P-adic Modular forms

Plan:
1) Moduli problems for Elliptic curves
2) Tate curve and the q-expansion principle.
3) The Igusa Curve.
4) Λ-adic forms / families of ordinary p-adic modular forms.

1) In what follows 6 is invertible.

R $\mathbb{Z}[1/6]$-algebra.

$$P_1(R) := [(E,\omega)_{/R}].$$

$E_{/R}$ then has a W. eqn of the form:

$$E: y^2 = x^3 - g_2 x - g_3. \qquad \Delta = g_2^3 - 27 g_3^2 \text{ unit}.$$

$$\omega = \frac{dx}{y} \qquad (x,y) \mapsto (u^2 x, u^3 y) = (x', y').$$

$$\frac{dx'}{y'} = u \frac{dx}{y}.$$

So fixing ω amounts at fixing the Weierstrass eqn.

$P_1(R)$ is representable by $\operatorname{Spec} \mathcal{R} = \mathcal{M}_1$.

$$\mathcal{R} = \mathbb{Z}\left[\frac{1}{6}, g_2, g_3, \frac{1}{\Delta}\right].$$

Over $\mathcal{M}_1$ there is a universal E-C $\mathbb{E}, \omega$

$$(\mathbb{E}, \omega) = \operatorname{Proj}\left(\mathcal{R}[X,Y,Z]/(\quad), \frac{dX}{Y} \right)$$

Now let's add level structure: $\Gamma_1(N)$

$$\mathcal{P}_{\Gamma_1(N)}(R) := \left[(E, \omega, \phi_N)_{/R} \right].$$

R $\mathbb{Z}[\frac{1}{6N}]$-alg

$\phi_N : \mu_N \hookrightarrow E[N]$.

or a point of order N in $E[N]$.

Giving a point P of order N in E.

$$\begin{tikzcd}
 & \mathcal{E} \arrow[r] & \mathcal{M}_1 \\
\operatorname{Spec} R \arrow[ur] \arrow[r] & E \arrow[u] \arrow[r] & \operatorname{Spec} R \arrow[u]
\end{tikzcd}$$

$\mathcal{P}_{\Gamma_1(N)}$ is represented by $\mathcal{M}_{\Gamma_1(N)} = \mathcal{E}[N] - \bigcup_{\substack{d \mid N \\ d \neq N}} \mathcal{E}[d]$.

$$\begin{array}{c} \mathcal{M}_{\Gamma_1(N)} \\ \downarrow \\ \mathcal{M}_1 \end{array}$$

and since N is invertible.

$\mathcal{M}_{\Gamma_1(N)}$ is a finite étale covering.

$$\mathcal{M}_{\Gamma_1(N)} = \operatorname{Spec}\left(\mathcal{R}_{\Gamma_1(N)}\right).$$

We have an action of $\mathbb{G}_m$ on $\mathcal{P}_{\Gamma_1(N)}(R)$.
functorial

$$\mathbb{G}_m(R) \times \mathcal{P}_{\Gamma_1(N)}(R) \longrightarrow \mathcal{P}_{\Gamma_1(N)}(R)$$

$$(\lambda, (E, \phi_N, \omega)) \longmapsto (E, \phi_N, \lambda\omega)$$

Jargon: $P_{\Gamma_1(N)}$ is a schematic rep of $\mathbb{G}_m$.

$$\mathcal{R}_{\Gamma_1(N)/R} = \bigoplus_{k \in \mathbb{Z}} \mathcal{R}_k(\Gamma_1(N); R)$$

$R^\times \circlearrowright$ acts by $x \mapsto \lambda^{-k}$.

