted piece is contained in the components $* \xrightarrow{\text{et rk 1, mul rk 2}} * \xrightarrow{\text{et}} *$ and $* \xrightarrow{\text{et rk 2, mul rk 1}} * \xrightarrow{\text{mul}} *$.

Remark 3.2. Despite the above estimates, the optimal p-divisibility of T^{naive} (namely, the naive cohomological correspondence coming from $X_{\mathrm{Kl}(p)} \times_{X_{par}} X_{\mathrm{Kl}(p)}$) is <u>not</u> $p^{\inf\{k_2,1\}+\inf\{k_2+2,k_1+k_2\}}$. As a composition of correspondences, the universal isogeny in concern is the composed isogeny $\omega_G \to \omega_{G/H^{\perp}} \to \omega_{(G/H^{\perp})/H'}$, and the composed isogeny has some extra constraint, that $G/H^{\perp}/H'$ is <u>principally polarized</u>.

- Firstly $H^{\perp} \subset G[p]$ where G is ordinary, so $H^{\perp}$ is etale locally isomorphic to either $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_p$ or $\mathbb{Z}/p\mathbb{Z} \oplus \mu_p^{\oplus 2}$. And H' is etale locally isomorphic to either μ_p or $\mathbb{Z}/p\mathbb{Z}$.
- So the kernel of $G \to G/H^{\perp}/H'$ is an extension of H' by $H^{\perp}$, and because of this constraint, not all extensions can appear. For example, for the component with $H^{\perp}$ etale rank 2 (=multiplicative rank 1) and H' multiplicative, we are led to consider extensions of μ_p by $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_p$, so etale locally two things are possible:

  $$0 \to (\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_p \to \begin{pmatrix} (\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_p^{\oplus 2} \\ (\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_{p^2} \end{pmatrix} \to \mu_p \to 0$$

  But the second possibility, $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_{p^2}$, is impossible, because $\frac{(\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus 2} \oplus \mu_{p^\infty}^{\oplus 2}}{(\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_{p^2}}$ is not principally polarized (its p^2-torsion is $\frac{(\mathbb{Z}/p^2\mathbb{Z})^{\oplus 2} \oplus \mu_{p^2}^{\oplus 2}}{(\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_{p^2}} = (\mathbb{Z}/p\mathbb{Z})^{\oplus 2} \oplus \mu_{p^2}$ which is not prinicpally polarizable). Thus the <u>composed universal isogeny has p-divisibility of $p^{k_1+k_2}$</u> for this component. If you compare this with what happened at T_1 and T_2, they had p-divisibilites of p^{k_2} and p^{k_2}, respectively, coming from the universal isogenies. This explains the calculation in [LPSZ19, p.13], and in the case of k_2 negative, the optimal p-divisibility is still obtained at the same component (namely, the component $* \xrightarrow{\text{et rk 2, mul rk 1}} * \xrightarrow{\text{et}} *$).

3.3. **U_p of Siegel threefold with Klingen level.** One would again like to construct a cohomological correspondence on $X_{\mathrm{Kl}(p)}$ related to the classical Hecke operator $U^{class} = [\mathrm{Kl}(p) \begin{pmatrix} p^2 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} \mathrm{Kl}(p)]$, and this is again not naively possible by a similar reason. We mimic the process of defining T_p but somewhat worse and more ad hoc. In fact, we can only define U_p as a cohomological correspondence over an open subscheme $X^{\geq 1}$ of multiplicative rank at least 1. The optimal divisibility is checked on the ordinary locus so this will anyways give you the optimal divisibility for U^{class}.

- Let $\mathfrak{X}$ etc. be the p-adic formal completion.

- Let $\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}$ be the moduli space of $H \subset G[p]$ where H is etale locally isomorphic to μ_p.
- Let $\mathfrak{X}^{\geq 1}_{par}$ be the same moduli space but with multiplicative rank of G being at least 1.
- Let $\mathfrak{X}^{\geq 1}_{par,\mathrm{Kl}(p)} \to \mathfrak{X}^{\geq 1}_{par}$ be the moduli space where it further parametrizes $H \subset G[p]$ with H etale locally isomorphic to μ_p.
- Similarly, let $\mathfrak{X}^{m}_{par}$ be the moduli space where the kernel of the degree p^2 polarization $\lambda : G' \to (G')^t$ contains multiplicative group (from the Klingen moduli, this corresponds to $G[p]/H^{\perp}$ being multiplicative), and define $\mathfrak{X}^{m}_{par,\mathrm{Kl}(p)}$ similarly as above.
- Now we mimic the construction of T_p by "base-changing over a part of $\mathfrak{X}_{\mathrm{Kl}(p)}$ where the moduli problem makes sense".
  - Let $\mathfrak{C}^1(p) \subset \mathfrak{X}_{\mathrm{Kl}(p)} \times_{\mathfrak{X}} \mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}$ be the open subscheme (here $\mathfrak{X}_{\mathrm{Kl}(p)} \to \mathfrak{X}$ sends $(G \to G') \mapsto G$) where the universal triple $(G \to G', H)$ satisfies $\ker(G \to G') \cap H = 0$. It admits a map $\mathfrak{C}^1(p) \to \mathfrak{X}^{m}_{par,\mathrm{Kl}(p)}$, $(G \to G', H) \mapsto (G'/(\ker(G \to G')^{\perp}), \mathrm{im}(H \to G \to G'))$. These admit a correspondence diagram

$$\begin{array}{ccc} & \mathfrak{C}^1(p) & \\ {\scriptstyle q_2} \swarrow & & \searrow {\scriptstyle q_1} \\ \mathfrak{X}^{m}_{par,\mathrm{Kl}(p)} & & \mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)} \end{array}$$

