AIP Classicality Notes

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1 Introduction

In these notes we seek to understand the generalization of the results of Coleman that small slope overconvergent modular forms are classical. Specifically Coleman proves the following result.

Theorem 1.1. *(Coleman) Let f be an overconvergent p-adic U_p-eigenform on $X_1(N)$ of weight k with $(N,p) = 1$, define the slope of f to be the $v_p(\lambda_p)$ where $U_P \cdot f = \lambda_p f$. If the slope of f for the action of the U_p operator is $< k-1$, then f is classical, i.e. it arises from a modular form on $X_{\Gamma(N;p)}$, where $\Gamma_{(N;p)} := \Gamma_1(N) \cap \Gamma_0(p)$.*

In fact this bound is very nearly optimal: ignoring the nebentypus, if f is a modular form for $X_0(N \cdot p)$ then if $f = \sum_n a_n q^n$ is old, then $f(\tau), f(p\tau)$ generate a U_p stable subspace of the modular forms of weight k and level $\Gamma(N;p)$, on this subspace U_p has the explicit form $\begin{pmatrix} a_p & p^{k-1} \\ -1 & 0 \end{pmatrix}$, so we see that the slopes of f must be bounded by $k-1$ and indeed that this bound is achieved by an ordinary oldform. On the other hand if f is new then it must therefore be Steinberg at p, an explicit computation with the Steinberg representation then shows us that $U_p \cdot f$ should be, possibly up to a root of unity, just $p^{\frac{k-2}{2}} \cdot f$.

Coleman also makes the following observation which handles the edge case of slope $k-1$.

Theorem 1.2. *(Coleman) Let f as in Theorem 1.1, but assume that f is an eigenform for the Hecke algebra and that its slope is $k-1$. Then f is classical as long as there is no overconvergent modular form g of weight $2-k$ such that $\theta^{k-1} g = f$.*

In this study Coleman makes the following very beautiful observation, which implies the above Theorems 1.1, 1.2 via an analysis of the commutation of the U_p and θ operators.

Theorem 1.3. *(Coleman) Let f as in Theorem 1.1, but assume that f is an eigenform for the Hecke algebra and that its weight is $k+2$ for $k \geqslant 0$. Then f is classical as long as there is no overconvergent modular form g of weight $-k$ such that $\theta^{k+1} g = f$.*

These theorems are the historical and ideological starting point of our study of classicality of Siegel modular forms.

2 A Prototype of Classicality

2.1 A Dual BGG Resolution

Here we will discuss the mainspring of the classicality arguments in AIP. We formulate a ‘classicality’ result for analytic vectors in associated representations of $G = \mathrm{Gl}_g$, and later we will show that this gives, in a precise sense, an etale local model for the classicality argument of AIP, from which the full statement will follow without much pain.

Let $G = \mathrm{Gl}_g(\mathbb{Q}_p)$, fix once and for all $B \subset G$ the Borel of upper triangular matrices, and within it the maximal split torus T of diagonal matrices. We will denote by N the unipotent radical of B, and by U^{op} its opposite with respect to T. With respect to this datum denote by $X^*(T)^+$ the set of dominant cocharacters of the torus T, and by Δ the set of positive simple roots. Let I denote the Iwahori subgroup associated to B, and let $N^{op} = I \cap U^{op}$. Let $L/\mathbb{Q}_p$ be some mostly irrelevant finite extension of $\mathbb{Q}_p$

Recall from Weibo’s talk that for $k \in X^*(T)^+$ a dominant classical weight we have defined the module V_k to be the classical algebraic induction of the weight k from the Borel. We also defined, for $w > 0$ the spaces $V_k^{w-\mathrm{an}}$, which we defined to be the subset of the set of functions $\{f : I \to L \mid f(ib) = f(i) \cdot k(b), b \in B(\mathbb{Z}_p), i \in I\}$, such that f is *w-analytic*, i.e. $f|_{N^{op}}$ is analytic on a tube of radius p^{-w} around $N^{op} \subset \mathbb{A}^{\frac{g(g-1)}{2}}$. We will denote by $V_k^{l-\mathrm{an}}$ the co-limit of all the $V_k^{w-\mathrm{an}}$, and by V_k^{an} the analytic version (with no restriction on the radius of analyticity). All of these spaces clearly admit the natural left-regular action of I.

We have, from the work of Jones, the following theorem.

Theorem 2.1. *There exists an I-equivariant exact sequence:*

$$0 \to V_k \to V_k^{an} \to \oplus_{\alpha \in \Delta} V_{s_\alpha \cdot k}^{an}$$

Where here s_α is the reflection associated to the simple root α and $\cdot$ denotes the dotted Weyl group action $s_\alpha \cdot k = s_\alpha(k + \rho) - \rho$, where ρ is the half sum of positive roots. The same holds with the condition V_k^{an} replaced by a w-analytic assumption.