$f \in \mathcal{R}_k(\Gamma_1(N); R)$

$$\mathcal{M}_{\Gamma_1(N)}(R) \longrightarrow \mathbb{A}^1_R.$$

f verifies the following properties:

(0) $(E, \phi_N, \omega)_{/R} \simeq (E', \phi_N', \omega')_{/R}$ then $f(\ ) = f(\ )'$

(1) $f((E, \phi_N, \lambda\omega)_{/R}) = \lambda^{-k} f(E, \phi_N, \omega)$

(2) compatibility with base change.

$$\mathbb{G}_m \backslash \mathcal{M}_{\Gamma_1(N)}(R) = [(E, \phi_N)_{/R}].$$

$$\mathbb{G}_m \backslash \mathcal{M}_{\Gamma_1(N)}(R) = \mathrm{Proj}(\mathcal{R}_{\Gamma_1(N)/R}).$$

In general if I have $\mathcal{A} = \bigoplus_{j \in \mathbb{Z}} \mathcal{A}_j$ with compatible $\mathbb{G}_m$ action.
and $\mathcal{A}$ has a unit in degree 1 $\quad \mathcal{A} = \mathcal{A}_0 \otimes \mathbb{Z}[u, u^{-1}]$.

$\mathrm{Spec}\,\mathcal{A} = \mathrm{Spec}\,\mathcal{A}_0 \times \mathbb{G}_m$

So we get that $\mathrm{Proj}(\mathcal{A}) = \mathrm{Spec}(\mathcal{A}_0)$.

When $\mathcal{A} = R_{\Gamma_1(N)/R}$.

$$\mathbb{G}_m \backslash \mathcal{M}_{\Gamma_1(N)}(R) = \operatorname{Spec}\left(R_0(\Gamma_1(N); R)\right) = Y_1(N)_{/R}.$$

$$\begin{array}{c} Y_1(N) \\ \downarrow j \\ M_1 = \operatorname{Proj}(\mathcal{R}) = \mathbb{P}^1(j) - \{\infty\} \end{array}$$

<u>Compactification of $Y_1(N)$</u>:

$\mathcal{R}$ we inverted Δ.

$$G(\mathbb{Z}) = \mathbb{Z}\left[\tfrac{1}{6}, g_2, g_3\right] \subseteq \mathcal{R} \subseteq R_{\Gamma_1(N)/\mathbb{Z}[1/6N]}.$$

$$G_{\Gamma_1(N)/\mathbb{Z}[1/6N]} = \text{integral closure } G(\mathbb{Z}) \text{ inside}.$$

$G_{\Gamma_1(N)/\mathbb{Z}[1/6N]}$ is also a graded ring.

$$G_{\Gamma_1(N)}(R) = G_{\Gamma_1(N)/\mathbb{Z}[1/6N]} \otimes R.$$

$$X_1(N)_{/R} = \operatorname{Proj}\left(G_{\Gamma_1(N)}(R)\right).$$

More geometrically. $X_1(N)$ is the normalization of $\operatorname{Proj}(G) = \mathbb{P}^1(j)$ in $Y_1(N)$.

$$\begin{tikzcd} X_1(N) \arrow[d, "j"] & \cdots \\ \mathbb{P}^1(j) & \infty \arrow[u] \end{tikzcd} \qquad \mathrm{Cusps}(X_1(N)) = j^{-1}(\infty)^{\mathrm{red}}.$$

How to describe this scheme : Next section -

2) AME Katz - Mazur.

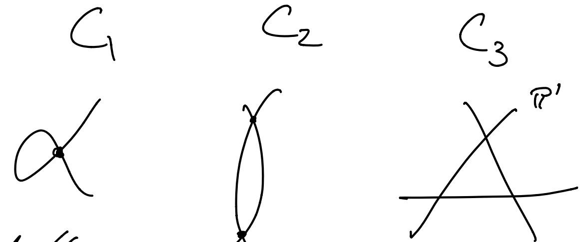
Facts : $\mathrm{Cusps}(X_1(N))$ is finite étale over R.

What are the moduli interpretation of the cusps. (Deligne - Rapoport).

These cusps correspond to generalized elliptic curves, specifically.

Néron n-gons. (with additional level N structure).

$\hookrightarrow$ Def : $\mathbb{P}^1 \times \mathbb{Z}/N\mathbb{Z}$



What does $X_1(N)$ look like "very close" to the cusps ?

