# TRIPLE PRODUCT $p$-ADIC $L$-FUNCTIONS

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This is the notes for a talk given at the student seminar on $p$-adic automorphic forms in Fall 2020. We will follow [AI19] and construct a triple product $p$-adic $L$-function for Coleman families.

## 1. Automorphic background

In the first section, we will develop the archimedean theory of the triple product $L$-function, culminating in Ichino's formula expressing its central value as a period integral times explicit local terms. Many of the details are taken from [Hsi20].

1.1. **Triple product $L$-functions.** Let $f = \sum_{n\geq 1} a_n q^n$ be a cuspidal eigenform of weight $k$, level $N$, and character $\chi$, then its $L$-function $L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has an Euler product

$$L(f,s) = \prod_{p\nmid N} \frac{1}{1 - a_p p^{-s} + \chi(p) p^{k-1-2s}} \prod_{p\mid N} L_p(f,s)$$

For $p \nmid N$, let $\alpha_{f,p}, \beta_{f,p}$ be the two roots of $T^2 - a_p T + p^{k-1}$. They are complex numbers of absolute value $p^{\frac{k-1}{2}}$ by the Ramanujan conjecture.

Now suppose for each $i = 1,2,3$, we have an eigenform $f_i \in S_{k_i}(N_i, \chi_i)$. Their triple product $L$-function can be defined by the Euler product

$$L(f_1 \times f_2 \times f_3, s) = \prod_p L_p(f_1 \times f_2 \times f_3, s)$$

where for $p \nmid N_f N_g N_h$, the local factor is

$$L_p(f_1 \times f_2 \times f_3, s) = \prod_{\square \in \{\alpha,\beta\}} (1 - \boxed{1}_{1,p} \boxed{2}_{2,p} \boxed{3}_{3,p} p^{-s})^{-1}$$

This defines a degree 8 $L$-function. On the Galois side, this is the $L$-function attached to the Galois representation $V_{f_1} \otimes V_{f_2} \otimes V_{f_3}$. For this tensor product to be self-dual, we need the condition $\chi_1\chi_2\chi_3 = 1$.

**Theorem 1.1** (Garrett, Piatetski-Shapiro–Ralis [PSR87])**.** *The $L$-function $L(f_1 \times f_2 \times f_3, s)$ has a meromorphic continuation to $\mathbf{C}$ with a functional equation interchanging $s$ and $k_1 + k_1 + k_3 - 2 - s$.*

We are interested in the value of the $L$-function at its centre $s = \frac{k_1+k_2+k_3-2}{2}$. Following Deligne's conjecture, this point is critical if it is an integer. The nature of the Deligne period depends on the relative sizes of the weights $k_i$. More precisely, let

$$\Omega_\infty = \begin{cases} \pi^{2k_i} \langle f_i, f_i \rangle^2 & \text{if } k_i \text{ is at least the sum of the other two weights} \\ \pi^{k_1+k_2+k_3-1} \langle f_1, f_1\rangle^2 \langle f_2, f_2\rangle^2 \langle f_3, f_3\rangle^2 & \text{otherwise} \end{cases}$$

The first case is called the unbalanced, and the second case is called balanced. It is then conjectured that $L\big(f_1 \times f_2 \times f_3, \frac{k_1+k_2+k_3-2}{2}\big)/\Omega_\infty$ is an algebraic number. We will later derive this conjecture from the works of Harris–Kudla [HK91] and Ichino [Ich08].

1.2. **Differential operator.** Let $\mathfrak{H}$ be the complex upper half plane. Let $C_k^\infty(N,\chi)$ denote the set of smooth slowly increasing functions on $\mathfrak{H}$ which transforms like a modular form of level $N$, weight $k$, and character $\chi$. In other words, we replace holomorphy by smooth.

**Definition 1.2.** Let $k \geq 2$. The *Maass–Shimura* differential operator of weight $k$ is

$$\delta_k = \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{k}{2iy}\right) : C_k^\infty(N,\chi) \to C_{k+2}^\infty(N,\chi)$$

For $m \geq 0$, let $\delta_k^m = \delta_{k+2m-2} \circ \cdots \circ \delta_{k+2m} \circ \delta_k$.

Now consider the space

$$N_k^{[m]}(N,\chi) = \bigcup_{i=0}^{m} \delta_{k-2m}^m M_{k-2m}(N,\chi) \subseteq C_k^\infty(N,\chi)$$

It is easy to check that the union is disjoint. We let $N_k(N,\chi) := \bigcup_{m\geq 0} N_k^{[m]}(N,\chi)$. This is the space of *nearly holomorphic* modular forms. The classical theory of Hecke operators extend to them.

1.3. **Ichino's formula.** We state Theorem 4.2 of [DR14], which is a consequence of the works of Harris–Kudla and Ichino using the theta correspondence and the integral representation of the triple product $L$-function.

**Theorem 1.3.** *Let $f_1, f_2, f_3$ be three modular forms. Suppose their levels are divisible by $N$, their weights $k_1, k_2, k_3$ satisfy $k_1 = k_2 + k_3 + 2t$ for some $t \in \mathbf{Z}_{\geq 0}$, and their characters satisfying $\chi_1\chi_2\chi_3 = 1$, then*

$$\frac{\prod_{q|N\infty} C_q}{\pi^{2k}\langle f_1, f_1\rangle_N^2} \cdot L\Big(f_1 \times f_2 \times f_3, \frac{k_1+k_2+k_3-2}{2}\Big) = \left(\frac{\langle f_1, \delta_{k_2}^t f_2 \times f_3\rangle_N}{\langle f_1, f_1\rangle_N}\right)^2$$

*where $\langle -,-\rangle_N$ is the Petersson inner product of level $\Gamma_1(N)$, and $C_q$ are explicit local constants.*

The right hand side can be interpreted as the projection of $\delta_{k_2}^t f_2 \times f_3$ onto the $f_1$-isotypic subspace. If all $f_i$ have algebraic Fourier expansions, then the right hand side is algebraic, and one can check that the local constants are also algebraic, which proves Deligne's algebraicity conjecture in the unbalanced case, *provided that* all $C_q$ are non-zero. However, in general, it is possible that $C_q$ is zero for some $q$.

