GEOMETRY OF SHIMURA VARIETIES

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Contents

We review geometric preliminaries regarding Shimura varieties before we dive into Pilloni's works. Topics are just loosely connected.

1. Siegel modular varieties, PEL Shimura varieties

For something like a modular form for higher dimensional groups, we would like an analogue of modular curves. For example, an analogue for $\mathrm{Sp}_{2g}(\mathbb{Q})$ is a **Siegel modular form** of genus n, which is something like holomorphic function on the Siegel upper half space

$$\mathbb{H}_g = \{\tau \in \mathrm{Mat}_g(\mathbb{C}) \mid \tau^T = \tau, \mathrm{Im}(\tau) \text{ positive definite}\}$$

satisfying transformation properties with respect to a congruence subgroup (using that $\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on $\mathbb{H}_g$) like $\mathrm{Sp}_{2g}(\mathbb{Z})$. Like modular curves, arithmetic quotients like $\mathrm{U}_g \setminus \mathrm{Sp}_{2g}(\mathbb{R}) / \mathrm{Sp}_{2g}(\mathbb{Z})$ can be thought as a moduli space of **principally polarized abelian varieties**:

- The notion of polarization suddenly appeared because higher-dimensional complex torus is not automatically algebraic (unlike 1-dimensional tori which are automatically algebraic), namely you need some kind of positivity to ensure algebraicity. To be more precise, a complex torus $\mathbb{C}^g/L$ is algebraic if and only if it admits a **polarization**. It can be thought as a Hermitian form H on $\mathbb{C}^g$ where $\mathrm{Im}(H)$ restricted to L is integer-valued, $\mathrm{Im}(H) : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$. It is called **principal** if $\mathrm{Im}(H)|_L$ is a perfect pairing, or equivalently there is a basis of L such that $\mathrm{Im}(H) = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. After thinking for a while one can deduce that such data is parametrized by $\mathrm{U}_g \setminus \mathrm{Sp}_{2g}(\mathbb{R}) / \mathrm{Sp}_{2g}(\mathbb{Z})$. Also it seems clear that algebraically a principal polarization of an abelian variety A, in algebraic language, is an isomorphism $\lambda : A \to \widehat{A}$ which is "symmetric" ($\widehat{\lambda}$ + double duality = λ) and "positive" ($\Gamma_\lambda : A \to A \times \widehat{A}$ pulls Poincare bundle $\mathscr{P}_{A\times\widehat{A}}$ back to an ample bundle). Details are not really important.

- There are deeper quotients, which can be also thought as parametrizing principally polarized abelian varieties **with a level structure**. For example, an analogue of $Y(N)$ is a **full level N structure**, which is a choice of symplectic basis of $A[N]$.

We know that such complex manifolds are algebraic. In general, Shimura proved that those are defined over a number field by some indirect method, but in our case we can see this in a much simpler way, because we have a moduli space and the moduli problem can be defined over a certain number field. Such moduli interpretation can be used to even define an **integral model** of a Shimura variety, with an appropriate integral moduli problem. A Shimura variety that can be defined as a moduli space of certain abelian varieties are called **PEL Shimura varieties**. We have much better understanding of integral model etc. of such moduli spaces, because we have moduli interpretation. P stands for polarization, L stands for level structure, and both seem completely natural. E stands for **endormorphism**, which seems somewhat odd. We just provide an example to justify it:

Example 1.1. As $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$, $\mathrm{SL}_2(\mathbb{R})^d$ acts transitively on $\mathbb{H}^d$. For a totally real field K of degree d, one can embed $\mathrm{SL}_2(K)$ into $\mathrm{SL}_2(\mathbb{R})^d$ using d real embeddings of K. Using this embedding, we can form a quotient $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathbb{H}^d$, which is also a Shimura variety, a **Hilbert modular variety**. It is an example of a PEL Shimura variety, where we need endomorphism structure. Namely, $\mathrm{SL}_2(\mathcal{O}_K)\backslash\mathbb{H}^d$ is a moduli space parametrizing the following data.

- A g-dimensional abelian variety A with a polarization H on $H_1(A, \mathbb{Z})$.
- $\mathcal{O}_K \subset \mathrm{End}(A)$.
- A $\mathcal{O}_K$-module isomorphism $H_1(A,\mathbb{Z}) \cong \mathcal{O}_K^2$, where the polarization H is sent to the pairing $H_{\mathrm{std}} : \mathcal{O}_K^2 \times \mathcal{O}_K^2 \to \mathbb{Z}$, $H_{\mathrm{std}}(\alpha_1, \alpha_2; \beta_1, \beta_2) = \mathrm{tr}_{K/\mathbb{Q}}(\alpha_1\beta_2 - \alpha_2\beta_1)$.

