THE OVERCONVERGENT SIEGEL MODULAR FORMS OF ANDREATTA, IOVITA AND PILLONI

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1. Algebraic and Analytic Inductions

Consider $\mathrm{GL}_g/\mathbb{Z}_p$. Let B be the Borel subgroup of upper triangular matrices, T the maximal torus of diagonal matrices, and U the unipotent radical of B. We let B^0 and U^0 be the opposite Borel of lower triangular matrices and its unipotent radical. We denote by $X(\mathrm{T})$ the group of characters of T and by $X^+(\mathrm{T})$ its cone of dominant weights with respect to B. For any $\kappa \in X^+(\mathrm{T})$, we set

$$V_\kappa = \{ f\colon \mathrm{GL}_g \to \mathbb{A}^1 \text{ morphism of schemes } /\mathbb{Q}_p \text{ s.t. } f(gb) = \kappa(b)f(g)\ \forall (g,b) \in \mathrm{GL}_g{\times}\mathrm{B}\}.$$

This is a finite dimensional $\mathbb{Q}_p$-vector space. The group GL_g acts on V_κ by left translation. If L is an extension of $\mathbb{Q}_p$ we set $V_{\kappa,L} = V_\kappa \otimes_{\mathbb{Q}_p} L$.

Let $\mathcal{W}$ be the weight space associated to $\mathrm{T}(\mathbb{Z}_p) \simeq \mathrm{T}(\mathbb{Z}/p\mathbb{Z}) \times (1+p\mathbb{Z}_p)^g$ as a rigid analytic space over $\mathbb{Q}_p$. We say that a character $\kappa \in \mathcal{W}(\mathbb{C}_p)$ is w-analytic if κ extends to an analytic map $\kappa : \mathrm{T}(\mathbb{Z}_p)(1+p^w\mathcal{O}_{\mathbb{C}_p})^g \to \mathbb{C}_p^\times$.

Let I be the Iwahori sub-group of $\mathrm{GL}_g(\mathbb{Z}_p)$ of matrices whose reduction modulo p is upper triangular. Let N^0 be the subgroup of $\mathrm{U}^0(\mathbb{Z}_p)$ of matrices which reduce to the identity modulo p. The Iwahori decomposition is an isomorphism: $\mathrm{B}(\mathbb{Z}_p) \times \mathrm{N}^0 \to \mathrm{I}$. We freely identify N^0 with $(p\mathbb{Z}_p)^{\frac{g(g-1)}{2}} \subset \mathbb{A}_{\mathrm{an}}^{\frac{g(g-1)}{2}}$, where $\mathbb{A}_{\mathrm{an}}$ denotes the rigid analytic affine line defined over $\mathbb{Q}_p$. For $\epsilon > 0$, we let N^0_ϵ be the rigid analytic space

$$\bigcup_{x \in (p\mathbb{Z}_p)^{\frac{g(g-1)}{2}}} B(x, p^{-\epsilon}) \subset \mathbb{A}_{\mathrm{an}}^{\frac{g(g-1)}{2}} (\text{Union of balls centered at } x \text{ with radius } p^{-\epsilon}).$$

Let L be an extension of $\mathbb{Q}_p$ and $\mathcal{F}(\mathrm{N}^0, L)$ the ring of L-valued functions on N^0. We say that a function $f \in \mathcal{F}(\mathrm{N}^0, L)$ is ϵ-analytic if it is the restriction to N^0 of a necessarily unique analytic function on N^0_ϵ. We denote by $\mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L)$ the set of ϵ-analytic functions. A function is analytic if it is 1-analytic.

Let $\epsilon > 0$ and $\kappa \in \mathcal{W}(L)$ be an ϵ-analytic character. We set

$$V^{\epsilon-\mathrm{an}}_{\kappa,L} = \{ f\colon \mathrm{I} \to L,\ f(ib) = \kappa(b)f(i)\ \forall (i,b) \in \mathrm{I}\times\mathrm{B}(\mathbb{Z}_p),\ f|_{\mathrm{N}^0} \in \mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L)\}$$

as a representation of I.