To remedy this, we need to modify $f_1, f_2, f_3$ to become linear combinations of old forms in their respective isotypic classes. Harris–Kudla [HK91] showed that there is always one combination which makes all local constants non-zero, resolving a conjecture of Jacquet. For the purpose of $p$-adic interpolation, it is necessary to know that such a modification process can be done in family. This is made explicit in Sections 3.4 and 3.5 of [Hsi20], and they depend only on the tame levels.

## 2. Interpolations

The goal of this section is to interpret $\delta^t$ acting on $p$-adic modular forms, where $t$ is allowed to be a $p$-adic weight. This was first used by [Kat78] to construct $p$-adic $L$-functions for Hecke characters over CM-fields, then used by [BDP13] to construct $p$-adic $L$-functions for modular forms twisted by Hecke characters, and finally by [DR14] to construct a triple product $p$-adic $L$-function in the ordinary case. We will begin by following them but switch to the approach of [AI19] and [Liu19] for later applications.

2.1. **Geometric differential operator.** We start with some general construction. Let $\pi : \mathcal{E} \to Y$ be a proper smooth morphism of relative dimension 1. Associated to this data, we can construct

– the relative de Rham sheaf $\mathcal{L} = H^1_{\mathrm{dR}}(\mathcal{E}/Y) := R^1\pi_*\Omega^1_{\mathcal{E}/Y}$.
– a Hodge filtration $0 \to \omega \to \mathcal{L} \to \omega^* \to 0$, where $\omega = \pi_*\Omega^1_{\mathcal{E}/Y}$. This splits non-functorially.
– the Gauss–Manin connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_Y$
– the Kodaira–Spencer map $\mathrm{KS} : \omega^{\otimes 2} \to \Omega^1_Y$.

Concretely, the Gauss–Manin connection over $\mathbf{C}$ can be defined by requiring its flat sections to be $R^1\pi_*\underline{\mathbf{C}}$.

Let $Y = Y_1(N)$ be the open modular curve, and let $\mathcal{E}$ be the universal elliptic curve over $Y$. The $\omega$ in the above setting is exactly the modular sheaf. In this case, the Kodaira–Spencer map is an isomorphism and corresponds to the familiar fact that modular forms of weight 2 correspond to differentials on the modular curve. For $k \geq 1$, let $\mathcal{L}_k = \mathrm{Sym}^k \mathcal{L}$. It has a filtration

$$\omega^{\otimes k} \subseteq \mathcal{L}_1 \otimes \omega^{\otimes(r-1)} \subseteq \cdots \subseteq \mathcal{L}_{k-1} \otimes \omega \subseteq \mathcal{L}_k$$

The Gauss–Manin connection and Kodaira–Spencer isomorphisms gives a weight raising map

$$\nabla : \mathcal{L}_k \xrightarrow{\nabla} \mathcal{L}_k \otimes \Omega^1_Y \xrightarrow{\mathrm{KS}^{-1}} \mathcal{L}_k \otimes \omega^{\otimes 2} \to \mathcal{L}_{k+2}$$

Now given a splitting $\Psi : \mathcal{L} \to \omega$ of the Hodge filtration, we can define

$$\theta_{\Psi,k} : \omega^{\otimes k} \to \mathcal{L}_k \xrightarrow{\nabla} \mathcal{L}_{k+2} \xrightarrow{\Psi} \omega^{\otimes(k+2)}$$

Taking global sections gives a map $\theta_{\Psi,k} : H^0(Y, \omega^{\otimes k}) \to H^0(Y, \omega^{\otimes(k+2)})$.

*Remark* 2.1. We need to extend this picture to the closed modular curve $X = X_1(N)$. This can be done by doing explicit computations over the Tate curve. The result is that in the Gauss–Manin connection and the Kodaira–Spencer isomorphism, we need to replace $\Omega^1_Y$ by $\Omega^1_X(\log Z)$, where $Z = X - Y$. Moreover, $\mathrm{KS}(\omega_{\mathrm{can}}^{\otimes 2}) = \frac{dq}{q}$ over a Tate curve.

In the analytic setting, the covering $\mathcal{E} \to Y$ is a quotient of $\mathcal{E} \to \mathfrak{H}$, where the fibre above $\tau \in \mathfrak{H}$ is $\mathcal{E}_\tau = \mathbf{C}/\langle 1, \tau\rangle$. Let $z$ be the local coordinate on $\mathbf{C}$, then $\omega$ has basis $dz$, and $H^1_{\mathrm{dR}}(\mathcal{E}_\tau) = \mathbf{C}dz \oplus \mathbf{C}d\bar{z}$. The Gauss–Manin connection is explicitly

$$\nabla dz = \left(\frac{dz - d\bar{z}}{\tau - \bar{\tau}}\right) \otimes d\tau$$

The first term on the right hand side is the dual of $\tau \in H_1(\mathcal{E}_\tau, \mathbf{C})$ in $H^1_{\mathrm{dR}}(\mathcal{E}_\tau)$. It is flat as expected since $\nabla$ is integrable. We call the resulting map $\Psi_{\mathrm{Hodge}} : \mathcal{L} \to \mathbf{C}dz$ the *Hodge splitting.*

**Definition 2.2.** The space of (geometric) *nearly holomorphic modular forms* of weight $k$ is the image of

$$\Psi_{\mathrm{Hodge}} : H^0(Y, \mathcal{L}_k^{\mathrm{an}}) \to H^0(Y, \omega_{\mathrm{an}}^{\otimes k})$$

Suppose $f$ is an analytic function on $\mathfrak{H}$ which is a nearly holomorphic modular form of weight $k$ in the classical sense, we can treat it in this framework as the section $\tau \mapsto f(\tau)(2\pi i dz)^k$. The key property is the following theorem, whose proof can be found in [BDP13, Lemma 1.5].