We have some data about endomorphism of A, and this is a typical "E part" of PEL datum.

2. Hasse invariants, canonical subgroups

We saw that canonical subgroups are important in overconvergence. As we are going to go beyond modular curves, we need a similar notion for something more other than elliptic curves.

2.1. **Hasse invariant.** Reference: [Fa]

Let S be an $\mathbb{F}_p$-scheme, and let G/S be a truncated p-divisible group of level 1 (something like $E[p]$). It has an easy definition that

$$G \xrightarrow{F} G^{(p)} \xrightarrow{V} G$$

is exact. Then $G \mapsto \omega_{G^D}$ (sheaf of differentials on the Cartier dual $G^D = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$; everything is clear by thinking as fppf sheaves over S) is a covariant functor, so the Frobenius map $F : G \to G^{(p)}$ induces a map $\psi_G : \omega_{G^D} \to \omega_{(G^D)^{(p)}} = (\omega_{G^D})^{(p)}$.

Definition 2.1. *The **Hasse invariant** $\mathrm{Ha}(G)$ is*

$$\mathrm{Ha}(G) = \det \psi_G \in H^0(S, (\det \omega_{G^D})^{-1} \otimes (\det \omega_{G^D})^{(p)}) = H^0(S, (\det \omega_{G^D})^{\otimes(p-1)})$$

because $p\mathcal{O}_S = 0$.

Remark 2.1. One might wonder this is a correct definition because there is equally well-known way of proceeding using Verschiebung and G. Fargues indeed proves that there is a **canonical isomorphism** $(\det \omega_G)^{\otimes(p-1)} \cong (\det \omega_{G^D})^{\otimes(p-1)}$ which gives an identification of Hasse invariants $\mathrm{Ha}(G) = \mathrm{Ha}(G^D)$ under this isomorphism.

Fixing a basis, one can take the p-adic valuation of $\mathrm{Ha}(G)$, the **Hodge height** $\mathrm{Hdg}(G) \in [0,1]$, which does not depend on the choice of basis.

2.2. **Canonical subgroups.** Let K be a complete valued field over $\mathbb{Q}_p$ with $v(p) = 1$, and let $\mathcal{O}_{K,w} = \mathcal{O}_K/\mathfrak{m}_w$ where $\mathfrak{m}_w = \{x \in K \mid v(x) \geq w\}$. For a (truncated) p-divisible group $G/\mathcal{O}_K$, we also define the Hodge height $\mathrm{Hdg}(G)$ to be the Hodge height of $G[p] \otimes \mathcal{O}_{K,1}$.

Theorem 2.1 (Fargues)**.** *Let $G/\mathcal{O}_K$ is a (truncated) p-divisible group. If $\mathrm{Hdg}(G) < \frac{1}{2p^{n-1}}$, then there is $H_n \subset G[p^n]$, the **canonical subgroup** of level n, which satisfies the following properties.*

(1) $H_n(\overline{K}) \cong (\mathbb{Z}/p^n\mathbb{Z})^d$ where d is the dimension of G (=rank of ω_G).

(2) Over $G \otimes \mathcal{O}_{K,1-\mathrm{Hdg}(G)}$, $H \otimes \mathcal{O}_{K,1-\mathrm{Hdg}(G)}$ is the kernel of Frobenius.

(3) H_k for $1 \leq k \leq n$ is the canonical subgroup of level k for G.

(4) H_n/H_k for $1 \leq k < n$ is the canonical subgroup of level $n-k$ for G/H_k.

One could already see from the properties that canonical subgroups can be constructed inductively, by showing that Hodge height doesn't get too large by modding out by its canonical subgroup (of level 1).

3. Generalized Hasse invariants

Reference: [Bo]

Whether Hasse invariant vanishes or not determines whether a given abelian variety/p-divisible group/etc. is ordinary or not. Among non-ordinary things, one has finer invariants that can detect more things, usually referred as **generalized Hasse invariant**. It uses more structures of p-divisible groups.