2. Canonical subgroups and Hodge-Tate map in families

Let $p \geq 5$ be a prime integer and K a complete valued extension of $\mathbb{Q}_p$ with $v(p) = 1$. For any $w \in v(\mathcal{O}_K)$ we set $\mathfrak{m}_w = \{x \in K, v(x) \geq w\}$ and $\mathcal{O}_{K,w} = \mathcal{O}_K/\mathfrak{m}_w$. Recall from last time that for a truncated p-divisible group $G/\mathcal{O}_K$ we have defined $\mathrm{Ha}(G)$ and $\mathrm{Hdg}(G) \in [0,1]$, and when $\mathrm{Hdg}(G) < \frac{1}{2p^{n-1}}$, there is a canonical subgroup $H_n \subset G[p^n]$ of level n. We have the Hodge-Tate map for H_n^D (viewed as a map of abelian sheaves on the $fppf$-topology):

$$\mathrm{HT}_{H_n^D} : H_n^D \to \omega_{H_n},$$

by sending an S-valued point $x \in H_n^D(S)$, i.e., a homomorphism of S-group schemes $x\colon H_{n,S} \to \mu_{p^n,S}$, to the pull–back $x^*(dt/t) \in \omega_{H_n}(S)$ of the invariant differential dt/t of $\mu_{p^n,S}$.

We let **Adm** be the category of admissible $\mathcal{O}_K$-algebras, i.e., flat $\mathcal{O}_K$-algebras which are quotients of rings of restricted power series $\mathcal{O}_K\langle X_1,\dots,X_r\rangle$, for some $r \geq 0$. We let **NAdm** be the category of normal admissible $\mathcal{O}_K$-algebras.

Let R be an object of **NAdm**, $S := \mathrm{Spec}(R)$, and S_{rig} is the rigid analytic space associated to $R[\frac{1}{p}]$. Let G be a semi-abelian scheme over S such that the restriction of G to a dense open sub-scheme U of S is abelian. We also fix a rational number v such that $v < \frac{1}{2p^{n-1}}$ with the property that for any $x \in S_{\mathrm{rig}}$, $\mathrm{Hdg}(x) < v$. Here $\mathrm{Hdg}(x) := \mathrm{Hdg}\big(G_x[p^\infty]\big)$. In applications R will come from the p-adic completion of an étale affine open sub-set of the toroidal compactification of the Siegel variety. If this open subset does not meet the boundary, then the semi-abelian scheme G will be abelian.

Proposition 2.1. *Under suitable assumptions, the canonical subgroup extends to a finite flat subgroup scheme $H_n \hookrightarrow G[p^n]$ over S.*

Let H_n denote the canonical subgroup of G of level n over S. If $w \in v(\mathcal{O}_K)$ we set $R_w = R \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w}$ and for any R-module M, M_w means $M \otimes_R R_w$. We also assume that $H_n^D(R) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$.

Proposition 2.2. *Let $w \in v(\mathcal{O}_K)$ with $w < n - v\frac{p^n-1}{p-1}$. The morphism of coherent sheaves $\omega_G \to \omega_{H_n}$ induces an isomorphism $\omega_{G,w} \to \omega_{H_n,w}$.*

Proposition 2.3. *There is a free sub-sheaf of R-modules $\mathcal{F}$ of ω_G of rank g containing $p^{\frac{v}{p-1}}\omega_G$ which is equipped, for all $w < n - v\frac{p^n}{p-1}$, with a map*

$$\mathrm{HT}_w\colon H_n^D(R[1/p]) \to \mathcal{F} \otimes_R R_w$$

deduced from $\mathrm{HT}_{H_n^D}$ which induces an isomorphism:

$$\mathrm{HT}_w \otimes 1\colon H_n^D(R[1/p]) \otimes_{\mathbb{Z}} R_w \to \mathcal{F} \otimes_R R_w.$$

Remark 2.4. The sheaf $\mathcal{F}$ is independent of $n \geq 1$, it is functorial in R and G.