  - Let $\mathfrak{C}^2(p) \subset \mathfrak{X}_{\mathrm{Kl}(p)} \times_{\mathfrak{X}_{par}} \mathfrak{X}^{m}_{par,\mathrm{Kl}(p)}$ be the open subscheme where the universal triple $(G' \to G, H' \subset G')$ satisfies that $\ker(G' \to G)$ is not a multiplicative group. It has a map $r_1 : \mathfrak{C}^2(p) \to \mathfrak{X}^{m}_{par,\mathrm{Kl}(p)}$ which is a tautology and $r_2 : \mathfrak{C}^2(p) \to \mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}$ defined by $(G' \to G, H' \subset G') \mapsto (G, \mathrm{im}(H' \to G' \to G))$. These admit a correspondence diagram

$$\begin{array}{ccc} & \mathfrak{C}^2(p) & \\ {\scriptstyle r_2} \swarrow & & \searrow {\scriptstyle r_1} \\ \mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)} & & \mathfrak{X}^{m}_{par,\mathrm{Kl}(p)} \end{array}$$

- Now we have

$$U_1^{naive} : R\Gamma(\mathfrak{X}^{m}_{par,\mathrm{Kl}(p)}, \omega(k_1,k_2)) \to R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2))$$

  using the universal isogeny $G \to G'$ over $\mathfrak{C}^1(p)$ and

$$U_2^{naive} : R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)) \to R\Gamma(\mathfrak{X}^{m}_{par,\mathrm{Kl}(p)}, \omega(k_1,k_2))$$

  using the universal isogeny $G' \to G$ over $\mathfrak{C}^2(p)$. To define optimal integral Hecke operators, we study the p-divisibilities of isogenies and trace maps.
  - For U_1^{naive}, the universal isogeny gives rise to the divisibility of p^{k_2}, and the trace map gives the divisibility of p^2. Thus $U_1 = \frac{1}{p^{k_2+2}} U_1^{naive}$.
  - For U_2^{naive}, the universal isogeny is actually etale, so there is no extra divisibility from it. The trace map has extra divisibility of p, so $U_2 = \frac{1}{p} U_2^{naive}$.

  The divisibility pattern is actually simpler than T because we are only defining the correspondence over a certain part of $\mathfrak{X}_{\mathrm{Kl}(p)}$.
- Define $U = U_1 \circ U_2 \in \mathrm{End}(R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)))$.

4. Integral Hecke operators and (generalized) Hasse invariants

Now we want to know how U_p and T_p interact with the Hasse invariant or the second Hasse invariant. This is about to which extent Hasse invariant satisfies functoriality. One knows that Hasse invariant satisfies "functoriality" along the isogeny $f : G \to G'$ when it satisfies any of the following conditions:

- when f is étale;
- when G and G' are multiplicative;
- when, over each geometric point, $\ker f \subset G[p]$ and there is an orthogonal complement of $\ker f$ in $G[p]$.

4.1. T_p.

Proposition 4.1. *For $k_1 > 2$ and $k_2 > 1$, the Hasse invariant commutes with T_p, namely the following diagrams commute over X_1 and $X_{par,1}$.*

$$\begin{array}{ccc} p_2^*\omega(k_1,k_2) & \xrightarrow{T_1} & p_1^!\omega(k_1,k_2) \\ \downarrow{\scriptstyle p_2^*\mathrm{Ha}} & & \downarrow{\scriptstyle p_1^!\mathrm{Ha}} \\ p_2^*\omega(k_1+p-1,k_2+p-1) & \xrightarrow{T_1} & p_1^!\omega(k_1+p-1,k_2+p-1) \end{array}$$

$$\begin{array}{ccc} p_1^*\omega(k_1,k_2) & \xrightarrow{T_2} & p_2^!\omega(k_1,k_2) \\ \downarrow{\scriptstyle p_1^*\mathrm{Ha}} & & \downarrow{\scriptstyle p_2^!\mathrm{Ha}} \\ p_1^*\omega(k_1+p-1,k_2+p-1) & \xrightarrow{T_2} & p_2^!\omega(k_1+p-1,k_2+p-1) \end{array}$$

Proof. Recall that over the two components of $X_{\mathrm{Kl}}^{\mathrm{ord}}$, the p-divisibilities of T_1 and T_2 are given by different formula. The inequalities $k_1 > 2$ and $k_2 > 1$ ensures that, for T_1, the component of multiplicative H is exactly where the optimal divisibility is achieved, and for T_2, the component of etale H is exactly where the optimal divisibility is achieved. So over mod p fiber, T_1 and T_2 just vanishes at the respective other component.

For T_1, the relevant isogeny is $G \to G/H^\perp$. Since $H^\perp$ has the maximal etale rank, and since the quotient is multiplicative, Hasse invariant is functorial under this isogeny. For T_2, the relevant isogeny is $G/H^\perp \to G$, and this is etale, so Hasse invariant is functorial under this isogeny. □

Since there is a distinguished triangle

$$R\Gamma(X_1,\omega(k_1,k_2)) \xrightarrow{\mathrm{Ha}} R\Gamma(X_1,\omega(k_1+p-1,k_2+p-2)) \to R\Gamma(X_1^{\leq 1},\omega(k_1+p-1,k_2+p-1)) \xrightarrow{+1}$$

one has T-action on $R\Gamma(X_1^{\leq 1},\omega(k_1+p-1,k_2+p-1))$ for $k_1 > 2, k_1 > 1$, as well as the long exact sequence of coherent cohomology groups is T-equivariant. Here superscript means condition on etale rank=multiplicative rank of truncated BT group.