Example 3.1. This is the case of [Pi]. For a ffgs G over a characteristic p scheme S, the connected-étale exact sequence $0 \to G^0 \to G \to G^{\text{ét}} \to 0$ is one of the most basic things one learns about ffgs. On the other hand, one can also consider the connected-étale exact sequence for the Cartier dual G^D:

$$0 \to (G^D)^0 \to G^D \to (G^D)^{\text{ét}} \to 0$$

Taking the Cartier dual of this, one gets another filtration of G, which is called the **multiplicative-unipotent exact sequence**

$$0 \to G^{\text{mul}} \to G \to G^{\text{un}} \to 0$$

Indeed a ffgs is called **multiplicative** if its Cartier dual is étale, etc. It is the same as locally being of form $\operatorname{Spec} R[\Gamma]$ for a finite abelian group Γ (e.g. μ_p when $\Gamma = \mathbb{Z}/p\mathbb{Z}$).

Now consider the case when G is a truncated p-divisible group of level 1, dimension 2, height 4 (in this case, this means G is finite of rank p^4 over S). Suppose further that the **étale rank** (=rank of $G^{\text{ét}}$) and the **multiplicative rank** (=rank of $G^{\text{mul}} \cap G^0$) are both 1. This is definitely not ordinary as ordinary p-divisible group should have étale rank = multiplicative rank = 2.

Example 3.2. For example, given a 2-dimensional principally polarized abelian variety A over a characteristic p field which has p points of order p over $\overline{\mathbb{F}}_p$, $A[p]$ will give rise to such an example of ffgs.

So the Hasse invariant is just zero over the whole base S. On the other hand, we have something called the **second Hasse invariant** in this case. This uses both etale-connected and multiplicative-unipotent exact sequence. We have a filtration of G as follows.

$$\begin{array}{c}
G_4 = G \\
\text{étale} \uparrow \\
G_3 = G^0 \\
\uparrow \\
G_2 = \ker F = \operatorname{im} V \\
\uparrow \\
G_1 = G^{\text{mul}} \cap G^0 \\
\text{connected-étale} \uparrow \\
G_0 = 0
\end{array}$$

Here each arrow has a description of corresponding subquotient, where **connected-étale** means the ffgs itself is connected and its Cartier dual is étale. Each successive subquotient G_i/G_{i-1} is of rank p, and $\omega_{G_i/G_{i-1}}$ is rank 1 for $i = 1, 2, 3$ and $\omega_{G_4/G_3} = 0$ because it's etale. Through elementary manipulations we see the following.

(1) $\det \omega_G = \otimes_{i=1}^3 \omega_{G_i/G_{i-1}}$.

(2) V induces isomorphisms

$$V : (G_1/G_0)^{(p)} \xrightarrow{\sim} G_1/G_0 \quad \text{(easy)}$$

$$V : (G_3/G_2)^{(p)} \xrightarrow{\sim} G_2/G_1$$

(dual to the below hard case)

(3) F induces isomorphisms

$$F : G_3/G_2 \xrightarrow{\sim} (G_2/G_1)^{(p)}$$

(connected means F nilpotent, so $F : G_3 \to G_2^{(p)}$, and $F = 0$ on G_2)

$$F : G_4/G_3 \xrightarrow{\sim} (G_4/G_3)^{(p)} \quad \text{(easy)}$$

(4) Taking determinants, we have isomorphisms

$$V^* : \omega_{G_1/G_0} \xrightarrow{\sim} \omega_{G_1/G_0}^{\otimes p}$$

$$V^* : \omega_{G_2/G_1} \xrightarrow{\sim} \omega_{G_3/G_2}^{\otimes p}$$

$$(F^*)^{-1} : \omega_{G_3/G_2} \xrightarrow{\sim} \omega_{G_2/G_1}^{\otimes p}$$

So we can cook up maps

$$\omega_{G_1/G_0} \xrightarrow{V^*,\sim} \omega_{G_1/G_0}^{\otimes p} \xrightarrow{(V^*)^{\otimes p},\sim} \omega_{G_1/G_0}^{\otimes p^2}$$

$$\omega_{G_2/G_1} \xrightarrow{V^*,\sim} \omega_{G_3/G_2}^{\otimes p} \xrightarrow{((F^*)^{-1})^{\otimes p},\sim} \omega_{G_2/G_1}^{\otimes p^2}$$

$$\omega_{G_3/G_2} \xrightarrow{(F^*)^{-1},\sim} \omega_{G_2/G_1}^{\otimes p} \xrightarrow{(V^*)^{\otimes p},\sim} \omega_{G_3/G_2}^{\otimes p^2}$$

Multiplying these together, we get a nowhere vanishing section

$$\mathrm{Ha}'(G) \in H^0(S, \omega_G^{\otimes(p^2-1)})$$

which is called the **second Hasse invariant**.