Let us refer to the first map in this sequence by d_0 (it is the obvious inclusion), and refer to the second by d_1. As our ‘classicality’ is simply lying in the image of d_0, this gives us an obvious path towards classicality: given an analytic function in V_k^{an} we simply apply d_1 to determine whether or not it is classical, using the exactness of the sequence.

We can start by being a little more explicit about d_1. Let $\iota : V_k^{an} \hookrightarrow L^{I,an}$ be the embedding from V_k^{an} into all analtyic functions on I with values in L.

This embedding is clearly I-equivariant, and so we can differentiate the action of I on V_k^{an} to obtain an action of $\mathfrak{g}$ on $L^{I,an}$, and thus an action of the universal enveloping algebra $U_L(\mathfrak{g})$. We can now let elements of the universal enveloping algebra act on V_k^{an}, viewed via its embedding ι. Let $\theta_\alpha = X_{-\alpha}^{\langle k,\alpha^\vee\rangle+1}$ be one of these operators, where $X_{-\alpha}$ is the infinitessimial generator of the $-\alpha$ root direction. It is our claim that $d_1 = \oplus_\alpha \theta_\alpha$, but first we must make sure this is well defined.

Proposition 2.1. *The map $\theta_\alpha : V_k^{an} \to V_{s_\alpha \cdot k}^{an}$ is well-defined.*

Proof. First let v be highest weight for V_k^{an}, then we have that $X_\beta \theta_\alpha v = [X_\beta, \theta_\alpha]v$ because v is annihilated by X_β, for any $\beta \in \Delta$. Now we note that $[X_\beta, \theta_\alpha] = 0$ unless $\alpha = \beta$, in which case

$$[X_\alpha, X_{-\alpha}^{\langle k,\alpha^\vee\rangle+1}] = X_{-\alpha}^{\langle k,\alpha^\vee\rangle+1} X_\alpha - X_{-\alpha}^{\langle k,\alpha^\vee\rangle+1} X_\alpha + \sum_{i=1}^{\langle k,\alpha^\vee\rangle+1} X_{-\alpha}^{i-1} H_\alpha X_{-\alpha}^{\langle k,\alpha^\vee\rangle+1-i} \tag{1}$$

$$= \sum_{i=1}^{\langle k,\alpha^\vee\rangle+1} -2\left(\langle k,\alpha^\vee\rangle + 1 - i\right) X_{-\alpha}^{\langle k,\alpha^\vee\rangle} \tag{2}$$

$$= \left(\langle k,\alpha^\vee\rangle + 1\right) X_{-\alpha}^{\langle k,\alpha^\vee\rangle} H_\alpha - \sum_{i=1}^{\langle k,\alpha^\vee\rangle+1} -2\left(\langle k,\alpha^\vee\rangle + 1 - i\right) X_{-\alpha}^{\langle k,\alpha^\vee\rangle} \tag{3}$$

$$= \left(\langle k,\alpha^\vee\rangle + 1\right) X_{-\alpha}^{\langle k,\alpha^\vee\rangle} \left(H_\alpha - \langle k,\alpha^\vee\rangle\right) \tag{4}$$

where here $H_\alpha = [X_\alpha, X_{-\alpha}]$ is the infinitesimal generator of the cartan subgroup corresponding to α. As the last factor in the last line annihilates v, $\theta_\alpha v$ is still highest weight.

To check that it has the correct eigenvalue for a $t \in T$, simply note that

$$H_\alpha \theta_\alpha v = \langle k,\alpha^\vee\rangle \theta_\alpha v - 2\left(\langle k,\alpha^\vee\rangle + 1\right) \theta_\alpha v \tag{5}$$

$$= (-2 - \langle k,\alpha^\vee\rangle) v \tag{6}$$

$$= \langle s_\alpha \cdot k, \alpha^\vee\rangle v \tag{7}$$

as desired. For $\beta \neq \alpha$ the action of H_β is unchanged. Upon exponentiation we are done. □

2.2 Classicality

We introduce here operators δ_i for $i = 1, \ldots, g-1$ on V_k^{an} which will, etale locally, play the role of the $U_{p,i}$ introduced in Weibo's talk. Let $\delta_i = \begin{pmatrix} p^{-1}\mathrm{Id}_{g-i} & 0 \\ 0 & \mathrm{Id}_i \end{pmatrix}$, δ_i acts by conjugation on Gl_g and preserves the borel subgroup B (but not the Iwahori!) so we have an induced action of δ_i on V_k. Using the isomorphism induced by $f \to f|_{N^{op}}$ for $f \in V_k^{an}$, we get $V_k^{an} \cong L^{N^{op},\mathrm{an}}$. Note that δ_i acts on the latter space, we use this isomorphism to pullback and obtain an action of δ_i on V_k^{an}. The map d_0 is equivariant for this action, and the operators induced by δ_i are norm decreasing and compact on V_k^{an}.

We now need to investigate the action of δ_i on $\theta_\alpha f$, this follows in a manner similar to our previous analysis of the θ operator. We have that: $\delta_i \cdot \theta_\alpha f = \alpha(\delta_i)^{\langle k,\alpha^\vee\rangle+1}\theta_\alpha \delta_i f(n)$, using the fact that the right regular action of the torus commutes with the left action of the Iwahori. Now we can prove our model classicality theorem.