Tate curves :

$$\mathrm{Tate}(q) = \mathrm{Proj}\left(\mathbb{Z}[[q]][1/6][X,Y,Z] \Big/ ZY^2 = X^3 - g_2(q)XZ^2 - g_3(q)Z^3 \right)$$

$\mathbb{Z}[[q]]$ at $q=0$ it is a nodal curve i.e. a Néron 1-gon.
But it is an E-C over $\mathbb{Z}[[q]][q^{-1}]$.

$\mathrm{Tate}(q^n)$ but the special fiber will be a Néron n-gon.

So we can see $\mathrm{Tate}(q^n)$ as a deformation of Néron's n-gons.

A few properties of the Tate-Curve :

* As a formal group : $\widehat{Tate(q)} \overset{\phi_{can}}{\simeq} \widehat{\mathbb{G}_m}$ $\qquad \frac{dt}{t}$.

$Tate(q) \qquad \omega_{can} := \phi_{can}^*\left(\frac{dt}{t}\right)$.

$0 \to \mu_N \xrightarrow{a_N} Tate(q)[N] \longrightarrow \mathbb{Z}/N\mathbb{Z} \to 0 \qquad \mathbb{Z}((q))$.

$\zeta \in \mu_N(R) \qquad \phi_{can}(a_N(\zeta)) = \zeta$.

$R \ni \zeta_N$. $\quad \cdot \ (Tate(q), \zeta_N)_{\mathbb{Z}[[q]]} : \operatorname{Spec} R[[q]] \longrightarrow X_1(N)$. $(*)$.

At the "∞" cusp there is a 1-gon : ζ_N primitive in R .

$(G_1, \zeta_N) \quad \alpha \qquad , \infty : \operatorname{Spec} R \longrightarrow X_1(N)$.

$\mathcal{O}_{X_1(N)}$

$(*)$ begets $\quad \widehat{\mathcal{O}}_{X_1(N),\infty} \longrightarrow R[[q]]$.

Theorem : This morphism is an <u>isomorphism</u> . at least when R is a field .

<u>What happens at the other cusps?</u> A very similar story .

d-gons (with additional level structure)

$Tate(q^d)$,

<u>Functions on $X_1(N)$</u> : $\quad X_1(N)_{/R} = \operatorname{Proj}\left(G_{\Gamma_1(N)}(R)\right)$.

$$G_{\Gamma_1(N)}(R) = \bigoplus_{k \in \mathbb{Z}} G_k(\Gamma_1(N); R) .$$

$f \in G_k(\Gamma_1(N), R)$ is the same as before on R_k but now we have also the cusps to use.

$$f(\mathrm{Tate}(q), \phi_N, \omega)_{/R[\zeta_N]} \in R[\zeta_N][[q]].$$

The q-expansion of f.

<u>Rmk</u>: It's subtle to see that this corresponds to the complex analytic picture of q-expansions.

It needs GAGA $\oplus$ $(E_{\mathbb{C}}^{\mathrm{sm}})^{\mathrm{an}}$ description in function of the Tate curve.

<u>Theorem</u>: $f \in G_k(\Gamma_1(N); R)$ is uniquely determined by its q expansion when R is an $\mathcal{O}$-algebra $\mathcal{O} = \mathbb{Z}[\frac{1}{6N}, \zeta_N]$.

$X := X_1(N)$ for simplicity

$$H^0(X_R, \omega_R^{\otimes k}) \longrightarrow R \otimes_{\mathbb{Z}} \mathbb{Z}[[q]]. \text{ is injective.}$$

<u>Proof</u>: $X_R \xrightarrow{f} X_{\mathcal{O}}$ this is affine.

$$H^0(X_R, \omega_R^{\otimes k}) \cong H^0(X_{\mathcal{O}}, f_* \omega_R^{\otimes k}) \cong H^0(X_{\mathcal{O}}, R|_{\mathcal{O}} \otimes \omega_{\mathcal{O}}^{\otimes k}).$$

We must show $H^0(X_{\mathcal{O}}, M \otimes_{\mathcal{O}} \omega_{\mathcal{O}}^{\otimes k}) \longrightarrow M \otimes \mathbb{Z}[[q]]$ is injective.