This seems like a cute trick but actually this works in general. For a truncated p-divisible group of level 1 over a characteristic p base, we can construct the **canonical filtration**. In the case of the above example, canonical filtration is the 4-step filtration we constructed. Using similar combinatorics, one deduces the following.

Theorem 3.1 (Boxer, Goldring–Koskivirta)**.** *If G is a truncated p-divisible group of level 1 over a characteristic p base S having constant canonical filtration over it, meaning that each part of canonical filtration has constant rank over the whole base, then there is some integer $N > 0$ (depending only on the combinatorics of canonical filtration type) and a nowhere vanishing section* $\mathrm{Ha} \in H^0(S, \omega_G^{\otimes N})$.

4. Stratification of special fiber

Reference: [Oo]

The above section is slightly unsatisfactory because the classical Hasse invariant is defined even for p-divisible groups of nonconstant type filtration so that vanishing locus = nonordinary locus. We want to do similar thing for generalized Hasse invariants.

One can think of defining moduli problem for integral model of Siegel modular variety over p-adic dvr, and defining such is not so problematic as long as "the level is prime to p". This enables us to talk about mod p fiber of such Siegel modular varieties. The picture is like:

- Over the mod p fiber, we have a universal abelian variety, and we can stratify the mod p fiber where over each stratum the canonical filtration is of constant type.
- Each generalized Hasse invariant lives over a stratum, and **it extends to the closure of the stratum**, so that nonzero locus of it is precisely the stratum over which the generalized Hasse invariant originally lives.

Such stratification is called **Ekedahl–Oort stratification**, and is parametrized by some set coming from Weyl group datum (denoted ${}^IW$ usually).

Theorem 4.1 (Boxer, Golding–Koskivirta)**.** *For each $w \in {}^IW$, some power of the generalized Hasse invariant $\mathrm{Ha}_w^{\otimes n}$ extends to the closure of the w-stratum, whose nonvanishing locus is precisely the w-stratum.*

Example 4.1. In the example of [Pi], $G = (\mathrm{G})\,\mathrm{Sp}_4(\mathbb{Q})$, the Ekedahl–Oort stratum can be described using a simpler invariant, multiplicative rank. Recall that the corresponding Shimura variety is a moduli space of principally polarized abelian varieties of dimension 2.

- Multiplicative rank 2 truncated level 1 p-divisible groups are ordinary.
- The complement of ordinary locus can be further stratified into two parts, multiplicative rank 1 and multiplicative rank 0. Using the theorem we can see that some power of the second Hasse invariant, which lives over the multiplicative rank 1 locus, can be extended to the whole nonordinary locus (the closure of multiplicative rank 1 locus contains the multiplicative rank 0 locus). Pilloni proves that in fact you don't need to take powers of it.

5. Compactification of Shimura varieties

So far we only worked with open Shimura variety. This means that Shimura varieties we worked are not proper. This is not good because open varieties lack finiteness etc. There are two main ways of compactifying Shimura varieties, **minimal compactification** and **toroidal compactification**. In general my feeling is that

- minimal compactification is singular but reflects open Shimura variety well,
- and toroidal compactification is smooth but there are choices involved and something is nontrivial.

I won't say much about how to construct minimal compactification, except that it is taken as the Proj of "graded ring of automorphic forms," like $\bigoplus_{i=0}^{\infty} H^0(\mathrm{Sh}, \omega^i)$. It is singular, but it is minimal in a sense that for any compactification $\mathrm{Sh} \hookrightarrow \overline{\mathrm{Sh}}$ with dense image and boundary a normal crossings divisor, there is a unique morphism $\overline{\mathrm{Sh}} \to \overline{\mathrm{Sh}}^{\min}$ compatible with embeddings of Sh. Note that the boundary of $\overline{\mathrm{Sh}}^{\min}$ itself is in most cases not of codimension 1 (except the case of modular curves where everything coincides).

We say a little more about toroidal compactification. Still the boundary strata are parametrized by the same set, denoted Cusp_K, which is a group-theoretic finite set depending on level K.

Example 5.1. In the case of Siegel modular varieties with level 1 (A_g), the boundary of minimal compactification is stratified by integers $0 \le i < g$ where in fact i-th stratum is isomorphic to A_i. In general, for a full level $\Gamma(N)$, $\mathrm{Cusp}_{\Gamma(N)} = \mathfrak{C}/\Gamma(N)$, where $\mathfrak{C}$ is, for $V = \oplus_{i=1}^{2g} \mathbb{Z} e_i$ with the standard symplectic form,

$$\mathfrak{C} = \{\text{totally isotropic direct factors } V' \subset V\}$$

Each boundary component is a Siegel modular variety (of smaller dimension) of appropriate level.