Proposition 2.2. *Let $k = (k_1, \ldots, k_g) \in X^*(T)^+$ be a weight, and let $v = (v_1, \ldots, v_{g-1})$ be such that $v_{g-i} = k_i - k_{i+1} + 1$ for all $1 \leqslant i \leqslant g-1$. Then if we denote $V_k^{an,<v}$ the sum of the generalized eigenspaces of δ_i on which the slopes of δ_i are bounded by v_i, then $V_k^{l\text{-}an,<v} \subset d_0(V_k)$.*

Proof. Because the δ_i are acting by nonzero eigenvalues, any v which is a mutual eigenvector of the δ_i is automatically analytic if it is locally analytic, because the contracting property of the δ_i allow us to analytically continue v to all of N^{op}.

For simplicity let v be a mutual eigenvector of the δ_i, then for all i we have that $\delta_i \theta_\alpha v = \alpha(\delta_i)^{\langle k,\alpha^\vee\rangle+1}\theta_\alpha\delta_i v = \lambda_i \alpha(\delta_i)^{\langle k,\alpha^\vee\rangle+1}\theta_\alpha v$ for all α. But if we let α be the simple root $(g-i, g-i+1)$ then $\theta_\alpha v$ is an eigenvector for δ_i with eigenvalue $\lambda_i \cdot \alpha(\delta_i)^{\langle k,\alpha^\vee\rangle+1} = \lambda_i p^{-(k_{g-i}-k_{g-i+1}+1)}$, which has negative valuation by our bound on the valuation of λ_i. This is impossible unless $\theta_\alpha v = 0$ as δ_i is contracting, and thus has operator norm less than or equal to 1. Repeating this for all i we see that $\theta_\alpha v = 0$ for all α, and thus v is classical. □

3 Sheaves of Overconvergent Modular Forms

Recall that for k a weight, w an analtyticity, and v an upper bound on the Hodge invariant, satisfying natural restrictions, we have constructed $\omega_w^{k,\dagger}$ a sheaf on $\mathcal{X}(p)(v)$ consisting of overconvergent Siegel modular forms of weight k and analyticity w. For the purposes of this talk k will be fixed and classical, and v will also be fixed. We summarize what we know about this from Weibo's talk.

Proposition 3.1. *The following are true when k is classical and k' is its opposite weight.*

1. *There is an embedding of sheaves $\omega^k|_{\mathcal{X}_I(p)(v)} \hookrightarrow \omega_w^{k,\dagger}$.*
2. *The above is etale locally modeled by $\mathcal{O}_{\mathcal{X}_I(p)(v)}\hat{\otimes}V_{k'} \hookrightarrow \mathcal{O}_{\mathcal{X}_I(p)(v)}\hat{\otimes}V_{k'}^{w-an}$.*

We also have the following pseudo-theorem, which is a simplification of many results in AIP, which is useful for the purpose of this talk.

Theorem 3.1. *The following are true.*

1. *There is a larger sheaf X and an I-equivariant embedding of sheaves $\omega_w^{k,\dagger} \hookrightarrow X$, where X is a large sheaf with an action of $\mathfrak{g}$, it roughly plays the role of all analytic functions on the Iwahori in this sheaf theoretic setting.*

2. *The action of $U_{p,i}$ on $\omega_w^{k,\dagger}$ for $1 \leqslant i \leqslant g-1$ is modeled on stalks by the action of $\delta_{g-i} \otimes 1$ on $\mathcal{O}_{\mathcal{X}_I(p)(v)} \hat{\otimes} V_{k'}^{w-an}$.*

4 Classicality on the Level of Sheaves

From the above pseudo-theorems we get an action of $U(\mathfrak{g})$ such that $X_{-\alpha}^{\langle k,\alpha^\vee \rangle+1}$ takes $\omega_w^{k,\dagger}$ to $\omega_w^{s_\alpha \cdot k,\dagger}$ as before, where the second map is etale locally modeled by $1 \otimes d_1$, and the first by $1 \otimes d_0$. So we can define a sequence:

$$0 \to \omega^k \to \omega_w^{\dagger,k} \to \oplus_{\alpha \in \Delta} \omega_w^{\dagger, s_\alpha \cdot k}.$$

Proposition 4.1. *There is an exact sequence of sheaves.*

Proof. Essentially this follows from the fact that this can be checked etale locally, and the fact that $\mathrm{im}(d_1)$ splits off inside of this sequence locally means that we don't have issues with completed tensor products. □

We can show the following now, again it will follow formally from our pseudo-theorems.