We can reduce to M finitely generated. (direct limits).

Since $\mathcal{O}$ is a Dedekind domain, we can use structure theorem of modules$/\mathcal{O}$.

With some dévissage, $M = \mathcal{O}, \mathcal{O}/\mathfrak{p}$.

$$\mathcal{O} \hookrightarrow K = \mathrm{Frac}(\mathcal{O})$$

We only need to prove that it is injective for fraction & residue field of $\mathcal{O}$.

But taking q-expansions $\Leftrightarrow$ taking the image of f in the completed stalk at ∞ty.

X_K is a smooth and connected variety so in particular integral.

This shows that the q-expansion map is <u>injective</u>.

$X_1(N)_{/\mathbb{Z}[1/N]}$ $\qquad G_k(\Gamma_1(N); \mathbb{Z}_p)$

base change to $\mathbb{Z}_p$ when $p \nmid N$.

The reason this does not work is because we would like a topological ring whose topology is compatible with q-expansions.

<u>The Hasse invariant</u>:

Let $E_{/R}$ an elliptic curve $\quad \mathrm{char}\, R = p$.

$\mathrm{Frob} : E_{/R} \longrightarrow E_{/R}$. $\quad \mathcal{T}_{E/R}$

$H^0(E_{/R}, \mathcal{T}_{E/R})$ is naturally dual to $H^0(E_{/R}, \Omega^1_{E/R}) \ni \omega$.

$\overset{\psi}{\eta(\omega)}$ dual to ω.

$$\mathrm{Frob}^* \eta(\omega) = \underbrace{A(E,\omega)} \eta(\omega)$$

this is the Hasse invariant

$$\eta(\lambda\omega) = \lambda^{-1}\eta(\omega)$$

$$\mathrm{Fabs}^*\eta(\lambda\omega) = A(E,\lambda\omega)\,\eta(\lambda\omega) = \lambda^{-1} A(E,\lambda\omega)\,\eta(\omega)$$

$$\mathrm{Fabs}^*\lambda^{-1}\eta(\omega) = \lambda^{-p} A(E,\omega)\,\eta(\omega) \qquad A(E,\lambda\omega) = \lambda^{1-p} A(E,\omega).$$

$$A \in G_{p-1}(\Gamma_1(1), \mathbb{F}_p)$$

what is the q-expansion of A?. $\rightsquigarrow$ the composition p-times.

$$H^0(E_{/R}, T_{E/R}) \cong \mathrm{Der}_{\mathcal{O}_{\mathrm{Spec} R}}(\mathcal{O}_{E/\mathcal{O}}, \mathcal{O}_{\mathrm{Spec} R}).$$

Fabs^*

$$A(\mathrm{Tate}(q), \omega_{can}) \qquad \omega_{can} = \varphi_{can}^*\left(\frac{dt}{t}\right) \qquad \frac{dt}{t} \quad \widehat{\mathbb{G}}_m,$$

$$D = t\frac{d}{dt}. \qquad D(t) = t \quad . \; D^p(t) = t.$$

$\mathrm{Fabs}^*\eta_{can} = \eta_{can}$. so this shows. the q-expansion of A is 1.

$$A(\mathrm{Tate}(q), \omega_{can}) = 1.$$

Can we lift A to char 0?. Yes $p \geq 5$.

$$X_1(N)_{\mathbb{Z}_p} \xleftarrow{\iota} X_1(N)_{\mathbb{F}_p} \qquad H^0(X_1(N)_{\mathbb{F}_p}, \omega_{\mathbb{F}_p}^{\otimes k}) = H^0(X_1(N)_{\mathbb{Z}_p}, j_*\omega_{\mathbb{F}_p}^{\otimes k}).$$

$$= H^0(X_1(N)_{\mathbb{Z}_p}, \omega_{\mathbb{F}_p}^{\otimes k}/p).$$

$$0 \to \omega_{\mathbb{Z}_p}^{\otimes k} \to \omega_{\mathbb{Z}_p}^{\otimes k} \to \omega_{\mathbb{Z}_p}^{\otimes k}/p \to 0.$$

To prove $H^0(X_1(N)_{\mathbb{Z}_p}, \omega_{\mathbb{Z}_p}^{\otimes k}) \longrightarrow H^0(X_1(N)_{\mathbb{Z}_p}, \omega_{\mathbb{F}_p}^{\otimes k}/p)$ surjective.

it suffices to prove that the $H^1(-)$ is trivial.