Now why **toroidal** compactification? This is because it uses the theory of torus embeddings. We will talk first about torus embeddings, and then about where the heck "torus" appears at the boundary, and finally some remarks about the relation between toroidal and minimal compactifications. We will just motivate so we will be extremely sketchy.

5.1. **Torus embeddings.**

(1) Given a split torus T over k, an **affine torus embedding** is an open embedding $T \hookrightarrow V$ into an affine k-variety V together with an action $T \times V \to V$ which extends the multiplication on T.

(2) It is determined by $k[V] \hookrightarrow k[T]$, and equivariance induces a grading $k[V] = \oplus_{\chi \in X^*(T)} k[V]_\chi$ where $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$.

(3) On the other hand, given a semigroup $S \subset X^*(T)$, one can consider $k[S]$.

(4) Such operations give a bijection

$$\left\{\begin{array}{c}\text{Finitely generated}\\ \text{semigroups}\\ S \subset X^*(T) \text{ which}\\ \text{generate } X^*(T) \text{ as}\\ \text{an abelian group}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Affine torus}\\ \text{embeddings of } T\end{array}\right\}$$

and, furthermore, $k[S]$ is normal iff S is **saturated**, namely $r^n \in S$ implies $r \in S$.

(5) Saturated semigroups appearing on the LHS arise from **rational polyhedral cones**, namely intersection of a finite set of rational half-spaces (passing through the origin). Given a rational polyhedral cone $\sigma \subset X_*(T)_{\mathbb{R}}$, the dual cone $\check{\sigma}$ is defined as

$$\check{\sigma} = \{\lambda \in X^*(T)_{\mathbb{R}} \mid \langle \lambda, x \rangle \geq 0 \text{ for all } x \in \sigma\}$$

Then, we have a bijective correspondence

$$\left\{ \begin{array}{c} \text{Polyhedral cones} \\ \text{in } X_*(T)_{\mathbb{R}} \text{ not} \\ \text{containing linear} \\ \text{subspaces} \end{array} \right\} \xrightarrow{\sigma \mapsto k[\check{\sigma} \cap X^*(T)] := V_\sigma} \left\{ \begin{array}{c} \text{Affine torus} \\ \text{embeddings of } T \\ \text{into normal} \\ \text{varieties} \end{array} \right\}$$

Both can be thought as categories, in a sense that $\sigma_1 \subset \sigma_2$ induces a morphism $V_{\sigma_1} \to V_{\sigma_2}$. It induces an open immersion iff $\sigma_1 \subset \sigma_2$ is a face.

(6) Now if you have a decomposition of $X_*(T)_{\mathbb{R}}$ into rational polyhedral cones, then you can patch to get a T-equivariant embedding of T into a proper variety.

Such construction makes sense over general base; we just lose the bijectivity.

5.2. **Structure at boundary.** Why does torus embedding have anything to do with compactification of Shimura varieties? It is because there is a torus torsor appearing at each boundary stratum.

Theorem 5.1 (Lan). *For each boundary component $Z \in \mathrm{Cusp}_K$, there are:*

- *an abelian scheme C over a finite étale covering of Z,*
- *a torus torsor $\Xi \to C$,*
- *given a choice of "decomposition into rational polyhedral cones", one has a relative torus embedding, and even formal scheme $\mathfrak{X}$ describing the boundary,*
- *and for $\Gamma \leq K$ the parabolic subgroup fixing the boundary component Z, $\mathfrak{X}$ admits a free action by Γ, and $\mathfrak{X}/\Gamma$ "is" the formal neighborhood of boundary divisor of M^{tor} corresponding to Z.*

It is this torus torsor that needs toroidal embedding.

Example 5.2. We just exhibit how it works for modular curve $Y(1)$, where we know the compactification is just $\mathbb{P}^1 = \mathbb{A}^1 \coprod \{\infty\}$. At $\{\infty\}$, the stabilizer of it inside $\mathrm{SL}_2(\mathbb{R})$ is $B(\mathbb{R}) = \{\begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}\}$, and it intersected with $\mathrm{SL}_2(\mathbb{Z})$ is $B(\mathbb{Z}) = \{\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}\}$. Consider $B(\mathbb{Z})\backslash\mathbb{H}$, which is via exponential map identified with a punctured open disk $\{0 < |z| < 1\}$. Then one fill the gap of puncture to get open unit disk $\{|z| < 1\}$, and it is glued with $Y(1)$.