Let $\mathcal{GR}_{\mathcal{F}} \to S$ be the Grassmannian parametrizing all flags $\mathrm{Fil}_0\mathcal{F} = 0 \subset \mathrm{Fil}_1\mathcal{F} \ldots \subset \mathrm{Fil}_g\mathcal{F} = \mathcal{F}$ of the free module $\mathcal{F}$ ([Ko, §I.1.7]). Let $\mathcal{GR}_{\mathcal{F}}^+$ be the T-torsor over $\mathcal{GR}_{\mathcal{F}}$ which parametrizes flags $\mathrm{Fil}_\bullet\mathcal{F}$ together with basis ω_i of the graded pieces $\mathrm{Gr}_i\mathcal{F}$.

We fix an isomorphism $\psi\colon (\mathbb{Z}/p^n\mathbb{Z})^g \simeq H_n^D(R[1/p])$ and call $x_1,\dots,x_g$ the $\mathbb{Z}/p^n\mathbb{Z}$-basis of $H_n^D(R[1/p])$ corresponding to the canonical basis of $(\mathbb{Z}/p^n\mathbb{Z})^g$. Out of ψ, we obtain a flag

$$\mathrm{Fil}_\bullet^\psi = \{0 \subset \langle x_1\rangle \subset \langle x_1,x_2\rangle \ldots \subset \langle x_1,\dots,x_g\rangle = H_n^D(R[1/p])\}.$$

We also have a basis $x_i \mod \mathrm{Fil}_{i-1}^\psi$ of the graded piece Gr_i^ψ.

Let R' be an object in $R-$**Adm**. We say that an element $\mathrm{Fil}_\bullet\mathcal{F}\otimes_R R' \in \mathcal{GR}_{\mathcal{F}}(R')$ is w-compatible with ψ if $\mathrm{Fil}_\bullet\mathcal{F}\otimes_R R'_w = \mathrm{HT}_w(\mathrm{Fil}_\bullet^\psi)\otimes_{\mathbb{Z}} R'_w$.

We say that an element $(\mathrm{Fil}_\bullet\mathcal{F}\otimes_R R', \{w_i\}) \in \mathcal{GR}_{\mathcal{F}}^+(R')$ is w-compatible with ψ if $\mathrm{Fil}_\bullet\mathcal{F}\otimes_R R'_w = \mathrm{HT}_w(\mathrm{Fil}_\bullet^\psi)\otimes_{\mathbb{Z}} R'_w$ and $w_i \mod p^w\mathcal{F}\otimes_R R' + \mathrm{Fil}_{i-1}\mathcal{F}\otimes_R R' = \mathrm{HT}_w(x_i \mod \mathrm{Fil}_{i-1}^\psi)$.

We now define functors

$$\begin{aligned}
\mathfrak{IW}_w : R - \mathbf{Adm} &\to SET\\
R' &\mapsto \big\{w - \text{compatible } \mathrm{Fil}_\bullet\mathcal{F}\otimes_R R' \in \mathcal{GR}_{\mathcal{F}}(R')\big\}\\
\mathfrak{IW}_w^+ : R - \mathbf{Adm} &\to SET\\
R' &\mapsto \big\{w - \text{compatible } (\mathrm{Fil}_\bullet\mathcal{F}\otimes_R R', \{w_i\}) \in \mathcal{GR}_{\mathcal{F}}^+(R')\big\}
\end{aligned}$$

These two functors are representable by affine formal schemes.

3. The overconvergent modular sheaves and modular forms

Let Y be the moduli space of principally polarized abelian schemes (A,λ) of dimension g equipped with a principal level N structure over Spec $\mathcal{O}_K$. Let X be a toroidal compactification of Y and $G \to X$ be the semi-abelian scheme extending the universal abelian scheme ([FC]).

The Siegel variety of Iwahori level. Let $Y_{\mathrm{Iw}} \to \operatorname{Spec} \mathcal{O}_K$ be the moduli space parametrizing principally polarized abelian schemes (A,λ) of dimension g, equipped with a level N structure and an Iwahori structure at p: this is the data of a full flag $\mathrm{Fil}_\bullet A[p]$ of the group $A[p]$ satisfying $\mathrm{Fil}_\bullet^\perp = \mathrm{Fil}_{2g-\bullet}$. Let X_{Iw} be a toroidal compactification of this moduli space ([Str]). We can choose the constructions of X and X_{Iw} in such a way that the forgetful map $Y_{\mathrm{Iw}} \to Y$ extends to a map $X_{\mathrm{Iw}} \to X$.