Proposition 4.2. *Let $v \leqslant \frac{p-2}{2p^2-p}$, then we have a commutative diagram for all $1 \leqslant i \leqslant g-1$.*

$$\begin{array}{ccc} H^0(\mathcal{X}_I(p)(v), \omega_w^{\dagger,k}) & \xrightarrow{\theta_\alpha} & H^0(\mathcal{X}_I(p)(v), \omega_w^{\dagger,s_\alpha \cdot k}) \\ \downarrow{\scriptstyle U_{p,i}} & & \downarrow{\scriptstyle \alpha(\delta_{g-i})^{\langle k,\alpha^\vee \rangle} U_{p,i}} \\ H^0(\mathcal{X}_I(p)(v), \omega_w^{\dagger,k}) & \xrightarrow{\theta_\alpha} & H^0(\mathcal{X}_I(p)(v), \omega_w^{\dagger,s_\alpha \cdot k}) \end{array}$$

Proof. Once again, this simply follows from computing locally, where it reduces to what we've already done. The condition on the Hodge height simply ensures that the appropriate comparison to the representation theory is permissible. □

Proposition 4.3. *Let $M_w^{\dagger,k,<\underline{v}}$ be the subspace of $M_w^{\dagger,k} = H^0(\mathcal{X}(p)(v), \omega_w^{\dagger,k})$ where all the generalized eigenspaces of the $U_{p,i}$ have finite slope for $U_{p,g}$ and have the slopes of the $U_{p,i}$ eigenvalues bounded by $v_i = k_{g-i} - k_{g-i+1} + 1$, then $M_w^{\dagger,k,<\underline{v}} \subset H^0(\mathcal{X}(p)(v), \omega^k)$.*

Proof. For simplicity we show the statement if $f \in M_w^{\dagger,k,<\underline{v}}$ is a mutual $U_{p,i}$ eigenform for all $1 \leqslant i \leqslant g$. In this case f is analytic by using the contracting property of $U_p = \prod_i U_{p,i}$ and the analytic continuation implied by $U_p f = Cf$ with $C \neq 0$. Define a norm on $M_w^{\dagger,k,<\underline{v}}$ by taking the sup norm on only the ordinary locus $\mathcal{X}_I^{ord} \cap \mathcal{X}_I(p)(v)$; it can be shown by comparison with Katz/Hida's space of overconvergent modular forms that the $U_{p,i}$ are norm-decreasing for this norm, although $M_w^{\dagger,k,<\underline{v}}$ is no longer complete.

As before we have:

$$U_{p,g-i}\theta_\alpha f = p^{k_{i+1}-k_i-1}\theta_\alpha U_{p,i} f \quad (8)$$
$$= a_i p^{k_{i+1}-k_i-1}\theta_\alpha f \quad (9)$$

which means that $\theta_\alpha f$ must be 0 as $U_{p,i}$ must be contracting. □

Classicality now follows from the following theorem of Bijakowski-Pilloni-Stroh.

Theorem 4.1. *Let* $v_g = k_g - \frac{(g+1)g}{2}$*, then* $H^0(\mathcal{X}_I(p)(v), \omega^k)^{<v_g} \subset H^0(X_I, \omega^k)$.

The above results technically allows us to win, but it is the main result of a very complex paper. We will spend the rest of these notes trying to explain this type of analytic continuation result for small slope overconvergent eigenforms.

5 Analytic Continuation

5.1 Coordinates on the Modular Curve

The advantage, to me, of working the results in this section only for Gl_2 is that it is possible to be much more explicit with our coordinates and precisely how they are effected by the U_p-operator. In addition probably everyone is much more familiar with the p-group schemes which occur inside of elliptic curves, than those that occur in general abelian varieties.

For the purposes of this section let $K/\mathbb{Q}_p$ be a finite extension, and $\mathcal{X}$ be the rigid analytic space over K associated to the (usual integral model X of) the compact modular curve at level $\Gamma_0(p) \cap \Gamma_1(N)$ for $(N, p) = 1$. The only thing we will need to know specifically about our choice of integral/formal model is that the special fiber $X_{\overline{\mathbb{F}}_p} = X_{\Gamma_1(N)} \coprod_{X^{ss}_{\Gamma_1(N)}} X_{\Gamma_1(N)}$ is two copies of the special fiber for the modular curve with only the tame level glued together over the supersingular locus to make simple nodes. We will only use this fact for illustrative purposes, but it adds a nice sense of geometric realism to the argument.

5.1.1 The Degree of a P-Torsion Group Scheme

We will introduce some new coordinates on our modular curve; they are related to the coordinates given by the Hodge invariant, but carry a bit more information at Iwahori level.

Definition 1. Let $\Lambda/\mathcal{O}_K$ be a finite torsion module. By the standard structure theorem Λ has a presentation $\Lambda = \oplus_i \mathcal{O}/(a_i)\mathcal{O}$, define $\deg(\Lambda) := \sum_i v_p(a_i)$, where $v_p(p) = 1$. If $H/\mathcal{O}_K$ is a finite flat group scheme then define $\deg(H) = \deg(\Omega_H)$.

We have the following easy consequence of this definition.