Where is a torus here? It is reflected at that $B(\mathbb{Z})\backslash\mathbb{H}$ is a punctured open disk, in that there is an obvious rotational action of S^1. What is this S^1? If you contemplate enough, you see that $S^1 = \mathbb{R}/\mathbb{Z} = B(\mathbb{R})/B(\mathbb{Z})$. This shows that at some part of group structure there is a torus.

Remark 5.1. Faltings–Chai showed that a toroidal compactification of A_g has a property relating to **degeneration of abelian varieties**. By degeneration of abelian varieties we mean a **semiabelian scheme**, an abelian scheme extended by a torus, over a complete dvr, such that the generic fiber is an abelian variety. Mumford's theory of degeneration of abelian variety gives rise to a combinatorial data, a quadratic form on the torus part of the semiabelian scheme over the special point.

From this viewpoint, a toroidal compactification $A_{g,\Sigma}$ carries an extension of a universal abelian scheme $X \to A_g$, a universal **semiabelian scheme** $X_\Sigma \to A_{g,\Sigma}$. It also has a nice universal/functorial property:

Theorem 5.2 (Faltings–Chai). *Let $G \to S$ be a semiabelian scheme where S is an irreducible normal scheme and over $U \subset S$ an open subset G_U is an abelian scheme. Let $f : U \to A_g$ be the map representing G_U/U. Then, it extends to $\overline{f} : S \to A_{g,\Sigma}$ if and only if, for every $s \in S$, there is a rational polyhedral cone Σ_α of the a priori fixed cone decomposition Σ of $C(\mathbb{Z}^g)$ ($\mathbb{Q}$-positive semidefinite bilinear forms on $\mathbb{Z}^g$) and a certain surjection $\mathbb{Z}^g \to X^*(\text{toric part of } G_s)$ such that, for any dvr $\operatorname{Spec} V \to S$ where the special point is sent to s, the associated quadratic form is pulled back via the map to Σ_α.*

In words, each $A_{g,\Sigma}$ is a moduli space parametrizing degenerations of abelian varieties of certain kinds (given by Σ).

5.3. **Vanishing of cohomology.** We will probably need several properties of toroidal/minimal compactification as we need in the development/proofs, so I won't dare to try to summarize what will be needed. I will just exhibit a particular result that illustrates a feeling of how these are used. We retain the following notation:

$$\begin{array}{ccc} \mathrm{Sh} & \xrightarrow{j^{\mathrm{tor}}} & \mathrm{Sh}_\Sigma^{\mathrm{tor}} \\ & \searrow^{j^{\min}} & \downarrow \eta \\ & & \mathrm{Sh}^{\min} \end{array}$$

and $D_\Sigma^{\mathrm{tor}} = \mathrm{Sh}_\Sigma^{\mathrm{tor}} - \mathrm{Sh}$.

Theorem 5.3 (Lan et al.). *(1) For each automorphic vector bundle $\mathcal{E}$ over Sh, there are two extensions, **canonical extension** $\mathcal{E}^{\mathrm{can}}$ and **subcanonical extension** $\mathcal{E}^{\mathrm{sub}}$. It satisfies $\mathcal{E}^{\mathrm{sub}} \subset \mathcal{E}^{\mathrm{can}}$ and morally speaking $\mathcal{E}^{\mathrm{can}}$ is the "sheaf of modular forms" whereas $\mathcal{E}^{\mathrm{sub}}$ is the "sheaf of cusp forms."*

(2) $j_*^{\mathrm{tor}}\mathcal{E} = \varinjlim_{n\geq 0} \mathcal{E}^{\mathrm{can}}(nD_\Sigma^{\mathrm{tor}})$, $R^i j_*^{\mathrm{tor}}\mathcal{E} = 0$ for $i > 0$.

(3) $R^i\eta_*\mathcal{E}^{\mathrm{sub}} = 0$, for $i > 0$.

(4) $R^i\eta_*\mathcal{E}^{\mathrm{can}}(-D_\Sigma^{\mathrm{tor}}) = 0$, for $i > 0$.

These are difficult results and have played important roles.

Remark 5.2. Over $\mathbb{C}$, the canonical extension $\mathcal{E}^{\mathrm{can}}$ coincides with the canonical extension of Deligne (extension as a vector bundle with connection with log singularities at the boundary.

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