Automorphic Reps of GSp_4:

Let $F/\mathbb{Q}$ be a totally real field and $\mathbb{A}$ denote the adeles of F. G/F conn. reductive group.

$\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ idèle class character.

$L^2_{disc}(G(F) \backslash G(\mathbb{A}), \chi)$ the space of χ-equivariant square integrable functions which decompose discretely under action of $G(\mathbb{A})$.

$\to$ $\Pi(G, \chi)$ be the space of irred. autom. reps occuring as constituents.

Goal: Describe $\Pi(G, \chi)$ for $G = GSp_4$.

Langland's tells us that we should be able to partition this by continuous maps:

$$\varphi : \Gamma_F = Gal(\bar{F}/F) \longrightarrow \check{G}(\mathbb{C}) \text{ modulo conjugation}$$

$$G = GSp_4 \Rightarrow \check{G} = GSpin_5 \cong GSp_4 .$$

How can we understand this?

$$\begin{tikzcd} \varphi : \Gamma_F \arrow[r] \arrow[rrr, bend right] & GSp_4(\mathbb{C}) \cong \check{G}(\mathbb{C}) \arrow[r] & Gl_4(\mathbb{C}) \cong \check{Gl_4}(\mathbb{C}) \end{tikzcd}$$

We can classify these compositions using the rep theory of finite groups.

Functorality tells us that these compositions should be assoc. to lifts to auto. reps of $Gl_4(\mathbb{A})$.

Idea (1) Classify which maps $\Gamma_F \to Gl_4(\mathbb{C})$ arise via this procedure.

(2) Describe the lifts to auto. reps of Gl_4.

Note any, arising from a map:

$\varphi : \Gamma \to Gl_4(\mathbb{C})$.

should inherit a Γ-invariant symplectic similitude form.

$\Rightarrow$ $\mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$, where $\mathbb{C}$ is acted on via an idele class character χ.

$$\Rightarrow \tilde{\varphi} : \Gamma \to \tilde{\check{G}} = Gl_4(\mathbb{C}) \times Gl_1(\mathbb{C})$$
$$(g \mapsto (\varphi(g), \chi(g)))$$

We should consider the set of all parameters $\tilde{\varphi} : \Gamma \to \tilde{G}$ satisfying some symplectic condition on the form.

$$\tilde{\varphi} = \varphi \oplus \chi.$$

We can apply the rep theory of finite groups

$$\varphi = l_1 \varphi_1 \oplus \cdots \oplus l_n \varphi_n \quad \varphi_i : \Gamma_F \to Gl(N_i, \mathbb{C})$$

(irred.)

$$\sum l_i N_i = 4.$$

$$\check{\alpha} : \check{G} \to \tilde{\check{G}}$$
$$(g, z) \mapsto (g^\vee z, z)$$
$$g^\vee = (g^{-1})^t$$

We say a parameter $\tilde{\varphi}$ is $\check{\alpha}$-stable if $\tilde{\varphi}$ is conjugate to φ.

Concretely, $\exists$ an involution $\{1, \ldots, n\} \to \{1, \ldots, n\}$
$i \longmapsto i^\vee$

of these indices s.t. $l_i = l_{i^\vee}$ $\varphi_i = \varphi_{i^\vee}$ $\varphi_i^\vee \otimes \chi \cong \varphi_{i^\vee}$.

$\Rightarrow \forall i \quad \varphi_i(\sigma)^\vee \cdot \chi(\sigma) = A_i^{-1} \varphi_i(\sigma) A_i$.

Applying $(-)^\vee$ to both sides and using Schur's lemma:

$${}^t A_i = c_i A \quad c_i^2 = 1,$$

if all the $c_i = -1$ we say the form is symplectic.

i.e. A_i defines a Γ-invariant symplectic similitude pairing, w. respect to χ on φ_i $\forall i = 1, \ldots, n$.

We say a parameter $\tilde{\varphi}$ is $\check{\alpha}$-discrete if all the $l_i = 1$.

Now $\Pi_a(G, \chi)$ should be partitioned by all $\check{\alpha}$-discrete symplectic parameters.