**Theorem 2.3** (Shimura, Katz)**.** *Under this identification, we have* $\theta_{\mathrm{Hodge},r} = \delta_r$.

In the $p$-adic setting, the papers [Kat78, BDP13, DR14] used the unit-root splitting coming from the slope filtration of the Frobenius action. This only works over the ordinary locus, which is enough for their application since they only need to evaluate the $p$-adic modular forms and take ordinary projection. We will need an overconvergent projector, which is a key part of [AI19].

2.2. **Nearly overconvergent modular forms.** Similarly to the transition from $\omega^{\otimes k}$ to $\omega^\kappa$, we will now define a sheaf $\mathbb{W}_\kappa$ with an increasing filtration $\mathrm{Fil}_\bullet \mathbb{W}_\kappa$, which should interpolate $\mathcal{L}_k$. This cannot be a coherent sheaf since we need $k \to \infty$ in the archimedean topology in order for $k \to \kappa$ in the $p$-adic topology. Eventually, we will see that truncation by slope produces makes it coherent.

We will recall some notations from my talk and Weibo's talk. Let $\mathfrak{X}(v)$ be a formal model of the rigid space $X(v) = \{x \in X_{\mathrm{rig}} \,|\, \mathrm{Hdg}(x) < v\}$. Given $n$, for $v$ sufficiently small, we have a partial Igusa tower $\mathfrak{Ig}_{n,v} \to \mathfrak{X}(v)$ parametrizing trivializations of $H^D_{n,\mathcal{E}}$. We constructed a formal scheme $\mathfrak{F}_{n,v}$ over $\mathfrak{Ig}_{n,v}$ which parametrizes sections of $\omega$ satisfying certain congruence conditions (this is $\mathfrak{IW}^+$ from Weibo's talk). The sheaf $\mathfrak{w}^\kappa$ is then $\pi_* \mathcal{O}_{\mathfrak{F}_{n,v}}[\kappa]$. The construction of $\mathbb{W}_\kappa$ replicates this using the rank 2 bundle $\mathcal{L} = H^1_{\mathrm{dR}}(\mathcal{E}/X)$.

**Theorem 2.4.** *There is a diagram of formal scheme* $\pi : \mathfrak{V}_{n,v} \to \mathfrak{Ig}_{n,v} \to \mathcal{X}(v)$ *characterized by the requirement that given a point* $x : \mathrm{Spf}\, R \to \mathfrak{X}(v)$ *corresponding to an elliptic curve* $E_{/R}$*, the fibre of* $\mathfrak{V}_{n,v}$ *above* $x$ *is in bijection with the set*

$$\left\{ (\psi, \rho) \;\middle|\; \begin{array}{l} \psi : H_{n,E_{/R}} \to \mu_{p^n},\ \rho : H^1_{\mathrm{dR}}(E_{/R}) \to \mathrm{Hdg}^{-\frac{p}{p-1}} R \\ \psi_{/R[\frac{1}{p}]} \text{ is an isomorphism} \\ \rho(\mathrm{HT}(\psi)) \equiv 1 \pmod{p^{n - \frac{p^n-1}{p-1}\mathrm{Hdg}}} \end{array} \right\}$$

*It carries the action of the formal group* $\mathfrak{I} = \mathbf{Z}_p^\times \left(1 + p^{n - \frac{p^n-1}{p-1}\mathrm{Hdg}} \mathbf{G}_a\right)$.

**Hypothesis.** In our construction of $\mathfrak{w}^\kappa$, we needed $n$ to grow as $\kappa$ approaches the boundary of the weight space, or equivalently that $\kappa$ extends to an analytic character on $\mathfrak{I}$. We will call those $\kappa$ compatible with $(n, v)$, and it will be a running hypothesis that this is satisfied.

**Definition 2.5.** Let $\mathbb{W} = \pi_* \mathcal{O}_{\mathfrak{V}_{n,v}}$. For a weight $\kappa$

$$\mathbb{W}_\kappa = \mathbb{W}[\kappa] = \pi_* \mathcal{O}_{\mathfrak{V}_{n,v}}[\kappa]$$

This is a formal Banach sheaf on $\mathfrak{X}(v)$. As usual, it is compatible between changing $n$ and $v$.

**Theorem 2.6.** *There is an increasing filtration on* $\mathbb{W}_\kappa$ *such that*

*(1)* $\mathrm{Fil}_0 \mathbb{W}_\kappa \simeq \mathfrak{w}^\kappa$, $\mathrm{gr}_r \mathbb{W}_\kappa \simeq \mathfrak{w}^\kappa \otimes \mathrm{Hdg}^r \omega^{-2r}$. *Both are locally free of rank 1.*

*(2)* $\mathbb{W}_\kappa$ *is the p-adic completion of* $\varinjlim_r \mathrm{Fil}_r \mathbb{W}_\kappa$.

*(3) If $\kappa = k$ is a classical weight, then* $\mathrm{Fil}_k \mathbb{W}_k\left[\frac{1}{p}\right] \simeq \mathrm{Sym}^k \mathcal{L}\left[\frac{1}{p}\right]$

*About the proof.* We will explain the above two theorems by giving a local description of $\mathbb{W}_\kappa$. This requires some more notation. Let $\delta = \mathrm{Hdg}^{\frac{1}{p-1}}$. It essentially follows from [AIP15, Proposition 4.3.1] that it is an ideal sheaf on $\mathfrak{X}(v)$, and $\delta\omega$ is locally free ($\delta\omega = \mathcal{F}$ from Weibo's talk). Let $H^{1,\sharp}_{\mathrm{dR}} = \delta^p H^1_{\mathrm{dR}}(\mathcal{E}/X) + \delta\omega$. This is still locally free of rank 2 over $\mathfrak{X}(v)$. Moreover, the Hodge–Tate map has image inside $H^{1,\sharp}_{\mathrm{dR}}$ with a free cokernel of rank 1. This modification is required to invoke the formalism of vector bundles with marked sections.

