Hida theory talk. ①

[Today we (almost) (probably) never use algebraic geometry. Just topology.]

Why does H work?

— Main idea is that the U_p-operator, the Hecke operator corresponding to $\begin{pmatrix}1&0\\0&p\end{pmatrix}=t$ ($\begin{pmatrix}p&0\\0&1\end{pmatrix}$ depending on how you write).

is constructed as a correspondence.

$$\begin{array}{ccc} & Y(\Gamma_0(p^n)\cap\Gamma^0(p)) \xrightarrow[\sim]{t(-)t^{-1}} Y(\Gamma_0(p^{n+1})) & \\ \swarrow & & \searrow \\ Y(\Gamma_0(p^n)) & & Y(\Gamma_0(p^n)) \end{array}$$

(Indeed, $\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&p^{-1}\end{pmatrix}$
$=\begin{pmatrix}a&p^{-1}b\\pc&d\end{pmatrix}$.)

So for ordinary part, where U_p operator is invertible,

→ this part of tower with going to deeper level structure "can be inverted," no effect.

"U_p operator is compact" (over convergence) "U_p increases rad. of conv."

- ~~There is still a bit more deep~~

You can still go deeper, namely to Γ_1-level instead of Γ_0-level.

This corresponds to "diamond operators",

e.g. $\Gamma_1(p^n) \trianglelefteq \Gamma_0(p^n)$, and it roughly corresponds to
(really $\Gamma_1(p) \cap \Gamma_0(p^n)$)

$\mathbb{Z}/p^n\mathbb{Z}$, so

$n \to \infty$ you get $\mathbb{Z}_p$, which is the source of Γ, Λ.
$\shortparallel$
Γ

You have action of Γ on the whole tower of modular curves, and this descends well to cohomology because

modular curves are curves, and H^0 and H^2 are boring so boring that they are not ordinary, so taking ordinary part nothing is lost.
"coh. is concentrated in middle degree"

- These two take care "varying the level" part.

The weight part almost automatically follows because you can find a modular form of low weight, higher level, congruent to a given mod form.
"(level & weight can be exchanged)"

§1. Varying the level

To simplify the exposition, we use singular homology of modular curves for this section. This has an advantage b/c $H_1 = \pi_1^{ab}$, so $H_1(Y(\Gamma), \mathbb{Z}) = \Gamma^{ab}$.

Now we consider

$$M^{ord} := \varprojlim H_1(Y_1(p^r), \mathbb{Z}_p)^{ord}$$

↑ will explain.

(adding a tame level doesn't change the argument. In fact $N=1$ is probably problematic b/c of stackiness issue)

This is about "varying level" only (weight 2)

Let $\Gamma = 1+p\mathbb{Z}_p$, $\Lambda = \mathbb{Z}_p[[\Gamma]]$.

<u>Thm</u> (Hida) M^{ord} is a free Λ-module of finite rank, actually

$$\mathrm{rank}_\Lambda M^{ord} = \dim_{\mathbb{F}_p} (H_1(X_1(p), \mathbb{F}_p)^{ord}). \quad (= \mathrm{rank}_{\mathbb{Z}_p} \cdots)$$

Also, there is an appropriate control theorem, namely

$$M^{ord}/\text{sth} \simeq H_1(X_1(p^r), \mathbb{Z}_p)^{ord}.$$

More precisely, let $\Gamma_r = 1+p^r\mathbb{Z}_p \leq \Gamma$, and then

$(\Gamma/\Gamma_r \simeq \mathbb{Z}/p^{r-1}\mathbb{Z})$

$\Lambda := \varprojlim \mathbb{Z}_p[\Gamma/\Gamma_r]$ let

Then diamond operator via Γ/Γ_r acts on $H_1(Y_1(p^r), \mathbb{Z})$.

($\rightsquigarrow$ Γ $\twoheadrightarrow \Gamma/\Gamma_r$ acts)

$$0 \to \mathfrak{a}_r \to \mathbb{Z}[\Gamma_r] \to \mathbb{Z} \to 0$$

augmentation ideal

The ultimate finite level statement we need is, for $r \geq s > 0$ (note!)