Over $X_1^{\leq 1}$, one has different kind of description of components, so the behaviour of T is not as obvious as above. Still one has similar kind of statement over $X_1^{\leq 1}$ with the generalized Hasse invariant Ha':

Proposition 4.2. *For $k_1 > 2p+2$ and $k_2 > p+1$, Ha' and T commute, namely the following diagrams commute over $X_1^{\leq 1}$ and $X_{par,1}^{\leq 1}$.*

$$\begin{array}{ccc} p_2^*\omega(k_1,k_2)|_{X_{par,1}^{\leq 1}} & \xrightarrow{T_1} & p_1^!\omega(k_1,k_2)|_{X_1^{\leq 1}} \\ \downarrow p_2^*\mathrm{Ha}' & & \downarrow p_1^!\mathrm{Ha}' \\ p_2^*\omega(k_1+p^2-1,k_2+p^2-1)|_{X_{par,1}^{\leq 1}} & \xrightarrow{T_1} & p_1^!\omega(k_1+p^2-1,k_2+p^2-1)|_{X_1^{\leq 1}} \end{array}$$

$$\begin{array}{ccc} p_1^*\omega(k_1,k_2)|_{X_1^{\leq 1}} & \xrightarrow{T_2} & p_2^!\omega(k_1,k_2)|_{X_{par,1}^{\leq 1}} \\ \downarrow p_1^*\mathrm{Ha}' & & \downarrow p_2^!\mathrm{Ha}' \\ p_1^*\omega(k_1+p^2-1,k_2+p^2-1)|_{X_1^{\leq 1}} & \xrightarrow{T_2} & p_2^!\omega(k_1+p^2-1,k_2+p^2-1)|_{X_{par,1}^{\leq 1}} \end{array}$$

Moreover, T_1 commutes with Ha' under $k_2 = p+1$ too.

Proof. One can check the commutativity over the dense open subscheme $Y_{\mathrm{Kl}(p),1}|_{Y_1^{=1}}$. Since the universal BT group has constant multiplicative kernel, $Y_{\mathrm{Kl}(p),1}|_{Y_1^{=1}}$ is actually consisted of three disjoint components, the etale/multiplicative/bi-connected component, where the adjectives describe the kernel of the universal isogeny $G' \to G$ (namely, over geometric points, the order p ffgs is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, μ_p, α_p, respectively). The correspondences T_1, T_2 then split into three parts. One prove similarly that T_1 and T_2 vanish over the multiplicative and bi-connected components under the given conditions, and over the etale component commutativity follows by the functoriality properties we've seen before. □

So again T_1 descends to a cohomological correspondence over the supersingular locus when $k_1 > p^2+2p+1$ and $k_2 \geq p^2+p$.

4.2. **Finiteness of T_p-ordinary cohomology.** Now one has a following crucial

Proposition 4.3. *There is a constant $C \geq 0$, independent of K^p, such that for any $k_1 - k_2 \geq C$ and $k_2 \geq p^2+p$, $T_1 : p_2^*\omega(k_1,k_2)|_{X_{par,1}^{=0}} \to p_1^!\omega(k_1,k_2)|_{X_1^{=0}}$ is zero.*

Proof. We just exhibit a heuristic why it could vanish. The correspondence T_1 uses the universal isogeny $G \to G/H^\perp$. Over $X_1^{=0}$, G has multiplicative rank 0, so H is necessarily killed by F, namely $H \subset \ker F$. Dually, $\ker F \subset H^\perp$, which means that the differential of the universal isogeny has yet another divisibility coming up.

That C does not depend on K^p follows from the fact that varying tame level gives rise to finite étale base change of situations, so if C works for one tame level, it works for all tame levels. □

Now we can show that T acts locally finitely on various cohomology groups.

Proposition 4.4.

(1) *For $k_1 > 2p+2$ and $k_2 \geq p+1$, T acts locally finitely on $H^0(X_1^{=1}, \omega(k_1,k_2)(-D))$.*
(2) *For $k_1 > 2p+3$ and $k_2 > p+1$, T acts locally finitely on $H^i(X_1^{=1}, \omega(k_1,k_2)(-D))$ for all i.*
(3) *There is a universal constant $C \geq 0$ (the one as above) such that for $k_1 - k_2 \geq C$ and $k_2 \geq p+1$,*

$$e(T_p)H^0(X_1^{\leq 1}, \omega(k_1,k_2)(-D)) = e(T_p)H^0(X_1^{=1}, \omega(k_1,k_2)(-D))$$

(4) *If* $k_2 > p+1$*, then*

$$e(T_p)H^{>0}(X_1^{\leq 1}, \omega(k_1,k_2)(-D)) = e(T_p)H^{>0}(X_1^{=1}, \omega(k_1,k_2)(-D)) = 0$$