The classical modular sheaves. Let ω_G be the co-normal sheaf of G along its unit section, $\mathcal{T} = \mathrm{Hom}_X(\mathscr{O}_X^g, \omega_G)$ be the space of ω_G and $\mathcal{T}^\times = \mathrm{Isom}_X(\mathscr{O}_X^g, \omega_G)$ be the GL_g-torsor of trivializations of ω_G. We define a left action $\mathrm{GL}_g \times \mathcal{T} \to \mathcal{T}$ by sending $\omega \colon \mathscr{O}_X^g \to \omega_G$ to $\omega \circ h^{-1}$ for any $h \in \mathrm{GL}_g$.

We define an automorphism $\kappa \mapsto \kappa'$ of $X(\mathrm{T})$ by sending any $\kappa = (k_1, \dots, k_g) \in X(\mathrm{T})$ to $\kappa' = (-k_g, -k_{g-1}, \dots, -k_1) \in X(\mathrm{T})$. This automorphism stabilizes the dominant cone $X^+(\mathrm{T})$. Let $\pi \colon \mathcal{T}^\times \to X$ be the projection. For any $\kappa \in X^+(\mathrm{T})$, we let $\omega^\kappa := \pi_* \mathscr{O}_{\mathcal{T}^\times}[\kappa']$ be the sub-sheaf of $\pi_*\mathscr{O}_{\mathcal{T}^\times}$ of κ'-equivariant functions for the action of B (with GL_g acting on the left on $\pi_*\mathscr{O}_{\mathcal{T}^\times}$ by $f(\omega) \mapsto f(\omega g)$ for any section f of $\pi_*\mathscr{O}_{\mathcal{T}^\times}$ viewed has a function over the trivializations ω, and any $g \in \mathrm{GL}_g$). The global sections $\mathrm{H}^0(X, \omega^\kappa)$ form the module of Siegel modular forms of weight κ over X.

We denote by $\mathfrak{X}$ the formal scheme obtained by completing X along its special fiber X_k and by X_{rig} the associated rigid space. We have a Hodge height function $\mathrm{Hdg} \colon X_{\mathrm{rig}} \to [0,1]$. Let $v \in [0,1]$, we set $\mathcal{X}(v) = \{x \in X^{\mathrm{rig}},\ \mathrm{Hdg}(x) \le v\}$, this is an open subset of X_{rig}. Let $v \in v(\mathcal{O}_K)$. There is a formal model $\mathfrak{X}(v)$ of $\mathcal{X}(v)$.

We have a canonical subgroup H_n of level n over $\mathcal{X}(v)$. Let $\mathcal{X}_1(p^n)(v) = \mathrm{Isom}_{\mathcal{X}(v)}((\mathbb{Z}/p^n\mathbb{Z})^g, H_n^D)$ be the finite étale cover of $\mathcal{X}(v)$ parametrizing trivializations of H_n^D. We let ψ be the universal trivialization over $\mathcal{X}(v)$. Let $\mathfrak{X}_1(p^n)(v)$ be the normalization of $\mathfrak{X}(v)$ in $\mathcal{X}_1(p^n)(v)$. The group $\mathrm{GL}_g(\mathbb{Z}/p^n\mathbb{Z})$ acts on $\mathfrak{X}_1(p^n)(v)$. We let $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$ be the quotient $\mathfrak{X}_1(p^n)(v)/\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$. It is also the normalization of $\mathfrak{X}(v)$ in $\mathcal{X}_1(p^n)(v)/\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$. The formal schemes $\mathfrak{X}_1(p^n)(v)$ and $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$ have nice modular interpretations away from the boundary.