Thm (Hida) $H_1(Y_1(p^r), \mathbb{Z}_p)^{\text{ord}}_{\Gamma_s\text{-coinv}} \xrightarrow{\sim} H_1(Y_1(p^s), \mathbb{Z}_p)^{\text{ord}}$.

$= /\mathfrak{a}_s$

The rest will follow from "the niceness of ring Λ".

As we've alluded before,

Γ-coinvariant becomes necessary for the diamond act'n, and in particular you expect to deepen the Γ_0-level w/o changing anything. Indeed, we will prove Thm by splitting into 2 iso's:

$$H_1(Y_1(p^r), \mathbb{Z}_p)^{ord}_{\Gamma_s} \underset{①}{\xrightarrow{\sim}} H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^r)), \mathbb{Z}_p)^{ord}$$

$$\underset{②}{\xrightarrow{\sim}} H_1(Y_1(p^s), \mathbb{Z}_p)^{ord}$$

Let's prove ② first, as on the level of H_1, a Hecke operator by t works as

precisely — for $[G:H] < \infty$, G/H rep's $g_1, \dots, g_n$, $y \mapsto \prod_{i=1}^{n} h_i$ where $y g_i = g_i h_i$ defines $G^{ab} \to H^{ab}$. (indep of choice of representatives)

$$\Gamma^{ab} \to (\Gamma \cap t^{-1}\Gamma t)^{ab} \underset{t(-)t^{-1}}{\xrightarrow{\sim}} (t\Gamma t^{-1} \cap \Gamma)^{ab} \to \Gamma^{ab}$$

~~transfer~~ = trace ~~corestriction~~ (dual of ~~~~ cores)

natural inclusion ~~restriction~~ (dual of res)

In this way one can reduce Γ_0-part one-by-one

$$H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^r)), \mathbb{Z}_p)^{ord}$$
$$\downarrow \wr$$
$$H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^{r-1})), \mathbb{Z}_p)^{ord}$$
$$\downarrow \wr$$
$$\vdots$$
$$\downarrow \wr$$
$$H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^{s+1})), \mathbb{Z}_p)^{ord}$$
$$\downarrow \wr$$
$$H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^s)), \mathbb{Z}_p)^{ord}$$
$$(Y_1(p^s))$$

ordinary part: since $U \curvearrowright H_1(\text{—}, \mathbb{Z}_p)$ is a finite $\mathbb{Z}_p$-module,

$\mathrm{Im}(\mathbb{Z}_p[U] \to \mathrm{End}_{\mathbb{Z}_p}(H_1(\text{—}, \mathbb{Z}_p)))$ is a finite $\mathbb{Z}_p$-alg $= \prod$ local rings

so among factors there are factors where U acts invertibly "Im^{ord}".

$H_1(\text{—}, \mathbb{Z}_p)^{ord} = H_1(\text{—}, \mathbb{Z}_p) \otimes_{\mathrm{Im}} \mathrm{Im}^{ord}$

exact.

~~More precisely, the~~

Now, what happens if $\Gamma = \Gamma_1(p^s) \cap \Gamma_0(p^r)$, $t = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$?

$$t \begin{pmatrix} a & b \\ c & d \end{pmatrix} t^{-1} = \begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix}, \quad \text{so}$$

$t\Gamma t^{-1} \cap \Gamma = \Gamma_1(p^s) \cap \Gamma_0(p^{r+1})$ $\qquad \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \bmod p.$

$\Gamma \cap t^{-1}\Gamma t = \Gamma_1(p^s) \cap \Gamma_0(p^r) \cap \Gamma^0(p)$

So the U-operator is the composition

$$(\Gamma_1(p^s) \cap \Gamma_0(p^r))^{ab} \xrightarrow{\text{transfer}} (\Gamma_1(p^s) \cap \Gamma_0(p^r) \cap \Gamma^0(p))^{ab} \underset{t(-)t^{-1}}{\xrightarrow{\sim}} (\Gamma_1(p^s) \cap \Gamma_0(p^{r+1}))^{ab} \xrightarrow{\text{natural}} (\Gamma_1(p^s) \cap \Gamma_0(p^r))^{ab}$$