Given a point $y : \mathrm{Spf}\, R \to \mathfrak{Ig}_{n,v}$, which determines a map $\psi : H_{n,E/R} \to \mu_{p^n}$. Choose a basis $\{e, f\}$ for $H^{1,\sharp}_{\mathrm{dR}}$ such that $e$ is a lift of $\mathrm{HT}(\psi)$, and let $\{e^\vee, f^\vee\}$ be its dual basis. The fibre of $\mathfrak{V}_{n,v}$ above $y$ can be identified with the set

$$\{ae^\vee + bf^\vee \mid a \in 1 + p^{n-\frac{p^n-1}{p-1}\mathrm{Hdg}}R,\ b \in R\}$$

Let $\beta_n$ be an element of $R$ of valuation $n - \frac{p^n-1}{p-1}\mathrm{Hdg}$. It always exists since the sheaf $\delta = \mathrm{Hdg}^{\frac{1}{p-1}}$ exists. Then we have

$$y^*\mathfrak{V}_{n,v} \simeq \mathrm{Spf}\, R\langle Z, Y\rangle, \quad Z \mapsto \frac{a-1}{\beta_n},\ Y \mapsto b$$

This carries an action of $1 + \beta_n R$, given by $t \cdot (a, b) = (t^{-1}a, t^{-1}b)$. Let $\pi_1$ be the projection to $\mathfrak{Ig}_{n,v}$, then it is easy to check that

$$y^*((\pi_1)_*\mathcal{O}_{\mathfrak{V}_{n,v}}[\kappa]) \simeq (1 + \beta_n Z)^\kappa R\left\langle \frac{Y}{1+\beta_n Z}\right\rangle$$

The filtration is by degree on the Tate algebra part. One can check that $y^*((\pi_1)_*\mathcal{O}_{\mathfrak{F}_{n,v}}[\kappa]) \simeq (1+\beta_n Z)^\kappa R$, so this explains Theorem 2.6 at the partial Igusa tower level. We needed to invert $p$ for the third point to account for the modification $H^{1,\sharp}_{\mathrm{dR}}$.

To actually prove the theorem, we need to further descent down to $\mathfrak{X}(v)$ using the $\mathbf{Z}_p^\times$-action. This is much more subtle than the rational version proven in my Coleman theory talk, even in the $\mathfrak{w}^\kappa$ case. The new input needed is a control of the ramification of the Igusa tower. The full proof can be found in [AIP18, Section 5.3.2] for $\mathfrak{w}^\kappa$ and [AI19, Section 3.3.3] for $\mathbb{W}_\kappa$. □

The ordinary locus is $\mathfrak{X}(0)$ in our notation. Over it, the ideal $\delta$ is the unit ideal. Therefore, our modification $H^{1,\sharp}_{\mathrm{dR}}$ is the same as the usual de Rham sheaf. Moreover, the unit root splitting gives a canonical decomposition

$$H^{1,\sharp}_{\mathrm{dR}} \simeq \omega \oplus \omega^{-1}$$

This induces a map $\mathbb{W}_\kappa \to \mathfrak{w}^\kappa$ over $\mathfrak{X}(0)$. The sheaf used to define Katz $p$-adic modular forms are the same as $\mathfrak{w}^\kappa$, so all together we have a map

$$H^0(\mathfrak{X}(v), \mathbb{W}_\kappa) \to H^0(\mathfrak{X}(0), \mathbb{W}_\kappa) \to H^0(\mathfrak{X}(0), \mathfrak{w}^\kappa) = M_\kappa^{\mathrm{Katz}}$$

**Definition 2.7.** The image of this map is the *nearly overconvergent* modular forms.

Applying this theory to the Tate curve gives $q$-expansions of nearly overconvergent modular forms. There are two such $q$-expansions, which fit into the following diagram

$$\begin{array}{ccc} H^0(\mathfrak{X}(v), \mathbb{W}_\kappa) & \hookrightarrow & H^0(\mathfrak{X}(0), \mathfrak{w}^\kappa) \\ \downarrow & & \downarrow \\ \mathbf{Z}_p[[q]](1+pZ)^\kappa\langle V\rangle & \longrightarrow & \mathbf{Z}_p[[q]] \end{array}$$

Here, $V = \frac{Y}{1+pZ}$ in the notation above. The bottom arrow is simply taking the constant term in $V$. The same diagram also appeared in [Urb14], except there he restricted to polynomials in $V$. The sheaves he constructed are essentially $\mathrm{Fil}_r \mathbb{W}_\kappa$. Following him, we call the value in $\mathbf{Z}_p[[q]]\langle V\rangle$ the *polynomial q-expansion*, and the other one simply $q$-expansion. We will often write $V_{\kappa,r} = Y^r(1+pZ)^{\kappa-r}$, so the polynomial $q$-expansion is written as $\sum_{r\geq 0} a_r(q)V_{\kappa,r}$.

As usual, the Hecke algebra acts on this space. The construction is similar to what we did for $\mathfrak{w}^\kappa$, and the result is an operator

$$U_p : H^0(\mathfrak{X}(v), \mathbb{W}_\kappa) \to H^0(\mathfrak{X}(v), \mathbb{W}_\kappa)\left[\frac{1}{p}\right]$$

which preserves the filtration, is compatible with the classical $U_p$-operator on the $\mathrm{Fil}_0$ piece, and has the expected behaviour on $q$ expansion. With more attention on integrality, we can show that the action of $U_p$

on the $r$-th graded piece is annihilated by $(p/\mathrm{Hdg}^{p+1})^r$. This is enough to apply Fredholm theory, which will later give us a slope decomposition.