By proposition 2.3, there is a rank g locally free sub-sheaf $\mathcal{F}$ of $\omega_G/_{\mathfrak{X}_1(p^n)(v)}$. It is equipped with an isomorphism:

$$(\mathrm{HT}_w \circ \psi) \otimes 1 \colon (\mathbb{Z}/p^n\mathbb{Z})^g \otimes_{\mathbb{Z}} \mathscr{O}_{\mathfrak{X}_1(p^n)(v)}/p^w \mathscr{O}_{\mathfrak{X}_1(p^n)(v)} \simeq \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w}.$$

We have a chain of formal schemes:

$$\mathfrak{IW}_w^+ \overset{\pi_1}{\to} \mathfrak{IW}_w \overset{\pi_2}{\to} \mathfrak{X}_1(p^n)(v) \overset{\pi_3}{\to} \mathfrak{X}_{\mathrm{Iw}}(p^n)(v) \overset{\pi_4}{\to} \mathfrak{X}_{\mathrm{Iw}}(p)(v). \tag{1}$$

Recall that $\mathfrak{IW}_w$ parametrizes flags in the locally free sheaf $\mathcal{F}$ that are w-compatible with ψ and $\mathfrak{X}_1(p^n)(v)$ parametrizes trivializations ψ of H_n^D, π_2 is just the forgetful map to ψ.

Let $\mathfrak{T}_w$ be the formal torus defined by

$$\mathfrak{T}_w(R) = \mathrm{Ker}\big(\mathrm{T}(R) \to \mathrm{T}(R/p^w R)\big)$$

for any object $R \in \mathbf{Adm}$. The formal scheme $\mathfrak{IW}_w^+$ is a torsor over $\mathfrak{IW}_w$ under $\mathfrak{T}_w$. We let $\mathfrak{B}_w$ be the formal group defined by

$$\mathfrak{B}_w(R) = \mathrm{Ker}\big(\mathrm{B}(R) \to \mathrm{B}(R/p^w R)\big)$$

for all $R \in \mathbf{Adm}$. There is a surjective map $\mathfrak{B}_w \to \mathfrak{T}_w$ with kernel the "unipotent radical" $\mathfrak{U}_w$.

The morphisms π_1, π_2, π_3 and π_4 are affine. Set $\pi = \pi_1 \circ \pi_2 \circ \pi_3 \circ \pi_4$. Let $\kappa \in \mathcal{W}(K)$ be a w-analytic character. The involution $\kappa \mapsto \kappa'$ of $X(\mathrm{T})$ extends to an involution of $\mathcal{W}$, mapping w-analytic characters to w-analytic characters. The character $\kappa' \colon \mathrm{T}(\mathbb{Z}_p) \to \mathcal{O}_K^\times$ extends to a character $\kappa' \colon \mathrm{T}(\mathbb{Z}_p)\mathfrak{T}_w \to \widehat{\mathbb{G}_m \mathcal{O}_K^\times}$ and to a character $\kappa' \colon \mathrm{B}(\mathbb{Z}_p)\mathfrak{B}_w \to \widehat{\mathbb{G}_m}$ with $\mathrm{U}(\mathbb{Z}_p)\mathfrak{U}_w$ acting trivially.

Let ϖ be a uniformizer in $\mathcal{O}_K$. If $\mathfrak{X}$ be a flat formal scheme of finite type over Spf $\mathcal{O}_K$, we denote by X_n the scheme over Spec $\mathcal{O}_K/\varpi^n\mathcal{O}_K$ deduced from $\mathfrak{X}$ by reduction modulo ϖ^n.

Definition 3.1. A formal Banach sheaf on $\mathfrak{X}$ is a family of quasi-coherent sheaves $\mathfrak{F} = (\mathfrak{F}_n)_{n\in\mathbb{N}}$ where:

(1) $\mathfrak{F}_n$ is a sheaf on X_n, flat over $\mathcal{O}_K/\varpi^n$.
(2) for all $n \geq m$, if $i : X_m \hookrightarrow X_n$ is the closed immersion, we have $i^*\mathfrak{F}_n = \mathfrak{F}_m$.

Definition 3.2. Let $\mathcal{X}$ be a rigid space and $\mathscr{F}$ be a sheaf on $\mathcal{X}$. We say that $\mathscr{F}$ is a Banach sheaf if:

(1) for all affinoid open sub-set $\mathcal{U}$ of $\mathcal{X}$, $\mathscr{F}(\mathcal{U})$ is a Banach $\mathscr{O}_{\mathcal{X}}(\mathcal{U})$-module,
(2) the restriction maps are continuous,
(3) there exists an admissible affinoid covering $\mathfrak{U} = \{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{X}$ such that for all $i \in I$ and all affinoid $\mathcal{V} \subset \mathcal{U}_i$, the map:

$$\mathscr{O}_{\mathcal{X}}(\mathcal{V})\hat{\otimes}_{\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)}\mathscr{F}(\mathcal{U}_i) \to \mathscr{F}(\mathcal{V})$$

is an isomorphism.