$(*)$

this is made invertible (bijective), so the natural map

$$H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^{r+1})), \mathbb{Z}_p)^{\text{ord}} \to H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^r)), \mathbb{Z}_p)^{\text{ord}} \text{ is surjective.}$$

Injectivity follows from that $(*)$ is "functorial", in that

$$\begin{array}{lcccc}
(s,r) & (\Gamma_1(p^s) \cap \Gamma_0(p^r))^{ab} \xrightarrow{\text{transfer}} & (\Gamma_1(p^s) \cap \Gamma_0(p^r) \cap \Gamma^0(p))^{ab} \underset{t(-)t^{-1}}{\xrightarrow{\sim}} & (\Gamma_1(p^s) \cap \Gamma_0(p^{r+1}))^{ab} \xrightarrow{\text{natural}} & (\Gamma_1(p^s) \cap \Gamma_0(p^r))^{ab} \\
& \uparrow & \uparrow & \uparrow & \uparrow \\
(s,r+1) & (\Gamma_1(p^s) \cap \Gamma_0(p^{r+1}))^{ab} \xrightarrow{\text{transfer}} & (\Gamma_1(p^s) \cap \Gamma_0(p^{r+1}) \cap \Gamma^0(p))^{ab} \underset{t(-)t^{-1}}{\xrightarrow{\sim}} & (\Gamma_1(p^s) \cap \Gamma_0(p^{r+2}))^{ab} \xrightarrow{\text{natural}} & (\Gamma_1(p^s) \cap \Gamma_0(p^{r+1}))^{ab}
\end{array}$$

where the vertical maps are all natural maps, is commutative. The only nontrivial part is that transfer commutes with natural maps, but for any choice of (s,r), $[\Gamma_1(p^s) \cap \Gamma_0(p^r) : \Gamma_1(p^s) \cap \Gamma_0(p^r) \cap \Gamma^0(p)] = p$ with coset representatives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & p-1 \\ 0 & 1 \end{pmatrix}$, so "the map does not depend on (r,s)" as well. So $\underset{\text{natural maps}}{\textcircled{\uparrow}} \to \to \; = \; \to \to \uparrow = U$ is injective.

Now it remains to prove ①. Let us define how diamond operator acts:

Note that we have exact sequence

(More generally, $1 \to \Gamma_1(p^r) \to \Gamma_1(p^s) \cap \Gamma_0(p^r) \to \Gamma_s/\Gamma_r \to 1$)

$$1 \to \Gamma_1(p^r) \to \Gamma_1(p^0) \cap \Gamma_0(p^r) \to \Gamma/\Gamma_r \to 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$$

So conjugation action gives an action of Γ/Γ_r on $\Gamma_1(p^r)^{ab}$. Obviously the tower $\to \Gamma_1(p^n)^{ab} \to \cdots \to \Gamma_1(p)^{ab}$ is Γ-equivariant.

Now the key observation is that b/c of this,

$\hookrightarrow Y_1(p^r) \to Y(\Gamma_1(p^s) \cap \Gamma_0(p^r))$ is a Γ_s/Γ_r-torsor, so at the level of singular chains / derived category we have

$$C_*(Y_1(p^r), \mathbb{Z}_p)_{\Gamma_s/\Gamma_r} \overset{\sim}{\to} C_*(Y(\Gamma_1(p^s) \cap \Gamma_0(p^r)), \mathbb{Z}_p).$$ So by

~~universal coefficient theorem~~ Künneth we have

$$1 \to H_1(Y_1(p^r), \mathbb{Z}_p)_{\Gamma_s/\Gamma_r} \to H_1(Y(\Gamma_1(p^s) \cap \Gamma_0(p^r)), \mathbb{Z}_p)$$

[same as Γ_s-coinv b/c Γ-act'n factors thru Γ/Γ_r]

$$\to \mathrm{Tor}_1^{\mathbb{Z}_p[\Gamma_s/\Gamma_r]}(H_0(Y_1(p^r), \mathbb{Z}_p), \mathbb{Z}_p) \to 0.$$

We have the ugly cokernel. But what if we take ordinary part?