Not quite:

L_F Langland's dual group of F.

$W_{F_v} \longrightarrow L_{F_v}$ if v is finite

$W_{F_v} \times SU(2)$ if v is archimedean.

We consider parameters ~~factoring~~

$$\tilde{\varphi} : L_F \times Sl(2, \mathbb{C}) \to \check{\tilde{G}} = Gl(4, \mathbb{C}) \times \mathbb{C}^\times$$

which are $\check{\alpha}$-discrete + symplectic, get similar classification, where the φ_i decompose as:

$\varphi_i = \mu_i \otimes \nu_i$, where $\mu_i : L_F \to Gl(m_i, \mathbb{C})$.

$\nu_i : Sl(2, \mathbb{C}) \to Gl(n_i, \mathbb{C})$ ind. rep. automatically self-dual.

ν_i is symplectic or orthogonal depending on whether it is an even or odd-dimensional

ϕ_i is symplectic iff μ_i is symplectic and ν_i is odd

μ_i is orthogonal and ν_i is even.

$\rightsquigarrow$ $\Psi_2(G, \chi)$ denote the space of such parameters which are α-discrete.
~~where the Gl_1 - fac~~
Write $\nu(n)$ be the ired rep. of dimension n.
Roughly: Expect a partition of $L^2_{disc}(G(F)\backslash G(\mathbb{A}), \chi)$

by these parameters.

To describe this, we need to understand some finer structure related to the sign of the functional-equation.

Given $\tilde{\phi} = \phi \oplus \chi \in \Psi_2(G, \chi)$

$\cancel{\mathcal{S}_\phi = S_\phi / S}$

$S_\phi = \mathrm{Cent}_{Gl_4}(\mathrm{Im}(\phi_\phi))$

$\mathcal{S}_\phi = S_\phi / Z(Gl_4) S_\phi^0$

If ϕ has r-components then:

$\mathcal{S}_\phi \cong (\mathbb{Z}/2\mathbb{Z})^{r-1}$

$r = 2, 1$

$\varepsilon_\phi : \mathcal{S}_\phi \to \mathbb{Z}/2\mathbb{Z}$ defined using symplectic root numbers.

Ex: $\phi = \phi_1 \oplus \phi_2$
ired 2-dim'l $\mathcal{S}_\phi = \{ (a, b) \in \mathbb{C}^* \times \mathbb{C}^* \subseteq Gl_2(\mathbb{C}) \times Gl_2(\mathbb{C}) \mid a^2 = b^2 \}$

We have a natural map

$$s \mapsto s_v$$
$$\mathcal{S}_\chi \to \mathcal{S}_{\chi_v}$$

$$\chi_v : W_{F_v} \times SL(2,\mathbb{C}) \to L_F \times SL(2,\mathbb{C}) \xrightarrow{\chi} GL_4(\mathbb{C}).$$
$$(W_{F_v} \times SU(2,\mathbb{R})).$$

Thm: (Gan - Takeda, for the finite places, Vogan-Mœglin for real places.)

$$\overset{\vee}{\mathcal{S}}_{\chi_v} \leftrightarrow \Pi_{\chi_v} \quad \text{local } L\text{-packet}.$$

$\vee \to$ character group

$$\xi_{\pi_v} \mapsto \pi_v \quad \pi_v \text{ smooth irred. rep of } G(F_v).$$

pinned down using a choice of Whittaker datum and character relations given by Langlands-Shelstad

$$\mathbb{1} \to \text{unique generic element}.$$

Have an explicit construction using θ-lifting.

global L-packet

$$\Pi_\chi = \left\{ \pi = \bigotimes_v \pi_v \;\middle|\; \pi_v \in \Pi_{\chi_v},\ \xi_{\pi_v} = 1 \text{ for almost all } v \right\}.$$

Thm: (Arthur 2004, Gee - Taibi 2018, using GT 2001

We have an explicit decomposition:

$$L^2_{disc}(G(F)\backslash G(\mathbb{A}), X) = \bigoplus_{\chi \in \Psi_2(G,X)} \bigoplus_{\{\pi \in \Pi_\chi,\ \xi_\pi = \varepsilon_\pi\}} \pi$$

$$\xi_\pi : s \mapsto \prod_v \xi_{\pi_v}(s_v).$$

s.t. (a) The global packets are disjoint
no irred. rep is a member of 2-packets

(b) If χ is trivial on $SL(2,\mathbb{C})$ then any $\pi \in \Pi_\chi$ occurs w/ multiplicity 1 or 0.)