Serre-duality :

The question is can we find an explicit lift ? . Yes :

$$E_k(q) = 2^{-1}\zeta(1-k) + \sum_{n\geq 1} q^n \sigma_{k-1}(n)$$

Von-Staudt-Clausen Bernoulli congruences : $p \mid$ denominator of a Bernoulli number iff

For $k = p-1$, we have that $p \mid$ denominator of B_{p-1}

$$\mathcal{E}_{p-1} = 1 + \underbrace{\frac{2}{\zeta(2-p)}\sum_{n\geq 1} q^n \sigma_{k-1}(n)}_{\equiv 0 \bmod p.} \equiv 1 \bmod p .$$

$\mathcal{E} \bmod p$ has the same q-expansion as A, $\mathcal{E} \bmod p = A$.

$X_1(N)_{\mathbb{Z}_p}$

$\mathcal{E} \equiv 1 \bmod p$. So $\mathcal{E}^{p^n}$ is closer and closer to 1 as $n \to \infty$.

$$\mathcal{E}^{p^n} \xrightarrow[n\to+\infty]{} 1$$

But then $\mathcal{E}^{p^n-1} \longrightarrow \mathcal{E}^{-1}$. So $\mathcal{E} \bmod p$ cannot vanish.
i.e. A cannot.

I.e. we need to cut out the supersingular, if we want to have any hope to define a good notion of p-adic modular form .

3) <u>The Igusa curve</u>. $p \geq 5$.

$M = X_1(N)_{\mathbb{Z}_p}$ - $S_0 = M[1/\mathcal{E}]$ = non vanishing locus of a very ample line bundle $\omega_{\mathbb{Z}_p}^{\otimes p-1}$.
= affine.

$W := \mathbb{Z}_p$ $W_m := \mathbb{Z}/p^m\mathbb{Z}$. $M_m := X_1(N)_{W_m}$. $S_m := M_m[1/\mathcal{E}]$.

$S_m = \mathrm{Spec}(V_{m,0})$.

$$\mathcal{E}_\alpha^{ord}(R) = \left[(E, \mu_{p^\alpha} \hookrightarrow E[p^\alpha], \phi_N)_{/R} \right]$$ R is a W_m algebra.

Just like we did for $P_{\Gamma_1(N)}$, $\mathcal{E}_\alpha^{ord}$ is represented by $\mathbb{E}[p^\alpha]^{ét}_{/S_m} - \mathbb{E}[p^{\alpha-1}]^{ét}_{/S_m}$. where

$$\begin{array}{c} \mathbb{E} \\ \downarrow \\ X_1(N)_{W_m} \end{array}$$ universal generalized E-C.

<u>Def</u>: The Igusa curve $T_{m,\alpha} := \left(\mathbb{E}[p^\alpha]^{ét} - \mathbb{E}[p^{\alpha-1}]^{ét} \right)_{/S_m}$.

Galois group $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$: $\begin{array}{c} T_{m,\alpha} \\ \downarrow \\ S_m \end{array}$.

<u>Igusa proved</u>: $T_{m,\alpha}$ is irreducible $m \geq 1$.

$T_{m,\alpha}$ is also reduced, so integral (étale part of $\mathbb{E}[p^\alpha]^{ét}$)

<u>Corollary</u>; the q-expansion principle also holds on $T_{m,\alpha}$.

$T_{m,\alpha} = \mathrm{Spec}(V_{m,\alpha})$

We have inclusions of W_m-algebras $V_{m,0} \subseteq V_{m,1} \subseteq$ -----

$$V_{m,\infty} := \bigcup_\alpha V_{m,\alpha}.$$

Finally we define what a p-adic modular form is.