<u>H_0 is basically about connected components, so the effect of U is to just multiply by p.</u> So taking ordinary part will kill the cokernel. □

Now how does this translate into the first thm?

- $M^{ord}_{\Gamma_r} \to H_1(X_1(p^r), \mathbb{Z}_p)^{ord}$, the natural map, is iso.

  * surjective because transition maps are surjective
  * injective because, if we choose $\Lambda \simeq \mathbb{Z}_p[[T]]$
    $\gamma_0 \mapsto 1+T$,
    (γ_0 top. gen of Γ)

  then

  $\alpha_s = (\gamma^{p^{s-1}} - 1)$

  and for $n \geq r > 0$, $\dfrac{\gamma^{p^{n-1}} - 1}{\gamma^{p^{r-1}} - 1} \in \mathfrak{m}^{n-r}$ where $\mathfrak{m} = (\alpha_1, p) \subset \Lambda$ max ideal.

So if there is a sequence

$(\ldots, m_n, \ldots, m_{r+1}, 0, 0, \ldots 0) = s_0$

then one can modify this, as $m_{r+1} = (\gamma^{p^{r-1}} - 1) e_{r+1}$,

by lifting e_{r+1} to $\tilde{e}_{r+1} \in M^{ord}$,

$(\ldots, m_n', \ldots, m_{r+2}', 0, 0, \ldots 0) = s_0 - (\gamma^{p^{r-1}} - 1)\tilde{e}_{r+1}$

we can do this inductively to kill one by one, and get

$$s = \sum_{i=r}^{\infty} (\gamma^{p^{i-1}} - 1)\tilde{e}_{i+1} = (\gamma^{p^{r-1}}-1) \sum_{i=r}^{\infty} \frac{\gamma^{p^{i-1}} - 1}{\gamma^{p^{r-1}} - 1} \tilde{e}_{i+1}$$

(convergent)

so it's injective.

- M^{ord} is free Λ mod of finite rank.

This ~~roughly~~ means that we can find coherent choice of bases for the whole tower

$\cdots \to H_1(Y_1(p^r), \mathbb{Z}_p)^{ord} \to \cdots$

so that each $H_1(Y_1(p^r), \mathbb{Z}_p)^{ord}$ is free $\mathbb{Z}_p[\Gamma/\Gamma_r]$-mod with that basis. But recognizing free mod. / $\mathbb{Z}_p[\Gamma/\Gamma_r]$ is hard. But it's not for Λ, b/c it's regular local of dim 2. In particular, free = reflexive,

ETS $M^{ord} \cong Hom_\Lambda(Hom_\Lambda(M^{ord}, \Lambda), \Lambda)$

Claim

This will follow when we develop the exact same thing for H^1 instead of H_1.

Namely, $H^1(Y_1(p^r), \mathbb{Z}_p) = Hom_{\mathbb{Z}_p}(H_1(X_1(p^r), \mathbb{Z}_p), \mathbb{Z}_p)$

$= Hom_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H_1(Y_1(p^r), \mathbb{Z}_p), \mathbb{Z}_p[\Gamma/\Gamma_r])$

so we can form an inverse system

$\cdots \to H^1(X_1(p^r), \mathbb{Z}_p)^{ord} \to H^1(X_1(p^{r-1}), \mathbb{Z}_p)^{ord} \to \cdots$

where transition map is (corestriction)

$$H^1(Y_1(p^r),\mathbb{Z}_p)^{ord} = \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H_1(Y_1(p^r),\mathbb{Z}_p)^{ord},\mathbb{Z}_p[\Gamma/\Gamma_r])$$

$$\to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H_1(Y_1(p^r),\mathbb{Z}_p)^{ord},\mathbb{Z}_p[\Gamma/\Gamma_r])/\omega_s$$

$$\to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H_1(Y_1(p^r),\mathbb{Z}_p)^{ord},\mathbb{Z}_p[\Gamma/\Gamma_r]/\omega_s)$$