2.3. **Gauss–Manin connection.** The other ingredient is a connection $\nabla$ on $\mathbb{W}_\kappa$. This follows quite formally from the corresponding objects on $H^1_{\mathrm{dR}}(\mathcal{E}/X)$. The main theorem is

**Theorem 2.8.** *There is an integrable connection*

$$\nabla_\kappa : \mathbb{W}_\kappa \to \mathbb{W}_\kappa \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}(v)}} \Omega^1_{\mathfrak{X}(v)}\left[\frac{1}{p}\right]$$

*It satisfies Griffith transversality, and the induced map*

$$\mathrm{gr}_r(\mathbb{W}_\kappa)\left[\frac{1}{p}\right] \to \mathrm{gr}_{r+1}(\mathbb{W}_\kappa)\hat{\otimes}_{\mathcal{O}_{\mathfrak{X}(v)}} \Omega^1_{\mathfrak{X}(v)}\left[\frac{1}{p}\right]$$

*is an isomorphism times $u_\kappa - r$, where $u_\kappa$ is the element in $\bar{\mathbf{Q}}_p$ such that $\kappa(t) = t^{u_\kappa}$ for all $t$ close to 1.*

We can compose the rational version of this connection with the Kodaira–Spencer isomorphism $\Omega^1_{\mathfrak{X}} \simeq \omega^{\otimes 2}$ to get a map

$$\nabla_\kappa : \mathbb{W}_\kappa \to \mathbb{W}_{\kappa+2}\left[\frac{1}{p}\right]$$

The denominator grows as $\kappa$ goes near the boundary, which will cause issue for $p$-adic iteration. In fact, a finer analysis shows that the denomiator in the image is bounded above by $\mathrm{Hdg}^{c_\kappa}$, where $c_\kappa$ is a constant depending only on the valuation of $u_\kappa$.

*About the proof.* Once again we do the calculations locally. For this, we will work with a further level of trivialization. Since we are inverting $p$, descent does not cause problem. Let $\mathfrak{Ig}'_{n,v}$ be a normal formal model of the rigid space which parametrizes trivializations $\mathcal{E}^\vee[p^n] \simeq (\mathbf{Z}/p^n\mathbf{Z})^2$. Suppose we have a point $y : \mathrm{Spf}\, R \to \mathfrak{Ig}'_{n,v}$. Fix a basis $\{\omega, \eta\}$ for $H^1_{\mathrm{dR}}(E_{/R})$ such that $\omega$ generates $\omega_{E_{/R}}$. If $\delta = \mathrm{Hdg}^{\frac{1}{p-1}}$, then a basis for $H^{1,\sharp}_{\mathrm{dR}}$ is $\{e = \delta\omega, f = \delta^p\eta\}$, and it is of the type considered in the local description of $\mathbb{W}_\kappa$.

The point $y$ determines a map $(\mathbf{Z}/p^n\mathbf{Z})^2 \to E^\vee_{/R}[p^n]$. This can be used to show that the Gauss–Manin connection restricts to a connection on $H^{1,\sharp}_{\mathrm{dR}}$. To describe this more explicitly, we use Grothendieck's description of integrable connections. Briefly, given a formal scheme $S$, let $P_S$ be the first infinitessimal neighbourhood of the diagonal $\Delta : S \hookrightarrow S \times S$, with two projections $\mathrm{pr}_i : P_S \to S$. An integrable connection $\nabla$ of a module $M$ over $S$ is identified with the map

$$\epsilon : \mathrm{pr}_2^* M \to \mathrm{pr}_1^* M,\ 1 \otimes x \mapsto x \otimes 1 + \nabla(x)$$

Here, we are using $\Omega^1_S \simeq I_\Delta/I_\Delta^2$ to identify $\nabla(x)$ as an element of $\mathrm{pr}_1^* M$. In our case, $S = \mathrm{Spf}\, R$ and $P_S = \mathrm{Spf}\, P_R$ for $P_R = (R\hat{\otimes}R)/I_\Delta^2$. With respect to the bases induced from $\{e, f\}$, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(P_R)$$

be the matrix of $\epsilon^{-1}$ for $M = H^{1,\sharp}_{\mathrm{dR}}$, so $a-1, b, c, d-1 \in I_\Delta$.

Using the notation of the local description for $\mathbb{W}_\kappa$, we have

$$\mathrm{pr}_i^* \mathbb{W}_\kappa = (1+\beta_n Z)^\kappa P_R\langle V\rangle, \quad V = \frac{Y}{1+\beta_n Z}$$

for $i = 1, 2$. Let $\epsilon_\kappa : \mathrm{pr}_2^* \mathbb{W}_\kappa \to \mathrm{pr}_1^* \mathbb{W}_\kappa$ be defined by

$$Y \mapsto b(1+\beta_n Z) + dY,\ 1+\beta_n Z \mapsto a(1+\beta_n Z) + cY$$

One can check that this defines an integrable connection $\nabla_\kappa$ on $\mathbb{W}_\kappa$. A generator of $\mathrm{gr}_r \mathbb{W}_\kappa$ is $V^r(1+\beta_n Z)^\kappa$, and we can compute

$$\begin{aligned}
\nabla_\kappa(V^r(1+\beta_n Z)^\kappa) &= \epsilon_\kappa(1 \otimes V^r(1+\beta_n Z)^\kappa) - V^r(1+\beta_n)^\kappa \otimes 1 \\
&= (1+\beta_n Z)^\kappa((a+cV)^{\kappa-r}(b+dV)^r - V^r) \\
&= (1+\beta_n Z)^\kappa((u_\kappa - r)(a-1+cV)\cdot(V^r + rV^{r-1}(b+(d-1)V)) - V^r) \\
&= (1+\beta_n Z)^\kappa(c(u_\kappa - r)V^{r+1} + ((d-1)r + (a-1)(u_\kappa - r))V^r + brV^{r-1})
\end{aligned}$$

Its image in $\mathrm{gr}_{r+1}\mathbb{W}_\kappa \otimes \Omega^1_R$ is therefore $(u_\kappa - r)V^{r+1}(1+\beta_n Z)^\kappa \otimes c$. Moreover, the description of the classical Kodaira–Spencer isomorphism shows that $\delta^{p-1}c$ is a basis of $y^*(\Omega^1_{\mathfrak{X}})$, which is isomorphic to $\Omega^1_R$ after inverting $p$. This proves the second statement. $\square$

The next theorem gives an explicit description of $\nabla_\kappa$ on polynomial $q$-expansions. Recall that they have the form $\sum_{r\geq 0} a_r(q)V_{\kappa,r}$, where $V_{\kappa,r} = Y^r(1+pZ)^{\kappa-r}$ in the notations we have been using.