Proof. For (1), note that as T commutes with Ha',

$$H^0(X_1^{=1}, \omega(k_1,k_2)(-D)) = \varinjlim_n H^0(X_1^{\leq 1}, \omega(k_1+n(p^2-1), k_2+n(p^2-1))(-D))$$

and $X_1^{\leq 1} = V(\mathrm{Ha})$ is projective, so done. Similarly for (2) (bound gets slightly worse because $T\,\mathrm{Ha}' = \mathrm{Ha}'\,T$ holds in higher cohomology under that worse bound). For (3), note that we have a morphism of long exact sequences

$$\begin{array}{ccccccc} 0 \to & H^0(X_1^{\leq 1}, \omega(k_1,k_2)(-D)) & \xrightarrow{\mathrm{Ha}'} & H^0(X_1^{\leq 1}, \omega(k_1+p^2-1, k_2+p^2-1)(-D)) & \to & H^0(X_1^{=0}, \omega(k_1+p^2-1, k_2+p^2-1)(-D)) \\ & \uparrow T_1 & & \uparrow T_1 & & \uparrow T_1 \\ 0 \to & H^0(X_{par,1}^{\leq 1}, \omega(k_1,k_2)(-D)) & \xrightarrow{\mathrm{Ha}'} & H^0(X_{par,1}^{\leq 1}, \omega(k_1+p^2-1, k_2+p^2-1)(-D)) & \to & H^0(X_{par,1}^{=0}, \omega(k_1+p^2-1, k_2+p^2-1)(-D)) \end{array}$$

The rightmost vertical map is zero under the bound, so if you apply $e(T_p)$ to the top row (which you can), then the rightmost term vanish, so we know that multiplying by Ha' induces an isomorphism

$$e(T_p)H^0(X_1^{\leq 1}, \omega(k_1,k_2)(-D)) \xrightarrow{\sim} H^0(X_1^{\leq 1}, \omega(k_1+p^2-1, k_2+p^2-1)(-D))$$

Taking the colimit of $\times(\mathrm{Ha}')^n$, we get the desired result. Similarly for (4). □

Proposition 4.5.

(1) *For* $k_1 > p+3$ *and* $k_2 \geq 2$*,* T *acts locally finitely on* $R\Gamma(X_1^{\geq 1}, \omega(k_1,k_2)(-D))$*.*

(2) *For* $k_1 - k_2 \geq C$ *and* $k_2 \geq 2$*,* $e(T_p)R\Gamma(X_1^{\geq 1}, \omega(k_1,k_2)(-D))$ *is a perfect complex of amplitude [0,1].*

  (a) *If* $k_2 \geq 3$*, then the natural map* $e(T_p)R\Gamma(X_1, \omega(k_1,k_2)(-D)) \to e(T_p)R\Gamma(X_1^{\geq 1}, \omega(k_1,k_2)(-D))$ *is a quasi-isomorphism.*

  (b) *If* $k_2 = 2$*, then* $e(T_p)H^0(X_1, \omega(k_1,k_2)(-D)) \xrightarrow{\sim} e(T_p)H^0(X_1^{\geq 1}, \omega(k_1,k_2)(-D))$ *and* $e(T_p)H^1(X_1, \omega(k_1,k_2)(-D)) \hookrightarrow e(T_p)H^1(X_1^{\geq 1}, \omega(k_1,k_2)(-D))$*.*

Proof. For (1), consider the resolution over $X_1^{\geq 1}$,

$$0 \to \omega(k_1,k_2)(-D) \xrightarrow{\mathrm{Ha}^n} \omega(k_1+n(p-1), k_2+n(p-1))(-D) \to \omega(k_1+n(p-1), k_2+n(p-1))(-D)/(\mathrm{Ha}^n) \to 0.$$

Since the higher direct image of subcanonical extension from toroidal compactification to minimal compactification vanishes, H^i of the second/third term is H^i of its pushforward to minimal compactification. Take the colimit over all n:

$$0 \to \omega(k_1,k_2)(-D) \xrightarrow{\varinjlim_n \mathrm{Ha}^n} \varinjlim_n \omega(k_1+n(p-1), k_2+n(p-1))(-D) \to \varinjlim_n \omega(k_1+n(p-1), k_2+n(p-1))(-D)/(\mathrm{Ha}^n) \to 0.$$

The second term is supported on the etale-rank 2 locus, whereas the third term is supported on the etale-rank 1 locus. As there are generalized Hasse invariants, one sees that any Ekedahl–Oort stratum in minimal compactification is affine. Thus, the colimit sequence is actually an acyclic resolution over $X_1^{\geq 1}$. Thus

$$R\Gamma(X_1^{\geq 1}, \omega(k_1,k_2)(-D)) \cong [H^0(X_1^{=2}, \omega(k_1,k_2)(-D)) \to \varinjlim_n H^0(X_1^{\geq 1}, \omega(k_1+n(p-1), k_2+n(p-1))(-D)/(\mathrm{Ha}^n))]$$

As

$$H^0(X_1^{=2}, \omega(k_1,k_2)(-D)) = \varinjlim_n H^0(X_1, \omega(k_1+n(p-1), k_2+n(p-1))(-D))$$

T acts locally finitely on this term. For the second term, note that $H^0(X_1^{\geq 1}, \omega(k_1 + (p-1), k_2 + (p-1))(-D)/(\mathrm{Ha})) = H^0(X_1^{=1}, \omega(k_1 + (p-1), k_2 + (p-1))(-D))$, on which we already know T acts locally finitely, and we have an exact sequence

$$0 \to H^0(X_1^{\geq 1}, \omega(k_1 + (n-1)(p-1), k_2 + (n-1)(p-1))(-D)/(\mathrm{Ha}^{n-1})) \to H^0(X_1^{\geq 1}, \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/(\mathrm{Ha}^n)) \to H^0(X_1^{\geq 1}, \omega(k_1 + (p-1), k_2 + (p-1))(-D)/(\mathrm{Ha}))$$

so by induction we get the result.