<u>Def'n</u>: We say π a rep. of $GSp_4(\mathbb{A})$ is globally generic if all $\xi_{\pi_v} = 1$ $\forall v$. (Have well defined L-functions by work of Shahidi)

We can classify reps using the possible decompositions of the parameter ϕ.

Before we do this, let's mention some things related to temperedness. ϕ.

$\phi_v : W_{F_v} \times Sl(2, \mathbb{C}) \to Gl(n, \mathbb{C})$ · v finite

$\nu \boxtimes V(n)$ where this is irred.

$W_{F_v} \to W_{F_v} \times Sl(2, \mathbb{C}) \to Gl(n, \mathbb{C})$

$g \longmapsto \left(g, \begin{pmatrix} |g|^{1/2} & \\ & |g|^{-1/2} \end{pmatrix}\right) \to$

$\phi_v^{ss}(w_v)$ is a ~~semi-simple matrix~~ diagonalizable matrix.

$c_v(\nu_i)$ denotes this diagonalizable matrix.

Similarly $c(v_i) = v_i \begin{pmatrix} |w_v|^{1/2} & \\ & |w_v|^{-1/2} \end{pmatrix}$

Given a symplectic i-discrete parameter $\phi : L_F \times Sl(2, \mathbb{C}) \to Gl(n, \mathbb{C})$.

we expect $\phi = \bigoplus_{i=1}^{\sim} \phi_i$. $\phi_i = \nu_i \otimes v(n_i)$ $N_i = m_i \cdot n_i$.

$\nu_i : L_F \to Gl(m_i, \mathbb{C})$
irred. χ self-dual symplectic

$\Updownarrow$ χ-self-dual.

unitary cuspidal auto reps. of $Gl(m_i, \mathbb{A})$ symplectic type.

We know that these reps should be classified by Satake parameters:

$$c(\nu_i) = \{ c_v(\nu_i) \mid v \notin V_{\nu_i} \}$$

which are given by $\begin{pmatrix} c_v(\nu_i) & & \\ & \ddots & \\ & & c_v(\nu_i) \end{pmatrix}$

$$W_{F_v} \longrightarrow L_F \xrightarrow{\nu_i} Gl(m_i, \mathbb{C}).$$

$\Downarrow$

~~diagonalize~~ eigenvalues of Frob

Rmk: The generalized Ramanujan conjecture tells us that these eigenvalues all lie on the unit-circle. It was observed by Piatetskii-Schapiro + Howe. (Counter-examples to Ramanujan conjecture) that For $G = GSp_4$ the $Sl(2, \mathbb{C})$ complicates things

The ~~eigenvalues~~ Satake parameters correspond to diagonalizable matrix given by:

$$W_{F_v} \to W_{F_v} \times Sl(2, \mathbb{C}) \to L_F \times Sl(2,\mathbb{C}) \xrightarrow{\chi_i} Gl(N_i, \mathbb{C})$$

$$g \longmapsto \left(g,\ \begin{pmatrix} |g|^{1/2} & \\ & |g|^{-1/2} \end{pmatrix} \right)$$

If $\chi_i = \nu_i \otimes r_i \qquad r_i = r(n_i)$ ← irred rep of $Sl(2,\mathbb{C})$ of dimension n_i.

$$c(r_i) = \left\{ c_v(r_i) = r_i \begin{pmatrix} |w_v|^{1/2} & \\ & |w_v|^{-1/2} \end{pmatrix} \right.$$

$$= \begin{pmatrix} |w|_v^{\frac{n-1}{2}} & & \\ & \ddots & \\ & & |w_v|^{-\frac{n-1}{2}} \end{pmatrix}$$

$$c(\varphi_i) = \left\{ c_v(\varphi_i) = c_v(\omega_i) \otimes_v c(v_i) \mid v \notin V_{\omega_i} \right\}$$

$$c(\varphi) = \left\{ \widetilde{\bigoplus_{i=1}} c_v(\varphi_i) \mid v \notin V_\varphi \right\}$$

These are the correct Satake parameters for the auto. reps associated to $G = GSp_4$.