**Theorem 2.9.** *Let $\partial = q\frac{d}{dq}$ be the Serre operator, then*

$$\nabla_\kappa(a(q)V_{\kappa,r}) = \partial(a(q))V_{\kappa+2,r} + a(u_\kappa - r)V_{\kappa+2,r+1}$$

*Proof.* Over the Tate curve, we have a canonical differential $\omega_{\mathrm{can}}$ such that $\mathrm{KS}(\omega_{\mathrm{can}}^{\otimes 2}) = \frac{dq}{q}$. The dual derivation of $\frac{dq}{q}$ is $\partial$, and define $\eta_{\mathrm{can}} = \nabla(\partial)(\omega_{\mathrm{can}})$. This gives a canonical basis $\{\omega_{\mathrm{can}}, \eta_{\mathrm{can}}\}$ of $H^1_{\mathrm{dR}}(\mathrm{Tate}(q))$. Using this basis, the matrix from the previous proof becomes

$$A = \begin{pmatrix} 1 & 0 \\ \frac{dq}{q} & 1 \end{pmatrix}$$

The expression for $\nabla_\kappa$ from the previous proof gives the required expansion since $c = \frac{dq}{q} \in \Omega^1_{\mathbf{Z}_p((q))}$ is identified with $\omega_{\mathrm{can}}^{\otimes 2} \in \mathbb{W}_2$, which is then identified with $(1+pZ)^2$ in our notation. $\square$

2.4. **$p$-adic iterations of differential operators.** Let $\nabla : \mathbb{W} \to \mathbb{W}[\frac{1}{p}]$ be the operator which restricts to $\nabla_\kappa$ on the $\kappa$-isotypic part. We now consider the question of defining $\nabla^s$, where $s$ is a general weight. We would hope to have a morphism $H^0(\mathfrak{X}(v), \mathbb{W}_\kappa) \to H^0(\mathfrak{X}(v) \times \Lambda_s, \mathbb{W}_{\kappa+2s})$. On $q$-expansion, we saw that if $g = \sum_{n=0}^\infty a_n q^n$, then

$$\nabla_\kappa g = \sum_{n=0}^\infty n a_n q^n$$

There is now the classical issue that $n^s$ does not interpolate in $s$ if $p|n$. Therefore, we need to restrict the domain to $H^0(\mathfrak{X}(v), \mathbb{W}_\kappa)^{U_p=0}$.

Now we run into some new issues, namely it is not clear that the tentative definition $\nabla^s g := \sum_{n=0}^\infty s(n)a_n q^n$ is nearly overconvergent. This requires us to actually interpolate the connection $\nabla^s$ before taking $q$-expansion.

**Definition 2.10.** Let $\kappa, \kappa' : \mathbf{Z}_p^\times \to \bar{\mathbf{Q}}_p^\times$ be two weights. We say they are *1-close* if there exists $u \in p\mathbf{Z}_p$ and a finite order character $\chi$ such that $\kappa = \kappa'\chi\exp(u\log(-))$.

**Theorem 2.11.** *Suppose $p$ is odd. Suppose $\kappa$ and $s$ are each 1-close to a classical weight, then for all $g \in H^0(\mathfrak{X}(v), \mathbb{W}_\kappa)^{U_p=0}$, the expression*

$$\nabla^s(g) := \exp\left(\frac{u_s}{p-1}\log(\nabla^{p-1})\right)(g)$$

*defines an element of $\mathrm{Hdg}^{-\gamma}H^0(\mathfrak{X}(v'), \mathbb{W}_{\kappa+2s})$ for some explicit $\gamma$ and $v'$ depending on $v$. Moreover, if $g$ has $q$-expansion $\sum_{n=0}^\infty a_n q^n$, then $\nabla^s(g)$ has $q$-expansion $\sum_{n=0}^\infty s(n)a_n q^n$.*

*About the proof.* If $s$ has finite order, then what we are actually constructing is a twist operator, which is a fairly classical object, constructed in this language in [AI19, Section 3.8]. Using this observation, it is easy to reduce the theorem to the case where $\kappa$ and $s$ are themselves of the form $\exp(u_\kappa\log(-))$ and $\exp(u_s\log(-))$, i.e. close to the centre of the weight space.

By iterating theorem 2.9, we get that

$$\nabla^N(a(q)V_{\kappa,r}) = \sum_{j=0}^N \binom{N}{j}\prod_{i=0}^j (u_\kappa - r + N - 1 - i)\partial^{N-j}(a(q))V_{\kappa+2N,r+j}$$

The proof now has two components: (1) Show that $(\nabla^{p-1}-1)^{Np}(a(q)V_{\kappa,r}) \in p^N H^0(\mathfrak{X}(0), \mathbb{W})$ by explicit estimation of the coefficients of the polynomial $q$-expansion. (2) Show that "$p$-divisibility is overconvergent". Given these facts, we show that the definition above, viewed as a formal power series in $\nabla^{p-1}-1$, is convergent in $H^0(\mathfrak{X}(v')\hat{\otimes}\Lambda_s, \mathbb{W})$. Considering the action of $\mathbf{Z}_p^\times$ shows that the result lies in $\mathbb{W}_{\kappa+2s}$. The $q$-expansion formula follows from the formula at classical points.