Let $\mathfrak{X}$ be a flat formal scheme locally of finite type over Spf $\mathcal{O}_K$. Let $\mathcal{X}$ be its rigid analytic fiber. Let $\mathfrak{F} = (\mathfrak{F}_n)_{n\in\mathbb{N}}$ be a formal Banach sheaf over $\mathfrak{X}$. We associate to $\mathfrak{F}$ a sheaf $\mathscr{F}$ on $\mathfrak{X}$, valued in the category of K -vector spaces, by setting

$$\mathfrak{U} \mapsto \mathscr{F}(\mathfrak{U}) := \mathfrak{F}(\mathfrak{U}) \otimes_{\mathcal{O}_K} K.$$

For every open subset $\mathfrak{U} \hookrightarrow \mathfrak{X}$, with rigid fiber $\mathcal{U}$, the $\mathscr{O}_{\mathcal{X}}(\mathcal{U})$-module $\mathscr{F}(\mathfrak{U})$ is a Banach module for the norm for which $\mathfrak{F}(\mathfrak{U})$ is the unit ball of $\mathscr{F}(\mathfrak{U})$.

The definition above attaches functorially to every flat formal Banach sheaf $\mathfrak{F}$ on $\mathfrak{X}$ a Banach sheaf $\mathscr{F}$, called the rigid analytic generic fiber of $\mathfrak{F}$.

Definition/Proposition 3.3. *Let $\kappa \in \mathcal{W}$ be a w-analytic character. The w-analytic, v-overconvergent modular forms of weight κ*

$$\mathfrak{w}_w^{\dagger\kappa} := \pi_*\mathscr{O}_{\mathfrak{IW}_w^+}[\kappa']$$

is a formal Banach sheaf.

Definition 3.4. The space of integral w-analytic, v-overconvergent modular forms of genus g, weight κ, principal level N is

$$\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(v)) = \mathrm{H}^0(\mathfrak{X}_{\mathrm{Iw}}(p)(v), \mathfrak{w}_w^{\dagger\kappa}).$$

The space of integral locally analytic overconvergent modular forms of weight κ and principal level N is the inductive limit:

$$\mathrm{M}^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)) = \lim_{v\to 0, w\to\infty} \mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(v)).$$

Taking analytification of (1), we can similarly define $\omega_w^{\dagger\kappa} := \pi_* \mathscr{O}_{\mathfrak{IW}_w^{+,rig}}[\kappa']$ to be the projective Banach sheaf of w-analytic, v-overconvergent weight κ modular forms over $\mathcal{X}_{\mathrm{Iw}}(p)(v)$. It is the Banach sheaf associated to the formal Banach sheaf $\mathfrak{w}_w^{\dagger\kappa}$.

Definition 3.5. The space of w-analytic, v-overconvergent modular forms of weight κ is:

$$\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v)) = \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa}).$$

The space of locally analytic overconvergent modular forms of weight κ is:

$$\mathrm{M}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)) = \lim_{v\to 0, w\to\infty} \mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v)).$$

The space $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$ is a Banach space, for the norm induced by the supremum norm on $\mathfrak{IW}_w^{+,rig}$. Its unit ball is the space $\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(v))$ of integral forms.

The space of w-analytic, v-overconvergent modular forms of weight κ interpolates the space of classical forms of weight κ as follows. If $\kappa \in X_+(\mathrm{T})$, then for the classical automorphic sheaf ω^κ there is a canonical restriction map:

$$\omega^\kappa|_{\mathcal{X}_{\mathrm{Iw}}(p)(v)} \hookrightarrow \omega_w^{\dagger\kappa}.$$

This map is locally for the étale topology isomorphic to the inclusion

$$V_{\kappa'} \hookrightarrow V_{\kappa'}^{w-\mathrm{an}}$$

of the algebraic induction into the analytic induction.