For (2), we see that from the same resolution, we have a morphism between exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(X_1^{\geq 1}, \omega(k_1, k_2)(-D)) & \longrightarrow & H^0(X_1^{=2}, \omega(k_1, k_2)(-D)) & \longrightarrow & \varinjlim_n H^0(X_1^{\geq 1}, \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/\mathrm{Ha}^n) & \longrightarrow & H^1(X_1^{\geq 1}, \omega(k_1, k_2)(-D)) & \longrightarrow 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & H^0(X_1, \omega(k_1, k_2)(-D)) & \longrightarrow & H^0(X_1^{=2}, \omega(k_1, k_2)(-D)) & \longrightarrow & \varinjlim_n H^0(X_1, \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/\mathrm{Ha}^n) & \longrightarrow & H^1(X_1, \omega(k_1, k_2)(-D)) & \longrightarrow 0
\end{array}$$

and also that $\varinjlim_n H^i(X_1, \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/\mathrm{Ha}^n) \cong H^{i+1}(X_1, \omega(k_1, k_2)(-D))$ for $i = 1, 2$.

For (a), we need to prove two things.

- $eH^{>0}(X_1, \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/\mathrm{Ha}^n) = 0$ for all $n \geq 1$.
  - For $n = 1$, this is just $eH^{>0}(X_1^{\leq 1}, \omega(k_1 + p - 1, k_2 + p - 1)(-D)) = 0$, which is proven already.
  - One can use an induction using a s.e.s.

$$0 \to \omega(k_1 + (n-1)(p-1), k_2 + (n-1)(p-1))(-D)/\mathrm{Ha}^{n-1} \xrightarrow{\mathrm{Ha}} \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/\mathrm{Ha}^n \to \omega(k_1 + p - 1, k_2 + p - 1)(-D)/\mathrm{Ha} \to 0$$

- $e(T_p)H^0(X_1, \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/\mathrm{Ha}^n) \to e(T_p)H^0(X_1^{\geq 1}, \omega(k_1 + n(p-1), k_2 + n(p-1))(-D)/\mathrm{Ha}^n)$ is an isomorphism.
  - For $n = 1$, this is just that

$$e(T_p)H^0(X_1^{\leq 1}, \omega(k_1 + p - 1, k_2 + p - 1)(-D)) \xrightarrow{\sim} e(T_p)H^0(X_1^{=1}, \omega(k_1 + p - 1, k_2 + p - 1)(-D)),$$

    which was already proven.
  - One can use the same s.e.s. and do induction. It's possible because we just proved that higher cohomology vanishes.

And (b) works similarly too. □

Remark 4.1. Here we proved something by, apparently, moving things to the minimal compactification. You might think that it's weird to transfer things to minimal compactification because toroidal compactification seems always better than minimal compactification. But one great advantage of minimal compactification is that it is constructed as $\mathrm{Proj} \bigoplus H^0(X, \omega^n)$. This means that there is an apparent extension of ω which is ample. This is not the case for toroidal compactification; namely, (sub)canonical extension is not ample. So the upshot is:

The nonvanishing locus of an automorphic form is affine in minimal compactification only.

4.3. U_p**.** The construction of U_p was much more horrible than T_p, but as we are only working on a "nice part", we have much better statements for commutativity with (generalized) Hasse invariant.

Proposition 4.6.

(1) Over $X_1^{\geq 1}$*,* U *commutes with* Ha.
(2) Over $X_1^{=1}$*,* U *commutes with* Ha'.

Proof is much easier, in fact. Note that we don't have any weight constraints.

5. Relation between U_p and T_p

We want analogous cohomological finiteness for U_p. We achieve it by relating U_p and T_p.

5.1. **Ordinary locus.** We want

Proposition 5.1.

(1) *For $k_1 - k_2 \geq 1$ and $k_2 \geq 2$, U acts locally finitely on $H^0((X^{=2}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D))$.*

(2) *For $k_1 - k_2 \geq 2$ and $k_2 \geq 2$, the natural pullback map*

$$e(T)H^0(X_1^{=2}, \omega(k_1,k_2)(-D)) \to e(U)H^0((X^{=2}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D))$$

is an isomorphism.

(3) *For $k_1 - k_2 \geq 2$ and $k_2 \geq 2$, the natural pullback map*

$$e(T)H^0(\mathfrak{X}^{=2}, \omega(k_1,k_2)(-D)) \to e(U)H^0(\mathfrak{X}^{=2}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)(-D))$$

is an isomorphism.

Proof. Note that under this condition, T_1 and T_2 mod p can be only accounted for half of the correspondence each. Namely what would matter is the correspondence

$$\begin{array}{ccc} & (X^{\mathrm{ord},m}_{\mathrm{Kl}(p)})_1 \times_{(X^{\mathrm{ord},\text{ét}}_{par})_1} (X^{\mathrm{ord}}_{\mathrm{Kl}(p)})_1 & \\ \swarrow & & \searrow \\ X_1^{\mathrm{ord}} & & X_1^{\mathrm{ord}} \end{array}$$

where the correspondence space parametrizes

$$\{(H \subset G[p], H' \subset G'[p]) \mid G, G' \text{ ordinary}, G/H^\perp \cong G'/(H')^\perp, H \text{ multiplicative}, H' \text{ étale}\}$$

