LEARNING SEMINAR ON p-ADIC AUTOMORPHIC FORMS

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Overview. The study of p-adic automorphic forms began with Serre's observation that there are many congruence relations between q-expansions of modular forms of different weight. This was immediately applied to deduce the existence of the Kubota–Leopoldt p-adic L-function.

The next development came from Hida, who constructed families of p-adic modular forms. This work was greatly extended by Coleman, resulting in the construction of the Coleman–Mazur eigencurve: a (rigid analytic) curve whose points parametrize overconvergent p-adic modular forms.

Since then, the theory has been developed for groups beyond GL_2 in many different directions, with far-reaching consequences throughout number theory. We will look at two such developments in this seminar:

- The construction by Andreatta–Iovita–Pilloni of an eigenvariety for GSp_{2n}.
- Pilloni's "higher Hida theory", which interpolates higher coherent cohomology classes in GSp_4.

We will also look at their applications towards constructing p-adic L-functions in the triple product case [AI19] and the GSp_4 case [LPSZ19].

Talks.

1. **Hida theory I**. We first try to understand why Hida theory works. We will unravel what is going on inside Hida's cohomological calculation. The argument is elementary and based on structure of level groups and basic commutative algebra. References: [Hid86a], [Hid86b], [Hid93], [Eme99].

2. **Hida theory II**. In this talk, we take a coherent point of view, namely modular forms are viewed as sections of sheaves on modular curves. This will give a geometric interpretation of the constructions from the previous talk and serve as a basis for the later talks. Finally, construct a two-variable Rankin–Selberg p-adic L-function. References: [Hid93], [Hid04], [Kat73], [Hid88].

3. **Coleman family**. The first goal is to construct Coleman families. We will use the approach of [Pil13] and [AIS14], which is a natural extension of the ideas from the previous talk and gives a good introduction to a more general construction later.

   Secondly, present Buzzard's axiomatic construction of eigenvarieties. In our case, the result is the Coleman–Mazur eigencurve. Explain some of its properties. References: [Pil13], [AIS14], [Buz07].

4. **Geometric background**. This talk consists of important background material as we move beyond the GL_2 case. It has two parts of different flavours.
   (a) Study the geometry of p-divisible groups which occur in Siegel moduli problems. Present the main results from the theory of canonical subgroups. References: [Far11].
   (b) Explain aspects of the geometry of Siegel moduli spaces, in particular their various compactifications. It may be good to concentrate on the Siegel threefold case, which parametrizes principally polarized abelian surfaces. References: [Nam80], [FC90].

5. **AIP I**. This is the first of two talks devoted to the paper [AIP15]. In this talk, we will construct the big sheaves interpolating the automorphic sheaves defined in the previous talk. They will carry Hecke actions, which allows us to define p-adic families. Reference: [AIP15].

6. **Triple product**. We will construct a triple product p-adic L-functions for overconvergent families. In the integral representation of the classical triple product L-functions, nearly holomorphic modular forms naturally show up. Their p-adic analogues are the "nearly overconvergent modular forms", obtained by applying differential operators to overconvergent forms. It is then necessary to interpolate powers of this differential operator. Reference: [AI19], [Urb14], [Liu19].

7. **AIP II**. We will show a classicality result, that small slope overconvergent automorphic form is classical. The proof is divided into two parts, where one proves "classicality at the level of sheaves", using simple representation theory. Extending an overconverging section to the whole Shimura

variety uses the analytic continuation technique of Buzzard and Kassaei. References: [AIP15], [BPS16], [Kas06].

8. **Automorphic background**: This talk consists of important background material on automorphic forms, in particular on GSp_4. The main topics are Arthur's classification, archimedean theory, and coherent cohomology of automorphic vector bundles. Reference: [Art04], [BW00], [Har90].
9. **Higher Hida theory I**: We study the higher Hida theory portion of [Pil20] in detail. The main goal is to state and prove the classicality theorem on the ordinary cohomology of $X^{\geq 1}(p)_{/\mathbb{F}_p}$. The preprint [BP20] may be helpful in clarifying aspects of it in the modular curve case.
10. **Higher Hida theory II**: This is a continuation of the previous talk.
11. **Higher Hida theory III**: We will first define the interpolation sheaves which appeared at the end of the previous talk and state the main theorem of higher Hida theory in the setting of [LPSZ19]. We then explain how this can be used to construct p-adic L-functions for GSp_4. In several parts, we need results from higher Coleman theory, so we will also discuss that briefly.

References

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