(coord is exact)

$$= \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_s]}(H_1(Y_1(p^r),\mathbb{Z}_p)^{ord}/\omega_s,\mathbb{Z}_p[\Gamma/\Gamma_s])$$

{ we did it before

$$= \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_s]}(H_1(Y_1(p^s),\mathbb{Z}_p)^{ord},\mathbb{Z}_p[\Gamma/\Gamma_s])$$

If we prove that

$$H^1(Y_1(p^r),\mathbb{Z}_p)^{ord}_{\Gamma_s} \xrightarrow{\sim} H^1(Y_1(p^s),\mathbb{Z}_p)^{ord}, \text{ then}$$

← can be done in the same topological argument

$N^{ord} := \varprojlim_r H^1(X_1(p^r),\mathbb{Z}_p)^{ord}$ will by the same proof have

$N^{ord}_{\Gamma_s} \cong H^1(X_1(p^r),\mathbb{Z}_p)^{ord}$. This is Λ-dual of M;

$$N^{ord} = \varprojlim_r H^1(X_1(p^r),\mathbb{Z}_p)^{ord} = \varprojlim_r \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H_1(X_1(p^r),\mathbb{Z}_p)^{ord},\mathbb{Z}_p[\Gamma/\Gamma_r])$$

$$= \varprojlim_r \mathrm{Hom}_{\Lambda}(M^{ord},\mathbb{Z}_p[\Gamma/\Gamma_r])$$

$$= \mathrm{Hom}_\Lambda(M^{ord},\Lambda)$$

So

$$\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(M^{ord},\Lambda),\Lambda) \simeq \varprojlim_r \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(M^{ord},\Lambda),\mathbb{Z}_p[\Gamma/\Gamma_r])$$

$$\simeq \varprojlim_r \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(N^{ord}_{\Gamma_r},\mathbb{Z}_p[\Gamma/\Gamma_r])$$

$$\simeq \varprojlim_r \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H^1(X_1(p^r),\mathbb{Z}_p)^{ord},\mathbb{Z}_p[\Gamma/\Gamma_r])$$

$$\simeq \varprojlim_r \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(\mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H_1(X_1(p^r),\mathbb{Z}_p)^{ord},\mathbb{Z}_p[\Gamma/\Gamma_r]),\mathbb{Z}_p[\Gamma/\Gamma_r])$$

$$\simeq \varprojlim_r H_1(X_1(p^r),\mathbb{Z}_p)^{ord} \simeq M^{ord}$$

(Here we used here that $H_1(X_1(p^r),\mathbb{Z}_p)^{ord}$ is $\mathbb{Z}_p[\Gamma/\Gamma_r]$-reflexive.

(B) This is easy, b/c $H_1(X_1(p^r),\mathbb{Z}_p)^{ord} = H$ is p-torsion free, so $\mathbb{Z}_p$-free, so

$$H \simeq \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Hom}_{\mathbb{Z}_p}(H,\mathbb{Z}_p),\mathbb{Z}_p)$$

$$\simeq \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(H,\mathbb{Z}_p[\Gamma/\Gamma_r]),\mathbb{Z}_p)$$

$$\simeq \mathrm{Hom}_{\mathbb{Z}_p[\Gamma/\Gamma_r]}(\text{———},\mathbb{Z}_p[\Gamma/\Gamma_r]).$$

)

□

§2. Varying the weight.