Proposition 5.1. *Let* $\lambda : A \to A'$ *be a p-power isogeny over* $\mathcal{O}_K$ *with kernel* H*, then* $deg(H) = v_p(det(\lambda^*))$*, where* $\lambda^* : \Omega_{A'} \to \Omega_A$*. Note that while the value of the determinant is not well-defined with this data, its valuation is. As a trivial consequence we see that* $deg(A[p]) = ng$*, where* $g = dim(A)$*.*

For instance $\mathbb{Z}/p\mathbb{Z}/\mathcal{O}_K$ has no tangent space to begin with, whereas μ_p does, but does not have a flat tangent space over $\mathbb{Z}/p^2\mathbb{Z}$. These schemes have degree $0, 1$ respectively, and these prototypical examples will be very important for us.

As a helpful point of comparison, if H_n is the level n canonical subgroup of an Abelian variety A of dimension g, then we have $\deg(H_n) = ng - \frac{p^n-1}{p-1}\mathrm{Hdg}(A)$. This clarifies the relationship between our new coordinates and the coordinates on the Siegel moduli space that we've been using in this seminar.

Definition 2. Let $P = (E, H)$[1] be a point on X. Define $\deg(P) = \deg(H)$ if E spreads out and reduces well modulo p. Otherwise let $\deg(E, H) = 0$ or 1 depending on whether E has additive or multiplicative reduction.

As is tradition, now that we have a new coordinate on the modular curve we have to spend some time explaining the results of Katz-Lubin in our context.

Proposition 5.2. *Let $P = (E, H)$ be a point on $\mathcal{X}$, then let $Alt(P) = \{(E, H')|H' \neq H\}$. We have the following statements, the first two of which are due to Katz-Lubin, the third and fourth to Fargues.*

1. *If $deg(P) > \frac{1}{p+1}$ then for any $P \neq P' \in Alt(P)$ we have $deg(P') = \frac{1-deg(P)}{p} < \frac{1}{p+1}$. Conversely if $deg(P) < \frac{1}{p+1}$ then there is a unique subgroup C_1 such that $deg(E, C_1) = 1 - pdeg(P) > \frac{1}{p+1}$, this is the canonical subgroup.*

2. *If $deg(P) = \frac{1}{p+1}$ then the same is true for all $P' \in Alt(P)$*

3. *If $0 \to G' \to G \to G'' \to 0$ is an exact sequence of finite flat group schemes then $deg(G) = deg(G') + deg(G'')$.*

4. *If $\phi : G \to G'$ is an morphism of finite flat group schemes which is generically an isomorphism, then $deg(G') \geqslant deg(G)$ with equality if and only if ϕ is an isomorphism.*

Further if $deg(P) < \frac{1}{p^{n-1}(p+1)}$, then (E, H) admits a first canonical subgroup C_1, taking $deg((E/C_1, \bar{H})) = 1 - deg(C_1) = pdeg(E, H) < \frac{1}{p^{n-2}(p+1)}$ so iterating E admits an nth order canonical subgroup.

5.1.2 A Geometric Picture: The Key Strategy

Given all this information we can form quasicompact admissible open subsets, for any $I = [p, q] \subset [0, 1]$ with $p, q \in \mathbb{Q}$, denote by $\mathcal{X}I$ the subset of $\mathcal{X}$ on which the deg function takes values in I. We can now give the following nice description of $\mathcal{X}$: $\mathcal{X}[0, 0], \mathcal{X}[1, 1]$ are the ordinary-etale and ordinary-multiplicative loci respectively, and they form two components which have the form of rigid tubes over $X_{\Gamma_1(N)}$ minus the supersingular discs. They are glued together across the supersingular annuli $\mathcal{X}(0, 1)$.

[1]Now, as always, we will suppress the tame level.

This geometric picture in hand, we can now sketch our strategy for analytic continuation of overconvergent eigenforms. Let f be a finite slope over convergent eigenform, then f will be defined on $\mathcal{X}[1-\epsilon, 1]$ for some $\epsilon > 0$. We will show that the U_p operator is contracting: i.e. on points it sends $P \in \mathcal{X}$ to a sum of points P' which are strictly 'closer' to $\mathcal{X}[1,1]$. As f is finite slope we can define a new function $F = \frac{1}{a_p} U_p(f)$ with a_p f's U_p eigenvalue. On the domain of definition of f F agrees with it, but because of the contracting property of U_p, F will be defined on a slightly larger region. Eventually we can get f to be defined on all of $\mathcal{X}(0,1]$.

Now assuming f has small slope we will then construct a formula for the value of f on all points of $\mathcal{X}[0,0]$ which f will have to satisfy if it is a classical U_p eigenform of small slope. We use this to get an analytic function F on $\mathcal{X}[0,0]$ which we will "glue", using a clever idea of Kassaei, to the extended f on $\mathcal{X}(0,1]$, despite the fact that these opens have empty intersection on the level of classical points.