$V := V_{\Gamma_1(N)} = \varprojlim V_{m,\infty}$ is taken with respect to reduction over the base $W_{m+1} \to W_m$.

V is the space of p-adic modular forms.

$\mathcal{V} = \varinjlim V_{m,\infty}$ is taken with respect to multiplication by p over the base $W_m \to W_{m+1}$.

$\mathbb{Q}_p/\mathbb{Z}_p = \varinjlim \mathbb{Z}/p^m\mathbb{Z}$.

P-ordinary modular forms:

U(p) - operator: $a(n; f\cdot U(p)) = a(np; f)$.

f is a Hecke eigen-form then. $a(p; f) =$ eigenvalue of $U(p)$ for f.
normalized $\qquad = \text{unit}$.

So $U(p)$ on ordinary forms must be invertible.

Hida's trick: p-ordinary projecto $e := \varinjlim_n U(p)^{n!}$.

$U(p) \circlearrowright \underbrace{X}_{\text{finite}}$. Then e exists ($(U(p)^n)$ is periodic).
and is an idempotent (i.e. projector).

and on the image of e, $U(p)$ is invertible.

$\mathcal{V}^{ord} = e\mathcal{V}$. $\quad V^{ord} = eV$

<u>Galois action</u>: $\begin{array}{c} T_{m,\alpha} \\ \downarrow \\ S_m \end{array}$ has Galois group $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$. $E[p^\alpha]^{et}$.

(E, ϕ_p, ϕ_N) is a point of $T_{m,\alpha}$ $\quad z \in (\mathbb{Z}/p^\alpha\mathbb{Z})^\times$.

$z \cdot (E, \phi_p, \phi_N) = (E, z\phi_p, \phi_N)$ $\qquad \mu_{p^\infty} \hookrightarrow E[p^\infty]$.

$V = \varprojlim V_{m,\infty}$.

$V_{m,\infty}[k] = \left\{ f \;\middle|\; f(E, z^{-1}\cdot\phi_p, \phi_N) \underset{w_m}{=} z^k f(E, \phi_p, \phi_N) \right\}$.

<u>Consider</u>:

$f \in H^0(S_m, \omega^{\otimes k})$ $\qquad \mu_{p^\infty}$ has a canonical differential $\frac{dt}{t}$.

(E, ϕ_p, ϕ_N) $\qquad \phi_p : \mu_{p^\infty} \hookrightarrow E[p^\infty] \hookrightarrow E$.

I can see f as a function on (E, ϕ_p, ϕ_N) by just declaring

$$f(E, \phi_p, \phi_N) := f\left(E, \phi_N, \phi_{p,*}\left(\frac{dt}{t}\right)\right).$$

$$f(E, z^{-1}\cdot\phi_p, \phi_N) = f\left(E, \phi_N, (\phi_p \circ z^{-1})_*\left(\frac{dt}{t}\right)\right) = f\left(E, \phi_N, z^{-1}\cdot \phi_{p*}\left(\frac{dt}{t}\right)\right)$$
$$= z^k f\left(E, \phi_N, \phi_{p*}\left(\frac{dt}{t}\right)\right).$$

$H^0(S_m, \omega^{\otimes k}_{w_m}) \subseteq V_{m,\infty}[k]$ is an inclusion.

and also in the other direction.

Morale: $V_{m,\infty}[k] = H^0(S_m, \omega^{\otimes k})$.

$$\mathcal{V}[k] = \varinjlim V_{m,\infty}[k] = H^0(S_{0/W}, \omega^{\otimes k} \otimes \mathbb{Q}_p/\mathbb{Z}_p) = H^0(S_{0/W}, \omega^{\otimes k}) \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

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4) Λ-adic forms

V^{ord} as the Pontryagin dual of $\mathcal{V}^{ord}$. $\curvearrowleft$ $\varprojlim (\mathbb{Z}/p^\alpha\mathbb{Z})^\times = \mathbb{Z}_p^\times$.