We have a similar "control theorem" for other weights, from Mod, and they are just formal algebraic consequences of finite level statements as before, so for clarity we will only talk about finite level (crucial) statements. The idea is the same, you can change the weight in exchange of level. To talk about modular forms of weight other than 2, we need to take twisted coefficients instead of $H^1(-, \mathbb{Z}_p)$.

Here it's easier to work with cohomology, b/c (open) modular curves have contractible universal cover $\mathbb{H}$, so sing coh = group coh of π_1.

$k = \mathbb{Z}, \mathbb{Z}_p, \mathbb{Z}/n\mathbb{Z}, \ldots$

(i) <u>Def</u> Let $\mathrm{Sym}^n k^2$ be a module where $M_2(\mathbb{Z})$ acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} e_1^i \otimes e_2^j = (ae_1 + ce_2)^i \otimes (be_1 + de_2)^j$$

(or $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x^n, \ldots, y^n)^{i+j=n} = ((ax+by)^n, (ax+by)^{n-1}(cx+dy), \ldots, (cx+dy)^n)$)

Let $k(j)$ be such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = d^j x$.

Although both are irr. rep.s when $k = \mathbb{Z}$, if you restrict your attention to $\Gamma_0(p^n)$, then if $p^n = 0$ in k (e.g. $k = \mathbb{Z}/p^n\mathbb{Z}$) then c has no effect on $\mathrm{Sym}^n k^2$ so it's no longer irr.

$\leadsto$ As $\Gamma_0(p^r)$-mod.s, we have morphisms

$$i_n : \mathrm{Sym}^k (\mathbb{Z}/p^r\mathbb{Z})^2 \to \mathbb{Z}/p^r\mathbb{Z}(k) \quad (x_0, \ldots, x_n) \mapsto x_n$$

$$j_n : \mathbb{Z}/p^r\mathbb{Z}(-k) \to \mathrm{Sym}^k (\mathbb{Z}/p^r\mathbb{Z})^2 \quad x \mapsto (x, 0, \ldots, 0)$$

Now ~~$H^1(X_0(p^r), \mathrm{Sym}^k(\mathbb{Z}/p^r\mathbb{Z})^2)$, $H^1(X_0(p^r$~~

for a $\Gamma_0(p^r)$-module M, there is a twisted coeff. sheaf $\tilde{M}$ on $Y_0(p^r)$ so that $H^1(Y_0(p^r), \tilde{M})$ makes sense. But really this is the group coh. $H^1(\Gamma_0(p^r), M)$ (i.e. 1-cocycles $u : \Gamma_0(p^r) \to M$ w/ $u(\alpha\beta) = \alpha u(\beta) + u(\alpha)$ mod 1-cob $u(\alpha) = \alpha m - m$)

So we use them interchangeably. In particular $H^1(Y_0(p^r), \mathrm{Sym}^n k^2) \sim$ weight $n+2$ mod fun w/ coeff.s in k.

On the level of cocycles, we have the following explicit description of Hecke op.s:

<u>Def</u> $\Phi, \Phi' \subseteq SL_2(\mathbb{Z})$, $\alpha \in GL_2(\mathbb{Q})$, then the Hecke operator

$$[\Phi\alpha\Phi'] : H^1(\Phi, M) \to H^1(\Phi', M) \text{ is def. by}$$

$$u \longmapsto \left(g \mapsto \sum_i \alpha_i^{\iota} u(g_i)\right)$$

$u : \Phi \to M$ 1-cocycle

where
- $\Phi\alpha\Phi' = \coprod_{i \in I} \Phi\alpha_i$
- $\forall i \in I$, $g_i \alpha_j = \alpha_i g$

$$- \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\iota} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Now what we aim to prove is that

$$H^1(\Gamma_0(p^r), \mathrm{Sym}^n(\mathbb{Z}/p^r\mathbb{Z})) \simeq H^1(\Gamma_0(p^r), \mathbb{Z}/p^r\mathbb{Z}(n)).$$