5.2 Extending Finite Slope Eigenforms

To make the above ideas rigorous we will usually ignore what is happening at the cusps: as the crux of the argument takes place over the very close to the supersingular annuli or in their interior this will not cause us much harm. So often we might define a procedure involving the moduli problem of $\mathcal{Y}$, the interior of the modular curve, and pretend that the construction goes through on $\mathcal{X}$.

Define the space $\mathcal{X}^p = \{(E,H,C) | H, X \subset E[p], H \neq C\}$. This space, as in previous talks, carries an etale correspondence over $\mathcal{X}$ given by $p_1(E,H,C) = (E,H)$ and $p_2(E,H,C) = (E/C, \bar{H})$. On a given sheaf $\mathcal{F}$ this comes from the morphism from $p_2^*\mathcal{F}$ to $p_1^*\mathcal{F}$ given by $\pi_C : E \to E/C$, the natural universal isogeny. A modular form, viewed as a section of the kth power of the Hodge bundle, has the following natural action of an operator U_p coming from this correspondence:

$$U_p \cdot f = \frac{1}{p} \sum_{(E,H,C)} \pi_C^*(f(E/C, \bar{H})),$$

obtained by pulling back along p_2 and taking the trace along p_1. We can do this with our overconvergent sheaves of modular forms, and for any $U_1, U_2 \subset \mathcal{X}$ such that $p_1^{-1}(U_1) \subset p_2^{-1}(U_2)$ we get a map $\omega^{\dagger,k}(U_2) \to \omega^{\dagger,k}(U_1)$.

To begin with: note that by part 4 of Proposition 5.2 U_p weakly increases degrees on the entire $\mathcal{X}$, in the sense that for $P \in \mathcal{X}$ we have the $\deg(E/C, \bar{H}) \geqslant \deg(P)$ for all P of good reduction, and any appropriate C which shows up in the definition of U_p. We will now show that this is strict on the supersingular locus.

Proposition 5.3. *This increase is strict and bounded below on any subset $\mathcal{X}[\alpha, 1-\epsilon]$, $\alpha, \epsilon > 0$ of the supersingular locus.*

Proof. Suppose it were not strict, then $\bar{H} \cong H$. This implies that $E[p] \cong H \oplus C$, this is impossible in the non-ordinary setting for all sorts of reasons: for one it cannot be true on the special fiber as there is no splitting of $0 \to \alpha_p \to E[p] \to \alpha_p \to 0$ when E is supersingular. For another reason it is known that the degree of a truncated Barsotti-Tate group over $\mathcal{O}_K$ is always an integer by a theorem of Pilloni, and if H were a summand of $E[p]$ this would be true.

Now take λ_p the universal p-isogeny over $\mathcal{X}^p$, and τ to be the universal p-isogeny over $\mathcal{X}$. This τ gives a canonical section δ of $\mathcal{L} := \Omega_E \otimes \Omega_{E/H}^{-1}$ over $\mathcal{X}$. This gives us a section $\delta_0 = p_1^*(\delta) \otimes p_2^*(\delta)^{-1}$ of $p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})^{-1}$ over $\mathcal{X}^p[\alpha, 1-\epsilon] := p_1^{-1}(\mathcal{X}[\alpha, \epsilon])$ with $\alpha, \epsilon \in \mathbb{Q}$. Because we know the degree increase induced by λ_p is strict, we have that $v_p(\delta_0) = v_p(p_1^*\delta) - v_p(p_2^*(\delta))$ which gives over any fiber (E, H, C): $\deg((E, H)) - \deg((E/C, \bar{H}) < 0$ for all $(E, H, C) \in \mathcal{X}^p[\alpha, 1-\epsilon]$, over this quasicompact domain this function must attain a maximum value t by the MMP, thus U_p strictly increases degree by t over this domain. □

Thus for any eigenform f of finite slope defined on $\mathcal{X}[1-\epsilon, 1]$ and any $\alpha > 0$ there is some $n > 0$ such that $U_p^n f$ is naturally defined on $\mathcal{X}[\alpha, 1]$, and thus so is f. So we can define any overconvergent finite slope modular form over $\mathcal{X}(0, 1]$.

5.3 A Formula for the Ordinary-Etale Locus

Now we need to understand how to construct f on $\mathcal{X}[0, 0]$, for this a new idea is required. We note the following.

Proposition 5.4. *Let f be classical with U_p eigenvalue a_p of slope $< k-1$, then on $(E, H) \in \mathcal{X}[0, 0]$, where we have all canonical subgroups, f can be represented as*

$$f(E, H) = \sum_{n=1}^{\infty} \sum_{\substack{C_{n-1} \subset D \subset E[p^n] \\ D \neq H, C_n}} \frac{1}{(pa_p)^n} pr_D^*(f(E/D, \bar{H})).$$

Proof. Let f be a classical U_p eigenform, and let (E, H) ordinary etale, this implies for instance that H is not the canonical subgroup. Then for all $C_1 \neq D \subset E[p]$ we have that $(E/D, \bar{H})$ is in $\mathcal{X}[1, 1]$, we can see this from the sequence $0 \to D \to E[p] \to \bar{H} \to 0$, which implies that $\deg(\bar{H}) = 1$. The case of C_1 is more complicated, which is why we separate it out. We thus get the expression $a_p \cdot f(E, H) = \frac{1}{p} \sum_{D \subset E[p], D \neq H, C_1} pr_D^* f(E/D, \bar{H}) + \frac{1}{p} pr_{C_1}^* f(E/C_1, \bar{H})$, continuing this process we get the desired expression.