Ex: (Howe; Piatetskii-Schapiro type.)

$$\varphi = \varphi_1 \boxplus \varphi_2 = (\lambda_1 \boxtimes v(2)) \boxplus (\lambda_2 \otimes v(2)).$$

λ_1 + λ_2 idèle class characters

$$\lambda_1^2 = \chi = \lambda_2^2$$

$$c(\varphi) = (c(\lambda_1) \otimes c(v(2)) \oplus (c(\lambda_2) \otimes v(2))$$

$$= \left\{ \begin{pmatrix} c_v(\lambda_1)|\omega|^{\frac{1}{2}} & & & \\ & c_v(\lambda_2)|\omega|^{1/2} & & \\ & & c_v(\lambda_2)|\omega_v|^{-1/2} & \\ & & & c_v(\lambda_1)|\omega_v|^{-1/2} \end{pmatrix} \right.$$

$\leadsto$ gives counter-examples to Ramanujan conjecture!

Now, as mentioned before, we are interested in globally generic reps.

~~We allow~~

~~Fact & Let G~~

Conjecture (Shahidi) (Asgari - Shahidi proven for $G = GSp_4$)

Let $\pi = \bigotimes \pi_v$ be a globally generic rep. Then π satisfies the generalized Ramanujan conjecture.

$\Rightarrow$ We should be looking at the reps, where the $Sl(2, \mathbb{C})$-factor is trivial. (i.e the tempered reps.)

Two types:

General type: $\psi = \nu \boxtimes 1$, ν is a self-dual unitary cuspidal auto. rep. of Gl_4 that is symplectic χ self-dual.

Yoshida-type: $\psi = \nu_1 \boxtimes 1 \oplus \nu_2 \boxtimes 1$

ν_1, ν_2 are distinct cuspidal auto. reps whose central characters satisfy $\chi_{\nu_1} = \chi_{\nu_2} = \chi$.

LSZ paper considers reps. of this form. $F = \mathbb{Q}$.

Fix an archimedean place v.

Let ξ be the highest weight of an algebraic rep of $GSp_4/\mathbb{Q}$.

$$\Pi_\xi \text{ discrete series L-packet} \leftrightarrow S_{\psi_v}^\vee$$

$\Pi_\xi = \{$smooth irreducible reps of $GSp_4(\mathbb{R})$ V whose $(\mathfrak{g}, K)$ cohomology w/ coefficients in V is non-zero.$\} \to C^\bullet(\mathrm{Hom}_K(\Lambda^\bullet \mathfrak{g}/\mathfrak{k}, V))$

non-holomorphic generic rep. $\leftrightarrow \mathbb{1}$

Rmk: Have an explicit construction of these using L^2-Dolbeault cohomology of a bundle on G/T defined by ξ.

Weights Ω, of $GSp_4(\mathbb{R})$ can be thought of as $(n_1, n_2; c)$ $n_1 + n_2 \equiv c\ (2)$ $\left(\begin{array}{l} V^{a,b} \\ (a+b)\chi_1 \\ +b\chi_2 \end{array} \right.$

$$\begin{pmatrix} sz_1 & & & \\ & sz_2 & & \\ & & sz_2^{-1} & \\ & & & sz_1^{-1} \end{pmatrix} \longmapsto z_1^{n_1} z_2^{n_2} s^c$$

$(a,b) = (k_2-3, k_1-k_2)$

(n_2, n_1-n_2)

This is dominant if $n_1 \geq n_2 \geq 0$.