We certainly have a natural morphism

$$(i_n)_*: H^1(\Gamma_0(p^r), \mathrm{Sym}^n(\mathbb{Z}/p^r\mathbb{Z})) \to H^1(\Gamma_0(p^r), \mathbb{Z}/p^r\mathbb{Z}(n)).$$

For the natural map in the other direction, define

$$t: H^1(\Gamma_0(p^r), \mathbb{Z}/p^r\mathbb{Z}(n)) \xrightarrow{[\tau]} H^1(\Gamma_0(p^r), \mathbb{Z}/p^r\mathbb{Z}(-n)) \xrightarrow{(j_n)_*} H^1(\Gamma_0(p^r), \mathrm{Sym}^n(\mathbb{Z}/p^r\mathbb{Z}))$$

$$\xrightarrow{[\Gamma_0(p^r)\delta\Gamma_0(p^r)]} H^1(\Gamma_0(p^r), \mathrm{Sym}^n(\mathbb{Z}/p^r\mathbb{Z}))$$

where $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- $(j_n)_*$ as before
- $\tau = \begin{pmatrix} 0 & -1 \\ p^r & 0 \end{pmatrix}$, so that $[\tau]$ is left conjugation by τ,

i.e. $u: \Gamma_0(p^r) \to \mathbb{Z}/p^r\mathbb{Z}(n)$ is sent to $[\tau](u): \Gamma_0(p^r) \to \mathbb{Z}/p^r\mathbb{Z}(-n)$

via $[\tau](u)(\alpha) = u(\tau\alpha\tau^{-1})$. Note

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau^{-1} = \begin{pmatrix} d & -c/p^r \\ -bp^r & a \end{pmatrix}$$ so indeed $(n) \rightsquigarrow (-n)$.

We will content ourselves with the following finite level statement (15)
and believe that this will give you that M^{ord} will also contain varying of wt.

Thm (Shimura.. actually)

$a \circ b$ and $b \circ a$ are U_p, i.e. $[\Gamma_0(p^r)\begin{pmatrix}1&0\\0&p^r\end{pmatrix}\Gamma_0(p^r)]$.

so for the ord. part the two modules are iso.

Control theorem would be proved by showing that the natural specialization map (which is surjective)

$M^{ord}_{k,\varepsilon} \to M_k(\Gamma_1(p)\cap\Gamma_0(p^r), \varepsilon)$ is bijective using

rank bounds, which will come out of the above finite rank statement comparison.

We only prove this for $b \circ a$.

pf) Note $\Gamma_0(p^r)\delta\Gamma_0(p^r) = \coprod_{u \bmod p^r} \Gamma_0(p^r)\delta\begin{pmatrix}1&u\\0&1\end{pmatrix}$ $= \delta_u$

so $b \circ a$ sends $u: \Gamma_0(p^r) \to \mathrm{Sym}^k(\mathbb{Z}/p\mathbb{Z})^2$

to

$g \mapsto \sum_{u \bmod p^r} \delta_u^{\iota} j_k(i_k(u(\tau g_u \tau^{-1})))$. — this is just $(x_1, \cdots, x_n) \mapsto (x_n, 0, \cdots 0)$ so $= \tau^{\iota} \cdot x$; we write in this way b/c

where $g_u \in \Gamma_0(p^r)$ is $\delta_u g \delta_v^{-1}$ for some $v \bmod p^r$. for given u,

$\sum_{u \bmod p^r} (\tau\delta_u)^{\iota} u(\tau g_u \tau^{-1}) = \sum_{u \bmod p^r} (\tau\delta_u)^{\iota} u((\tau\delta_u) g (\tau\delta_v)^{-1})$.

So $b \circ a = U_p^r$ if $\Gamma_0(p^r)\begin{pmatrix}1&0\\0&p^r\end{pmatrix}\Gamma_0(p^r) = \coprod_{u \bmod p^r} \Gamma_0(p^r)\tau\delta_u$ where $\tau\delta_u = \begin{pmatrix}1&u\\0&p^r\end{pmatrix}$

"Iwahori factorization"