We will now elaborate on point (2). The key statement is the following: suppose $s \geq 0$ and we are given a section $w \in \mathrm{Hdg}^{-s}H^0(\mathfrak{X}(v), \mathbb{W}_\kappa)$. If its restriction to the ordinary locus is divisible by $p^N$, then there exists

$v' < v$ depending only on $v, s, \kappa$ such that $w|_{\mathfrak{X}(v')} \in H^0(\mathfrak{X}(v'), p^{\lfloor N/2 \rfloor}\mathbb{W}_\kappa)$. For our purpose, we need to make the dependency explicit. The statement can be proven locally, so let $V = \operatorname{Spf} S \subseteq \mathfrak{X}(v)$ be an affine open subset, and let $U = \operatorname{Spf} R \subseteq \mathfrak{Ig}_{n,v}$ be the pre-image of $V$. Here, $n$ is chosen based on $\kappa$, or more precisely based on the valuation of $u_\kappa$. For $V$ small enough, $\mathbb{W}_\kappa(U) = R\langle Z, Y\rangle[\kappa]$. Adding a superscript ord to indicate restriction to the ordinary locus, we have similarly $\mathbb{W}_\kappa^{\mathrm{ord}}(U^{\mathrm{ord}}) = R^{\mathrm{ord}}\langle Z^{\mathrm{ord}}, Y^{\mathrm{ord}}\rangle[\kappa]$, with restriction given by $Y \mapsto Y^{\mathrm{ord}}$ and $Z \mapsto p^n\beta_n^{-1}Z^{\mathrm{ord}}$. The factor on $Z$ comes from the fact that over $\mathfrak{X}(v)$, the congruence condition is modulo $\beta_n$, but over $\mathfrak{X}^{\mathrm{ord}}$, the $n$-th level of the Igusa tower is a congruence condition modulo $p^n$.

The kernel of $\mathbb{W}_\kappa(U)/p^N\mathbb{W}_\kappa(U) \to \mathbb{W}_\kappa^{\mathrm{ord}}(U)/p^n\mathbb{W}_\kappa^{\mathrm{ord}}(U)$ is therefore equal to a Tate algebra over the kernel of $R/p^nR \to R^{\mathrm{ord}}/p^nR^{\mathrm{ord}}$. By [AI19, Lemma 3.4], it is annihilated by

$$I_{n,v,N} = \mathrm{Hdg}^{Nv^{-1} + \frac{p^n - p}{p-1}}$$

Descending down to $V \subseteq \mathfrak{X}(v)$ involves taking invariants, so the analogous result holds. It follows that $I_{n,v,N}\mathrm{Hdg}^s w \in p^N\mathbb{W}_\kappa(V)$. To deduce the statement, we just need to choose $v'$ sufficiently small so that over $\mathfrak{X}(v')$, the valuation of $I_{n,v,N}\mathrm{Hdg}^s$ is not too large. More precisely, we want $(s + Nv^{-1} + \frac{p^n - p}{p-1})v' < \frac{N}{2}$. After rearranging, this is implied by the bound

$$v' < \frac{1}{2}\left(\frac{s}{N} + v^{-1} + \frac{p^n - p}{p-1}\right)^{-1}$$

The key point to note is that the dependency on $s$ comes via $\frac{s}{N}$, so we are allowed to have growing denominators provided we can establish better congruence. This is exactly covered by point (1). $\square$

## 3. $p$-adic $L$-function

3.1. **Goal.** Let $\mathcal{W}$ be the weight space, and let $\mathcal{U} = \operatorname{Sp} A$ be an affinoid open subset of $\mathcal{W}$. Over $\mathcal{U}$, we have a universal weight $\boldsymbol{\kappa} : \mathbf{Z}_p^\times \to A^\times$. We can repeat the construction from before to produce a sheaf $\mathbb{W}_{\boldsymbol{\kappa}}$ on $X(v) \times \mathcal{U}$ for $v$ sufficiently small. A family of nearly overconvergent $p$-adic modular forms is a global section $\boldsymbol{f}$ of $\mathbb{W}_{\boldsymbol{\kappa}}$. Given a $\mathbf{Z}_p$-point of $\mathcal{U}$, we can specialize by pulling the section back along the inclusion $X(v) \to X(v)\times\mathcal{U}$. Note that these families are not Coleman families, in the sense that the specializations need not be eigenforms. This can be fixed by keeping track of a finite Hecke algebra action, cf. the construction of the eigencurve. It has the effect of replacing $A_i$ by some finite flat cover. We will not discuss this relatively minor complication here.

Suppose for each $i = 1, 2, 3$, we have a family $\boldsymbol{f}_i$ defined over $\operatorname{Sp} A_i \subseteq \mathcal{W}$. Ideally, a $p$-adic $L$-function should be an element $\mathcal{L}_p \in A_1\hat{\otimes}A_2\hat{\otimes}A_3$ such that for each triple of classical weights $(k_1, k_2, k_3)$, we have

$$\mathcal{L}_p(k_1, k_2, k_3) = L\Big(f_{1,k_1} \times f_{2,k_2} \times f_{3,k_3}, \frac{k_1 + k_2 + k_3 - 2}{2}\Big) \cdot \Omega_\infty^{-1}\mathcal{E}_p(k_1, k_2, k_3)$$

where the left hand side is defined to be $(k_1 \otimes k_2 \otimes k_3)(\mathcal{L}_p)$. On the right hand side, we first divide the $L$-value by a transcendental period to make it algebraic, and then we multiply by an interpolation factor (as usual, we have chosen embeddings of $\bar{\mathbf{Q}}$ into $\mathbf{C}$ and $\mathbf{C}_p$). However, this picture is too simplistic for many reasons:

- The behaviour of $\Omega_\infty$ depends on the relative sizes of the weights, and we expect a different $\mathcal{L}_p$ to exist for each separate regions.
- There could be denominators in $\mathcal{L}_p$ in general, and we will see some from our construction.
- If all $\boldsymbol{f}_i$ are Hida families, then Coates and Perrin-Riou made detailed conjectures on the shape of $\mathcal{E}_p$ [CPR89]. This has been verified by Hsieh [Hsi20]. However, in the generality of Coleman families, there is no prediction on what $\mathcal{E}_p$ should be.
- The archimedean period $\Omega_\infty$ is defined up to a rational multiple. For many arithmetic applications, it is necessary to choose it correctly, which is also sometimes related to the issue of denominators. In the Hida family case, there is the notion of Hida canonical period, which is essentially the Petersson norm divided by the size of a congruence ideal. This does not yet have a good generalization to the finite slope case.
- One can sometimes remove the ambiguity in choosing $\Omega_\infty$ by inserting a $p$-adic period $\Omega_p$.