We ~~consider~~ ~~Θ_f~~ assume Π is an automorphic rep as before where Π_∞ is the generic/non-holomorphic discrete series rep. of weight $(n_1, n_2; n_1+n_2)$ ~~k~~

L_λ, $0 \leq i \leq 3$ are the irred. $M_\circ$-reps w/ the following highest weights:

$\rightsquigarrow$ lift of $\pi_1 \boxplus \pi_2$ where π_1 + π_2 are generated by hol. modular forms of weight $n_1 + n_2 + 4$ $n_1 - n_2 + 2$.

We want to realize Π_f these in the coherent cohomology of ~~some~~ a Shimura variety.

Let $K \subseteq G(\mathbb{A}_f)$ be a ~~suitable variety~~ sufficiently small compact open.

We consider the Siegel 3-fold:

2×2 symmetric matrices whose det is positive def.

$$Y_{G,\mathbb{Q}}(\mathbb{C}) \cong G(\mathbb{Q})_+ \backslash \mathcal{H}_2 \times G(\mathbb{A}_f) / K.$$

$Y_{G,\mathbb{Q}}$ canonical integral model.

We let $Y_G \hookrightarrow X_G$ be a suitable toroidal compactif.
we saw in Gyujin's talk that $\exists$ a canonical semi-abelian variety A_G extending the usual abelian variety over X_G.

~~We have a natu~~

Let P_S be the Siegel parabolic w/ Levi factor M.

Get a P_S-torsor T_G parametrizing ~~trivializations~~

We get a P-torsor over Y_G parametrizing trivializations $H^1_{dR}(A/Y_G)$ ~~as a~~ w/ its Hodge filtration. (logarithmic de-Rham complex.)

Gyujin. Get an extension of this filtered bundle to X_G.

Let T_G denote the extension of this ~~vector bundle~~ torsor to all of X_G.

Given an algebraic rep V of M_S we get via the map:

$$P_S \to M_S \quad \text{a vector bundle:} \quad (1,0,0.$$

$$[V] := V \times^{P_S} T_G. \qquad V = (1,0;1)\ldots$$

$\to$ KS iso.

$$\left(\Omega^i_{X_G}(\log D_G) \;\middle|\; \begin{array}{cccc} 0 & 1 & 1 & 2 \\ \omega_6(0,0;0) & (2,0;0) & (3,1;0) & 3 \\ & & (3,3,0) & \end{array} \right.$$

We consider

$$\underset{\circlearrowleft\, G(\mathbb{A}_f).}{H^i(X_{G,\mathbb{Q}}^{(\infty)}, [V])} := \varinjlim_{K \to \{1\}} H^i(X_{G,\mathbb{Q}}, [V])$$

Now we consider for $r_1 \geq r_2 \geq 0$ as before the fixed weights:

$$L_0 : \lambda(r_1+3, r_2+3; m) \qquad L_1 : \lambda(r_1+3, 1-r_2; m)$$
$$L_2 : \lambda(r_2+2, -r_1; m) \qquad L_3 : \lambda(-r_2, -r_1; m)$$

We get the following theorem: $\Pi'_f = \Pi_f \otimes \|\nu\|^{\frac{-(r_1+r_2)}{2}}$ similitude char.

<u>Theorem</u>: If Π is of general type $\forall\ 0 \leq i \leq 3$, Π'_f appears w/ mult 1 in $G(\mathbb{A}_f)$-reps:

$$H^i(X_{G,\mathbb{Q}}, [L_i](-D)) \otimes \mathbb{C} \quad \text{and} \quad H^i(X_{G,\mathbb{Q}}, [L_i])$$

it appears as a direct sum of both reps.
If Π is of Yoshida type then the proceeding statements apply for $i = 1, 2$. and it doesn't appear for degree $i = 0, 3$.

<u>Pf</u>: ? Roughly this should be given by some $(\mathfrak{g}, K)$-cohom. calculation.

$$H^i(X_{G,\mathbb{Q}}, \mho^{can}[L_i]) \cong H^i_{(\mathfrak{g}_H, K_H)}(\mathcal{A}(G) \otimes L_i)$$

? described by Arthur's multiplicity formula.