The only thing left to check is that this sum converges. But this is also fairly clear: $v_p(\det(pr_{C_n}^*)) = \deg(C_n) \cdot k = nk$, thus p^{nk} divides the term coming from D which contain C_n. Thus the valuation of each term is $nk - n - nv_p(a_p) = n[(k-1) - v_p(a_p)]$ so this series converges when $v_p(a_p) < k-1$. □

Our statement of "Classicality on the level of Sheaves" gave us functions which are defined on an analogue of $\mathcal{X}(0, 1]$, to conclude we want to replicate this procedure to obtain an expression for f on the locus $\mathcal{X}[0, 0]$ if f is non-classical but defined on this region. The sum above requires arbitrarily deep canonical subgroups to make sense, so a priori it only coheres on $\mathcal{X}[0,0]$ and does not overconverge, however each nth partial sum is well defined on $\mathcal{X}[0, \frac{1}{p^n(p+1)}]$, so we can define the entire sum modulo p^{nk} on a small collared neighborhood around $\mathcal{X}[0,0]$, and by a clever argument this will be enough to glue.

First note the following: if one has an etale correspondence $\mathcal{X}^p$, then if $p_1^{-1}(U)$ breaks up as $p_1^{-1}(U) = V_1 \coprod V_2$ with $V_i = p_2^{-1}(U_i)$ then we can decompose our correspondence into a sum of two correspondences, each supported on V_i. Carrying this out explicitly with $U_1 = \mathcal{X}[1,1], U_2 = \mathcal{X}[0,0]$, with $U = \mathcal{X}[0,0]$. Using this we write $U_p|_{\mathcal{X}[0,0]} = U_p^{sp} + U_p^{nsp}$ where U_p^{sp} is the isogeny correspondence coming from the canonical subgroup, and U_p^{nsp} corresponds to the less interesting subgroups. Note that on sections $U^{nsp} : \omega^k(\mathcal{X}[1,1]) \to \omega^k(\mathcal{X}[0,0])$ whereas $U^{sp} : \omega^k(\mathcal{X}[0,0]) \to \omega^k(\mathcal{X}[0,0])$, and U_p is the sum of these two maps $U_p : \omega^k(\mathcal{X}[0,0]) \oplus \omega^k(\mathcal{X}[1,1]) \to \omega^k(\mathcal{X}[0,0])$.

Let f a slope $< k-1$ overconvergent U_p eigenform defined on $\mathcal{X}(0,1]$ Define

$$F = \sum_{n=1}^{\infty} \frac{1}{(pa_p)^n} (U^{sp})^{n-1} U^{nsp} f|_{\mathcal{X}[1,1]},$$

this gives a well-defined expression for f on $\mathcal{X}[0,0]$, which converges for the same reason as before. Let F_m denote its mth partial sum.

5.4 A Gluing Lemma of Kassaei

The following lemma is a first step on our journey towards gluing.

Proposition 5.5. *We have the following analytic facts which will allow us to glue.*

1. *F_m are uniformly bounded.*

2. *$|f - F_m|$ goes to zero on $\mathcal{X}(0, \frac{1}{p^m(p+1)}]$*

3. *As m goes to ∞, $|F_{m+1} - F_m| \to 0$.*

Proof. We bound f successively and use the control of f to control the F and F_m. Fix $1 > \epsilon > 0$ a very small rational number, let $U_n = \mathcal{X}[0, \frac{1-\epsilon}{(p+1)p^n}]$, and let $V_n = U_n - U_{n+1}$. Note that each U_n is quasicompact and f is defined on U_n, so f can be bounded on any U_n individually, note also that F_n overconverges to U_n for all n. Note that $U_p(V_0)$ is entirely contained in $\mathcal{X}[\frac{1}{p+1}, 1]$ from the equation $\deg(E/D, \bar{H}) = 1 - \deg(D)$, and the fact that $\deg(D) = \deg(E,H)$ unless D is canonical. If D is canonical then $\deg(D) = \frac{1-\deg(E,H)}{p} < \frac{1}{p}(1 - \frac{1-\epsilon}{p^2(p+1)}) = \frac{1}{p+1}(1 + \frac{1}{p} - \frac{1}{p^2} + \frac{\epsilon}{p^2}) \leqslant \frac{p}{p+1}$ easily, whence $\deg(E/D, \bar{H}) \geqslant \frac{1}{p+1}$. For any $U \subset U_1$ let d_U be a uniform lower bound for the degree of the canonical subgroup on this region.