Inspired by this, we construct a yet another correspondence

$$\begin{array}{ccc} & \mathfrak{D} & \\ {}^{g_2}\swarrow & & \searrow^{g_1} \\ \mathfrak{X}^{\mathrm{ord}} & & \mathfrak{X}^{\mathrm{ord}} \end{array}$$

where $\mathfrak{D}$ parametrizes $(L \subset G[p^2])$, where G is ordinary and L is a totally isotropic subgroup scheme of $G[p^2]$, which is an extension

$$0 \to L^m \to L \to L^{\text{ét}} \to 0$$

where L^m is etale locally isomorphic to μ_p and $L^{\text{ét}}$ is etale locally isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$. Let $g_1((G,L)) = G$ and $g_2((G,L)) = G/L$. Then the correctly normalized cohomological correspondence for this (using $G \to G/L$) is the same as T modulo p; the two constructions are related as

$$0 \to \underset{\text{etale 2, mul 1}}{H^\perp} \to L \to \underset{\text{etale}}{G'[p]/(H')^\perp} \to 0$$

From this, we can see that g_2 actually factors through $g_2 : \mathfrak{D} \to \mathfrak{X}^{\mathrm{ord}}_{\mathrm{Kl}(p)}$, $(G, L) \mapsto (G/L, G[p]/L)$ ($G[p]/L$ is actually rank p in G/L because $L \cap G[p]$ is rank p^3, "$H^{\perp}$"). So T factors through

$$\begin{array}{ccc} H^0((X^{\mathrm{ord}}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D)) & & \\ \uparrow & \searrow & \\ H^0(X^{\mathrm{ord}}_1, \omega(k_1,k_2)(-D)) & \xrightarrow[T]{} & H^0(X^{\mathrm{ord}}_1, \omega(k_1,k_2)(-D)) \end{array}$$

Complete this into a square and name the new correspondences:

$$\begin{array}{ccc} H^0((X^{\mathrm{ord}}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D)) & \xrightarrow{T'''} & H^0((X^{\mathrm{ord}}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D)) \\ i\uparrow & \overset{T''}{\searrow} & \uparrow i \\ H^0(X^{\mathrm{ord}}_1, \omega(k_1,k_2)(-D)) & \xrightarrow[T=T']{} & H^0(X^{\mathrm{ord}}_1, \omega(k_1,k_2)(-D)) \end{array}$$

We can then note several things:

- $(T''')^n = iT^{n-1}T''$, so actually T''' acts locally finitely on $H^0((X^{\mathrm{ord}}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D))$ (as T acts locally finitely).
- We see that U mod p has a correspondence diagram

$$\left\{ \begin{array}{c} (G \to G', H \subset G, \underline{G'} \to \underline{G}, \underline{H'} \subset \underline{G'}), \text{ where} \\ \ker(G \to G') \cap H = 0, \ker(\underline{G'} \to \underline{G}) \text{ is} \\ \text{multiplicative, and} \\ (G'/(\ker G \to G')^{\perp}, \operatorname{im}(H \to G')) = (\underline{G'}, \underline{H'}) \end{array} \right\}$$

$$(\underline{G}, \operatorname{im}(\underline{H'} \to \underline{G})) \swarrow \qquad \searrow (G \to G')$$

so this is the same as the correspondence for T''', except that there is extra condition on $\ker(G \to G') \cap H = 0$. So $T''' = U + F$ where F accounts for nontrivial intersection. In this way T''' and U are related, and by studying moduli problem one can show that $U \circ T''' = U \circ U$. At this point (1) is deduced. Also, the natural pullback

$$e(T)H^0(X^{=2}_1, \omega(k_1,k_2)(-D)) \to e(U)H^0((X^{=2}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D))$$

is surjective, because given G in the target, $e(U)ie(T)T''U^{-1}G = G$. By Nakayama, the surjectivity of (3) is deduced.
- Now consider $\mathrm{Tr} \circ U^n \circ i \operatorname{End}(H^0(\mathfrak{X}^{\mathrm{ord}}, \omega(k_1,k_2)(-D)))$, which is associated with the correspondence

$$\left\{ \begin{array}{c} (G, H, L_n), \text{ where } G \text{ is ordinary, } H \subset G[p] \text{ is multiplicative} \\ \text{order } p \text{ and } L_n \subset G[p^{2n}] \text{ totally isotropic, sitting inside} \\ 0 \to L^m_n \to L_n \to L^{\text{ét}}_n \to 0 \text{ where } L^{\text{ét}}_n \text{ is etale locally} \\ \text{isomorphic to } \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^{2n}\mathbb{Z} \text{ and } L^m_n \text{ is etale locally} \\ \text{isomorphic to } \mu_{p^n}, \text{ such that } L_n \cap H = 0 \end{array} \right\}$$

By moduli interpretation, we have $\mathrm{Tr} \circ U^n \circ i = pT^{n,\text{ét}}$ where $T^{n,\text{ét}}$ is the part of T^n where $G \to G'$ has maximally etale kernel. There is p because of extra datum of H where there are exactly p choices. Then as before $T^{n,\text{ét}}$ has the least divisibility under the given condition, and in fact $T^n \equiv T^{n,\text{ét}} (\operatorname{mod} p^{k_1-k_2})$. Thus, $\operatorname{Tr} \circ U^n \circ i \equiv pT^n (\operatorname{mod} p^{k_1-k_2})$.

- Now injectivity of (2) follows from injectivity of (3) (the modules are complete flat $\mathbb{Z}_p$-modules). The injectivity of (3) uses that $\operatorname{Tr} \circ U^n \circ i = pT^n (\operatorname{mod} p^{k_1-k_2})$ which says that if it's in the kernel then it's a p-multiple (here we use $k_1 - k_2 \geq 2$). By induction we get the vanishing.