For any $\kappa \in X_+(\mathrm{T})$, we have an inclusion:

$$\mathrm{H}^0(X_{\mathrm{Iw}}, \omega^\kappa) \hookrightarrow \mathrm{M}_w^{\dagger,\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$$

from the space of classical forms of weight κ into the space of w-analytic, v-overconvergent modular forms of weight κ.

Moreover, we have the following independence result for the space of overconvergent modular forms.

Proposition 3.6. $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$ *is independent on the choice of the toroidal compactification.*

4. Overconvergent Hecke algebra and Siegel Eigenvariety

We now define an action of the dilating Hecke algebra at p. For $i = 1, \ldots, g$, let C_i be the moduli scheme over K parametrizing principally polarized abelian schemes A, a level N structure ψ_N, an self-dual flag $\mathrm{Fil}_\bullet A[p]$ of subgroups of $A[p]$ and a lagrangian sub-group $L \subset A[p^2]$ if $i = 1, \ldots, g-1$ or $L \subset A[p]$ if $i = g$, such that $L[p] \oplus \mathrm{Fil}_i A[p] = A[p]$. There are two projections $p_1, p_2 \colon C_i \to Y_{\mathrm{Iw},K}$. The first projection is defined by forgetting L. The second projection is defined by mapping $(A, \psi_N, \mathrm{Fil}_\bullet A[p])$ to $(A/L, \psi'_N, \mathrm{Fil}_\bullet A/L[p])$ where ψ'_N is the image of the level N structure and $\mathrm{Fil}_\bullet A/L[p]$ is defined as follows:

- For $j = 1, \ldots, i$, $\mathrm{Fil}_j A/L[p]$ is simply the image of $\mathrm{Fil}_j A[p]$ in A/L,
- For $j = i+1, \ldots, g$, $\mathrm{Fil}_j A/L[p]$ is the image in A/L of $\mathrm{Fil}_j A[p] + p^{-1}(\mathrm{Fil}_j A[p] \cap pL)$.

As before we consider the analytifications $p_1, p_2 \colon C_i^{\mathrm{an}} \to Y_{\mathrm{Iw}}^{\mathrm{an}}$.

Proposition 4.1 (Fargues, [Far])**.** *Let G be a semi-abelian scheme of dimension g over $\mathcal{O}_K$, generically abelian. Assume that $\mathrm{Hdg}(G) < \frac{p-2}{2p-2}$. Let H_1 be the canonical sub-group of level 1 of G and let L be a sub-group of $G_K[p]$ such that $H_1 \oplus L = G_K[p]$. Then $\mathrm{Hdg}(G/L) = \frac{1}{p}\mathrm{Hdg}(G)$, and $G[p]/L$ is the canonical sub-group of level 1 of G/L.*

Let $\mathcal{C}_g(v) = C_g^{\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw}}^{\mathrm{an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. If $v < \frac{p-2}{2p-2}$, by the previous proposition, we have a diagram:

$$\begin{array}{ccc} & \mathcal{C}_g(v) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(\frac{v}{p}) \end{array}$$

We see the operator $U_{p,g}$ corresponding to the above diagram improves the radius of overconvergence. For $U_{p,i}$, $1 \leq i \leq g-1$, each of them improves analyticity corresponding to a simple root of GL_g, we will go not into the details of that but conclude:

The composite operator $\prod_{i=1}^{g} U_{p,i}$ induces a map from $\mathrm{M}_{\underline{w}'}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(\frac{v}{p})) \to \mathrm{M}_{w}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(v))$ where $\underline{w}' = (w'_{i,j})$ is defined by:

$$w'_{i,j} = i - j + w_{i,j}.$$

The natural restriction map $res \colon \mathrm{M}_{w}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(v)) \to \mathrm{M}_{\underline{w}'}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(\frac{v}{p}))$ is compact. We let $U = \prod_{i=1}^{g} U_{p,i} \circ res$. This is a compact endomorphism of $\mathrm{M}_{w}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(v))$.