By our understanding of the canonical subgroup, we know that if $U^{sp}Q \in U \subset U_1$, then for a form f we have $|U^{sp}f(Q)| = |\frac{1}{p}pr^*_{C_1}f(U^{sp}(Q))| \leqslant p^{1-kd_{U^{sp,-1}(U)}}|f|_U$, this bound is the key input for our estimates.

First note that we can bound f on V_0 by bounding f on $U_p(V_0)$ and using $\frac{1}{a_p}(U_p(f)) = f$. Further since F_1 is defined on U_1 which is quasicompact, we can bound this as well. Let M be a mutual bound for these functions. So f is bounded by M on V_0 in particular.

Assume for induction that f is bounded on V_{n-1} by $Mp^{k(\frac{1-\epsilon}{p(p+1)}+\cdots+\frac{1-\epsilon}{p^{n-1}(p+1)})}$. One can see by a variant of our above analysis that on V_n the degree of the canonical subgroup is bounded below by $1-\frac{1}{(p+1)p^n}$, plugging this in to the above bound we see that

$$|\frac{1}{a_p}U^{sp}f|_{V_n} \leqslant p^{v_p(a_p)+1-k(1-\frac{1-\epsilon}{(p+1)p^n})}|f|_{V_{n-1}} \tag{10}$$

$$\leqslant Mp^{k(1-\epsilon)(\frac{1}{p(p+1)}+\cdots+\frac{1}{p^n(p+1)})} \tag{11}$$

Here we derive that $|f|_{V_n} \leqslant \mathrm{Max}(|f-\frac{1}{a_p}U^{sp}f|_{V_n}, |\frac{1}{a_p}U^{sp}f|_{V_n}) \leqslant Mp^{k(\frac{1}{p(p+1)}+\cdots+\frac{1}{p^n})}$ as desired. Whence $|f|$ is bounded on $\mathcal{X}(0,\frac{1}{(p+1)}]$ by $Mp^{\frac{k}{(p+1)(p-1)}}$, by writing this region as $\bigcup_n V_n$ and using the bounds on each term of this coproduct.

Obviously since $F_m = f-\frac{1}{a_p^n}U^{sp,n}f$ we can easily bound F_m by noticing[2] that $|\frac{1}{a_p^n}U^{sp,n}f|_{\mathcal{X}(0,\frac{(1-\epsilon)}{(p+1)p^n}]} \leqslant p^{nv_p(a_p)+n-k(n-\frac{p(1-\epsilon)}{(p+1)(p-1)})} = p^{n(v_p(a_p)-(k-1))}C$, since this goes to 0 in n we have our uniform bound on U_n, by noting that the F_n converge on $\mathcal{X}[0,0]$. Even more clearly this also bounds $|f-F_n|$ on U_n as desired. The statement about the difference $|F_n - F_{n+1}|$ clearly follows from the above estimates in a similar fashion. □

We now sketch the coup de grâce of Kassaei's argument. Since everything in sight is bounded we can restrict f, F_n to small open U around the boundary of the supersingular annuli and perform the gluing there. By rescaling we can assume that f, F_m are sections of the sheaf $\mathcal{O}^{\leqslant 1}$ of sections of the structure sheaf with norm less than or equal to 1. Choosing a subsequence of F_n we can assume that on $(U_n-\mathcal{X}[0,0])\cap U$ that $|f-F_n| < \frac{1}{p^n}$, and thus f, F_n glue to form a section $\hat{f}_n$ of $\mathcal{O}^{\leqslant 1}/p^n$ on $U^\dagger := \mathcal{X}[0,\frac{1-\epsilon}{(p+1)}] \cap U$. A powerful theorem of Bartenwerfer tells us that if Z is a smooth quasicompact rigid space then there is a constant c with $|c| \leqslant 1$ such that $cH^1(Z,\mathcal{O}^{\leqslant 1}) = 0$, thus $c\hat{f}_n \in \mathcal{O}^{\leqslant 1}(U^\dagger)/p^n\mathcal{O}^{\leqslant 1}(U^\dagger)$ gives a compatible family, whence we obtain $f \in \mathcal{O}^{\leqslant 1}(U^\dagger)$, this is our desired section. So we obtain $f \in H^0(\mathcal{X},\omega^k)$ and by rigid GAGA f is classical.

As near as I can tell this is exactly the same technique, modulo much more elaborate bookkeeping, that Bijakowski-Pilloni-Stroh use to prove their aforementioned theorem, so hopefully this summary does a good job of demystifying the technique required.

[2] We can use induction and the same technique we used to bound $|\frac{1}{a_p}U^{sp}f|$ above.

References

[1] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. p-adic families of siegel modular cuspforms. *Annals of Mathematics*, page 623–697, 2015.

[2] Robert F. Coleman. Classical and overconvergent modular forms. *Journal de Théorie des Nombres de Bordeaux*, 7(1):333–365, 1995.

[3] Payman L Kassaei. A gluing lemma and overconvergent modular forms. *Duke Mathematical Journal*, 132(3):509–529, 2006.