□

5.2. **Rank 1 locus.** Note that over rank 1 locus, $(X^{=1}_{\mathrm{Kl}(p)})_1 \to X^{=1}_1$ is an isomorphism. By definition, over $X^{=1}_1$, $U = T^{\text{ét}}_1 \circ T^{\circ}_2$. So by the same reasoning we have

Proposition 5.2. *$U = T$ if $k_1 > 2p+2$ and $k_2 \geq p+1$*

5.3. **Finiteness of U_p-ordinary cohomology.**

Theorem 5.1.

(1) If $k_2 \geq 2$ and $k_1 - k_2 > p+1$, U acts locally finitely on $R\Gamma((X^{\geq 1}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D))$. Also, the natural pullback map

$$e(T)R\Gamma(X^{\geq 1}_1, \omega(k_1,k_2)(-D)) \to e(U)R\Gamma((X^{\geq 1}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D))$$

is a quasi-isomorphism.

(2) If $k_2 \geq 2$ and $k_1 - k_2 \geq C$ (as before), $e(U)R\Gamma(X^{\geq 1}_1, \omega(k_1,k_2)(-D))$ is a perfect complex of $\mathbb{F}_p$-vector spaces of amplitude $[0,1]$.

We can combine this with $e(T)R\Gamma(X_1, \omega(k_1,k_2)(-D)) \xrightarrow{\sim} e(T)R\Gamma(X^{\geq 1}_1, \omega(k_1,k_2)(-D))$ for $k_2 \geq 3$, $k_1 - k_2 \geq C$.

Proof. One uses similar exact sequence to show that

$$R\Gamma((X^{\geq 1}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D)) \cong [H^0((X^{=2}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D)) \to \varinjlim_n H^0((X^{\geq 1}_{\mathrm{Kl}(p)})_1, \omega(k_1+n(p-1), k_2+n(p-1))(-D)/\operatorname{Ha}^n)]$$

and argue as before. □

One can use a similar acyclic resolution but with p^n-congruences:

$$0 \to \omega(k_1,k_2)(-D) \to \varinjlim_{l, \times \operatorname{Ha}^{p^{n-1}}} \omega(k_1+lp^{n-1}(p-1), k_2+lp^{n-1}(p-1))(-D) \to \varinjlim_l \omega(k_1+lp^{n-1}(p-1), k_2+lp^{n-1}(p-1))(-D)/\operatorname{Ha}^{lp^{n-1}} \to 0$$

and this can be seen as a s.e.s on $X^{\geq 1}_n$, as $\operatorname{Ha}^{p^{n-1}}$ lifts to X_n. By the same reason this is an acyclic resolution of $\mathbb{Z}/p^n\mathbb{Z}$-sheaves, and one can argue exactly in the same way to gain the following.

Theorem 5.2.

(1) For $k_2 \geq 2$ and $k_1 - k_2 > p+1$, T and U acts locally finitely on $R\Gamma(\mathfrak{X}^{\geq 1}, \omega(k_1,k_2)(-D))$ and $R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)(-D))$, respectively. These complexes have cohomologies concentrated in degree $[0,1]$.

(2) The natural pullback map

$$e(T)R\Gamma(\mathfrak{X}^{\geq 1}, \omega(k_1,k_2)(-D)) \to e(U)R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)(-D))$$

is a quasi-isomorphism.

(3) If $k_1 - k_2 \geq C$, then $e(U)R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)(-D))$ is a perfect complex of $\mathbb{Z}_p$-modules with amplitude $[0,1]$. If furthermore $k_2 \geq 3$,

$$e(T)R\Gamma(\mathfrak{X}, \omega(k_1,k_2)(-D)) \to e(U)R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)(-D))$$

is a quasi-isomorphism. If $k_2 = 2$, then its bijective on H^0 and injective on H^1.

6. Interpolation

Now one can use the Igusa tower over $\mathfrak{X}^{\geq 1}$.

- Let $\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^n)}$ be the moduli of $H_m \subset G[p^m]$ where H_m is etale locally isomorphic to μ_{p^m}. Taking the inverse limit we have a formal scheme $\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^\infty)}$ too.
- Let $\mathfrak{IG}(p^m) = \mathrm{Isom}(\mu_{p^m}, H_m)$ over $\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^m)}$. This is a $(\mathbb{Z}/p^m\mathbb{Z})^\times$-torsor. Also $\mathfrak{IG}(p^\infty) \to \mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^\infty)}$ is a $\mathbb{Z}_p^\times$-torsor.
- Let $\pi : \mathfrak{IG}(p^\infty) \to \mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}$ and let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Let $\kappa : \mathbb{Z}_p^\times \to \Lambda$ be the universal character.
- Let $\mathfrak{F}^\kappa = (\pi_* \mathcal{O}_{\mathfrak{IG}(p^\infty)} \widehat{\otimes}_{\mathbb{Z}_p} \Lambda)^{\mathbb{Z}_p^\times}$, where $\mathbb{Z}_p^\times$ acts diagonally via the torsor structure and κ.
- For any $\chi : \mathbb{Z}_p^\times R^\times$, $\mathfrak{F}^\chi := \mathfrak{F}^\kappa \otimes_{\Lambda,\chi} R$.
- We have truncated and modulo p-power version. Namely $\mathfrak{F}_{m,n}$ means it's modulo p^n and Igusa tower is truncated at the m-th level.
- $\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^n)} \to \mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}$ is affine,
- Quite obviously, over $(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^n)})_n$, $\mathfrak{F}_{n,n} \xrightarrow{\sim} \omega_{H_n}$, where H_n is the universal order p^n subgroupscheme.
- Let $K\omega(k_1, 0) = \ker(\omega(k_1,0) \to \omega_{H_n}^{k_1})$, and $K\omega(k_1,k_2) = K\omega(k_1 - k_2, 0) \otimes \omega^{k_2}$. The map is "projection to the highest weight vector."
- We have

  $$0 \to K\omega(k_1,k_2) \to \omega(k_1,k_2) \to \mathfrak{F}^{k_1-k_2}_{n,n} \otimes \omega^{k_2} \to 0$$