V^{ord} (and $\mathcal{V}^{ord}$) are naturally $W[[\mathbb{Z}_p^\times]]$-modules.

$p \geq 5$: $\mathbb{Z}_p^\times \simeq \Delta \times \Gamma$ $\quad \underset{\gamma \text{ top generator}}{\Gamma \simeq \mathbb{Z}_p}$. $\Lambda = W[[\Gamma]] \simeq W[[X]]$.

$\gamma \mapsto 1+X$.

Theorem: (VCT) $V^{ord}_{[\chi]}$ is a free Λ-module of finite rank.

$\chi: \Delta \to \mathbb{Z}_p^\times$

$V^{ord} = \bigoplus_{\chi \in \hat{\Delta}} V^{ord}[\chi]$ is also free of finite rank.

$$V^{ord} \underset{W[[\mathbb{Z}_p^\times]], k}{\otimes} W \simeq \begin{cases} \mathrm{Hom}_W(G_k^{ord}(\Gamma_1(N), W), W) & k \geq 3 \\ \smile & k = 2 \end{cases}$$

$g^k - 1$

$$G_k^{ord}(\Gamma_1(N), W) := eH^0(X_1(N)_{W}, \omega^{\otimes k})$$

One can bound this rank: Trying to understand $G_k^{ord}(\Gamma_1(N), W)$.

Theorem: $\dim_{\mathbb{Q}_p}(G_k^{ord}(\Gamma_1(N), W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ depends only on $k \bmod p-1$.

Λ-adic forms :

$$G(\chi, \Lambda) = \mathrm{Hom}_{\Lambda}\left(V^{ord}[\chi], \Lambda\right).$$

$$G(\chi, \Lambda) \otimes_{\Lambda, k} W \simeq G_k^{ord}(\Gamma_1(N), W) \quad \text{when } k \geq 3.$$

Consider $V_{/\Lambda}$ equivalent definition to V $\quad T_{m,\infty} \times_{\mathrm{Spec}\, \mathbb{Z}_p} \mathrm{Spec}\, \Lambda$.

$$V_{/\Lambda} = V \hat{\otimes}_{\mathbb{Z}_p} \Lambda.$$

$V_{/\Lambda}$ has two Λ-module structures. * One comes from the Galois action.

* The other is the natural one.

$$G(\Lambda) := \{ f \in V_{/\Lambda} : \underbrace{f|\langle z \rangle}_{\text{Galois action}} = \nu(z) f \} \qquad \nu : \Gamma \to \Lambda^{\times}, \; = \mathbb{Z}_p[[\Gamma]].$$

$\Phi \in G(\Lambda)$ has a q-expansion of the form :

$$\Phi(T, q) = \sum_{n \geq 0} \underset{\substack{\uparrow \\ \Lambda}}{a(n, \Phi)(T)}\, q^n.$$

$$V_{/\Lambda} \otimes_{\Lambda, s} \mathbb{Z}_p \longrightarrow V_{/\mathbb{Z}_p}. \qquad \gamma^s - 1$$

$\Phi(\gamma^s - 1)$ a a p-adic modular form.

This is true because the structures as Λ-module and the Galois side coincide.

Theorem : $\bigoplus_{\chi \in \hat{\Delta}} G(\chi, \Lambda) \cong e(\mathbb{G}(\Lambda))$.

Φ is in $G(\chi, \Lambda)$ $\qquad \{\Phi(\gamma^s - 1)\}_{s \geq 2}$ a family of p-adic modular forms.

This is a Hida family.

Recap : * Understood q-expansions determine the modular form.

* p-adic modular forms : we need to cut out the supersingular locus.

* Defined p-adic mod forms over the Igusa curves

$\hookrightarrow$ q-expansion principle holds.

$\to U(p)$.

$\to$ a notion of ordinary p-adic modular forms.

* Enjoy nice properties V^{ord} is a Λ-module of finite rank. with "bounded rank".

* We can patch-up together p-adic forms in a Λ-adic one.

$\hookrightarrow$ This gives the notion of a family of ordinary p-adic modular forms