For all $q \nmid pN$, let $\mathbb{T}_q$ be the spherical Hecke algebra

$$\mathbb{Z}[\mathrm{GSp}_{2g}(\mathbb{Q}_q)/\mathrm{GSp}_{2g}(\mathbb{Z}_q)].$$

Let $\mathbb{T}^{Np}$ be the restricted tensor product of the algebras $\mathbb{T}_q$. There is an action of $\mathbb{T}^{Np}$ on the Fréchet space $\mathrm{M}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p))$. Consider the dilating Hecke algebra, $\mathbb{U}_p$, defined as the polynomial algebra over $\mathbb{Z}$ with indeterminates $X_1, \dots, X_g$. There is also an action of $\mathbb{U}_p$, sending X_i to $U_{p,i}$. We claimed that the operator $U = \prod_i U_{p,i}$ is compact. Let us denote by $\mathbb{T}^{\dagger\kappa}$ the image of $\mathbb{T}^{Np} \otimes_{\mathbb{Z}} \mathbb{U}_p$ in $\mathrm{End}\big(\mathrm{M}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p))\big)$ and call it the overconvergent Hecke algebra of weight κ.

For any affinoid open subset $\mathcal{U}$ of $\mathcal{W}$, there exists $w_U > 0$ such that there is a universal character $\kappa^{\mathrm{un}} \colon \mathrm{T}(\mathbb{Z}_p) \times \mathcal{U} \to \mathbb{C}_p^{\times}$ extends to an analytic character $\kappa^{\mathrm{un}} \colon \mathrm{T}(\mathbb{Z}_p)(1 + p^{w_{\mathcal{U}}}\mathcal{O}_{\mathbb{C}_p}) \times \mathcal{U} \to \mathbb{C}_p^{\times}$.

The construction discussed in last section works in families:

Proposition 4.2. *Suppose $w \geq w_U$. There exists a sheaf $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ on $\mathcal{X}_{\mathrm{Iw}}(p)(v) \times \mathcal{U}$ such that for any weight $\kappa \in \mathcal{U}$, the fiber of $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ over $\mathcal{X}_{\mathrm{Iw}}(p)(v) \times \{\kappa\}$ is $\omega_w^{\dagger\kappa}$.*

Let A be the ring of rigid analytic functions on $\mathcal{U}$. Let $M_{v,w}$ be the A-Banach module $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v) \times \mathcal{U}, \omega_w^{\dagger\kappa^{\mathrm{un}}})$. Passing to the limit on v and w we get the A-Fréchet space $M^{\dagger} = \lim\limits_{v \to 0, w \to \infty} M_{v,w}$.

The geometric definition of Hecke operators works in families. We have an action of the Hecke algebra of level prime to Np, $\mathbb{T}^{Np}$ on the space $M_{v,w}$. We also have an action of $\mathbb{U}_p$, the dilating Hecke algebra at p, on $M_{v,w}$ for v small enough.

Let D be the boundary in $\mathcal{X}_{\mathrm{Iw}}(v)$. We let $\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)$ be the cuspidal sub-sheaf of $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ of sections vanishing along D. Let $M_{v,w,\mathrm{cusp}}$ be the A-Banach module $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v) \times \mathcal{U}, \omega_w^{\dagger\kappa}(-D))$ and $M^{\dagger}_{\mathrm{cusp}} = \lim\limits_{v \to 0, w \to \infty} M_{v,w,\mathrm{cusp}}$. All these modules are stable under the action of the Hecke algebra.

Using Buzzard's eigenvariety machine [Buz], we can construct a local piece of eigenvariety $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ out of the discussed eigenvariety data above. We conclude by the following theorem of [AIP].

Theorem 4.3 (Andreatta, Iovita and Pilloni)**.** *There is a equidimensional eigenvariety $\mathcal{E}$ and a locally finite map to the weight space $w \colon \mathcal{E} \to \mathcal{W}$. For any $\kappa \in \mathcal{W}$, $w^{-1}(\kappa)$ is in bijection with the eigensystems of $\mathbb{T}^{Np} \otimes_{\mathbb{Z}} \mathbb{U}_p$ acting on the space of finite slope locally analytic overconvergent cuspidal modular forms of weight κ.*

If f is a weight κ cuspidal eigenform of Iwahori level at p, then there is an affinoid neighbourhood $\mathcal{U}$ of $\kappa \in \mathcal{W}$, a finite surjective map of rigid analytic varieties $w\colon \mathcal{E}_f \to \mathcal{U}$, and $\mathcal{E}_f$ is equidimensional of dimension g.

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