  On the level of sheaves, one proves easily that U acts with extra p divisibility on non-highest weight vector, namely

  $$U \in p\,\mathrm{End}(R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^n)})_n, K\omega(k_1,k_2)))$$

Theorem 6.1. *Let $k_2 \geq 2$, and $M_{\kappa,k_2} \in D(\mathrm{Mod}(\Lambda))$ be defined as*

$$M_{\kappa,k_2} = R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \mathfrak{F}^\kappa \otimes \omega^{k_2}(-D))$$

(1) U acts locally finitely on M_{κ,k_2}.
(2) $e(U)M_{\kappa,k_2}$ is quasi-isomorphic to a perfect complex of Λ-modules with amplitude $[0,1]$.
(3) There is a quasi-isomorphism

$$e(U)R\Gamma(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)}, \omega(k_1,k_2)) \xrightarrow{\sim} e(U)M_{\kappa,k_2} \otimes^{\mathbb{L}}_{\Lambda,k_1-k_2} \mathbb{Z}_p$$

Here in (3), there is an improvement of previously proven statement, that we have local finiteness of U on $\omega(k_1,k_2)$ for $k_1 \geq k_2$.

Proof. Proof is really similar. For $m \geq n$, one has an acyclic resolution of $\mathfrak{F}^\kappa_{m,n} \otimes \omega^{k_2}(-D)$ over $(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_n$,

$$0 \to \mathfrak{F}^\kappa_{m,n} \otimes \omega^{k_2}(-D) \to \varinjlim_{l, \times \mathrm{Ha}^{p^{n-1}}} \mathfrak{F}^\kappa_{m,n} \otimes \omega^{k_2 + lp^{n-1}(p-1)}(-D) \to \varinjlim_l \mathfrak{F}^\kappa_{m,n} \otimes \omega^{k_2 + lp^{n-1}(p-1)}(-D)/\mathrm{Ha}^{lp^{n-1}} \to 0$$

Also using $0 \to K\omega(k_1,k_2) \to \omega(k_1,k_2) \to \mathfrak{F}_{n,n}^{k_1-k_2} \otimes \omega^{k_2} \to 0$, one easily shows that

- U acts locally finitely on $R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D))$,
- there is a natural quasi-isomorphism

$$e(U)R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \omega(k_1,k_2)(-D)) \xrightarrow{\sim} e(U)R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D))$$

Now we will be done if we prove that the natural map is a quasi-isomorphism,

$$e(U)R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) \xrightarrow{\sim} e(U)R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{m,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)),$$

namely one can climb up the Igusa tower without changing cohomology. This will give the local finiteness of M_{κ,k_2} by Nakayama applied twice (mod p and specific weight). Now specializing at any $k_1 \geq k_2$, U acts locally finitely on M_{k_1,k_2} mod p, the cohomology of $\mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)$, and as $K\omega$ has divisibility by p, this means U acts locally finitely on $\omega(k_1,k_2)(-D)$ mod p. This means U acts locally finitely on $\omega(k_1,k_2)(-D)$ without mod p.

To climb up the Igusa tower, what we want to show is that “Igusa tower contains the U^m-correspondence”. What’s the correspondence for U^m?

$$\left\{\begin{array}{c}(G,H_1,G_m) \text{ where } H_1 \subset G[p] \text{ is etale-locally } \mu_p \\ \text{and } G \to G_m \text{ is an isogeny where} \\ L_m = \ker(G \to G_m) \text{ has } L_m \cap H_1 = 0 \text{ and } L_m \text{ is} \\ \text{an extension of etale group scheme locally} \\ \text{isomorphic to } \mathbb{Z}/p^{2m}\mathbb{Z} \text{ by truncated BT group of} \\ \text{level } m\text{, height 2 dimension 1}\end{array}\right\}$$

$$(G_m, \mathrm{im}(H_1)) \xleftarrow{z_2} \{\cdots\} \xrightarrow{z_1} (G, H_1)$$

But z_2 actually lifts to a map z_2 : whatever $\to (\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p^m)})_1$, $(G,H_1,G_m) \to (G_m, \mathrm{im}(G[p^m] \to G_m))$ (something like $G[p^m]/H^{\perp}$, which is multiplicative). So U^m on $(\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1$ factors through

$$\begin{array}{ccc} R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{m,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) & \searrow & \\ \uparrow & & \\ R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) & \xrightarrow[U^m]{} & R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) \end{array}$$

which can be completed into a square of commuting U’s in the Igusa tower,

$$\begin{array}{ccc} R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{m,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) & \xrightarrow{U^m} & R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{m,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) \\ \uparrow & \searrow & \uparrow \\ R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) & \xrightarrow[U^m]{} & R\Gamma((\mathfrak{X}^{\geq 1}_{\mathrm{Kl}(p)})_1, \mathfrak{F}_{1,1}^{k_1-k_2} \otimes \omega^{k_2}(-D)) \end{array}$$

Thus, local finiteness of U on $\mathfrak{F}_{m,1}$ as well as isomorphism of ordinary part along Igusa tower follows. $\square$