In this talk, we will focus on the unbalanced case $k_1 \geq k_2 + k_3$. Using Ichino's formula, one key component is the interpolation of

$$\frac{\langle f_1, \delta_{k_2}^t f_2 \times f_3\rangle}{\langle f_1, f_1\rangle}$$

We know there are compatible modifications of $f_i$ such that this is non-zero if and only if the triple product $L$-function is non-zero, so if we can interpolate this, then we get an $\mathcal{L}_p$ which vanishes exactly when the central critical value vanishes. This is enough for some applications, as in [DR14].

3.2. **Overconvergent projectors.** We already have a good $p$-adic analogue for $\delta_{k_2}^t f_2 \times f_3$ as a nearly overconvergent modular form. This section will be concerned with constructing the overconvergent projector, which should be a section of

$$H^0(X(v) \times \mathcal{U}, \omega^{\boldsymbol{\kappa}}) \to H^0(X(v) \times \mathcal{U}, \mathbb{W}_{\boldsymbol{\kappa}})$$

The construction is based on the following theorem.

**Theorem 3.1.**

(1) *The groups $H^0(X(v) \times \mathcal{U}, \mathrm{Fil}_r \mathbb{W}_{\boldsymbol{\kappa}})$ for $r \leq \infty$ admit slope decompositions with resepct to $U_p$. Moreover, given a slope $h$, if $r$ is sufficiently large, then the natural map*

$$H^0(X(v) \times \mathcal{U}, \mathrm{Fil}_r \mathbb{W}_{\boldsymbol{\kappa}})^{\leq h} \to H^0(X(v) \times \mathcal{U}, \mathbb{W}_{\boldsymbol{\kappa}})^{\leq h}$$

*is an isomorphism.*

(2) *For each $r$, the map*

$$H^0(X(v) \times \mathcal{U}, \omega^{\boldsymbol{\kappa}}) \otimes_A A' \to \frac{H^0(X(v) \times \mathcal{U}, \mathrm{Fil}_r \mathbb{W}_{\boldsymbol{\kappa}})}{\nabla_{\boldsymbol{\kappa}-2} H^0(X(v) \times \mathcal{U}, \mathrm{Fil}_{r-1} \mathbb{W}_{\boldsymbol{\kappa}-2})} \otimes_A A'$$

*is an isomorphism, where $A' = A[\frac{1}{u_\kappa - i} \,|\, 0 \leq i \leq 2r]$.*

*Proofs.* The first statement follows easily from the earlier remark that the action of $U_p$ on the $r$-th graded piece is divisible by $(p/\mathrm{Hdg}^{p+1})^r$. The second is [Urb14, Proposition 3.5.4] written in our language. His proof was based on defining the $\epsilon$-operator on polynomial $q$-expansions. Section 3.9 of [AI19] presents a different formulation of the proof. □

From these two isomorphisms, we get an overconvergent projector

$$H^{\dagger, \leq h} : H^0(X(v) \times \mathcal{U}, \mathbb{W}_{\boldsymbol{\kappa}}) \otimes_A A' \to H^0(X(v) \times \mathcal{U}, \omega^{\boldsymbol{\kappa}})^{\leq h} \otimes_A A'$$

*Remark* 3.2. In [AI19], it is shown that Coleman's analysis of the $\theta$-operator can be used to explicitly identify the error terms at the poles.

3.3. **Interpolation properties.** The final ingredient is something [Urb14] and [AI19] called "$p$-adic Petersson inner product". Let $M = M_{\boldsymbol{\kappa}}^{\dagger, \leq a}$ be the space of overconvergent $p$-adic families of modular forms of slope $\leq a$, where $\boldsymbol{\kappa} : \mathbf{Z}_p^\times \to A^\times$ is a universal weight. Let $\mathbb{T}$ be the image of the Hecke algebra in $\mathrm{End}(M)$. We have the classical pairing $M \times \mathbb{T} \to A$ sending $(\boldsymbol{f}, T)$ to $a_1(\boldsymbol{f}|T)$, which defines an $A$-linear map $M \to \mathbb{T}^\vee$. The space $\mathrm{Sp}\,\mathbb{T}$ defines a finite generically étale open subset of the eigencurve, so we have a trace map $\mathbb{T}^\vee \to \mathbb{T} \otimes_A \mathfrak{d}$, where $\mathfrak{d}$ is the ramification divisor of $\mathrm{Sp}\,\mathbb{T} \to \mathcal{W}$. The map $\mathrm{pr}_{\boldsymbol{f}}$ is defined to be the image of $\boldsymbol{f}$ under the composite $M \to \mathbb{T}^\vee \to \mathbb{T} \otimes_A \mathfrak{d} \to M^\vee \otimes_A \mathfrak{d}$. By Coleman's classicality theorem, $\mathfrak{d}$ does not contain classical points of large weight.

We are now ready for the construction. For $i = 1, 2, 3$, let

$$\boldsymbol{f}_i \in H^0(X(v) \times \mathcal{U}_i, \omega^{\boldsymbol{\kappa}_i})$$

be a $p$-adic family of modular forms. We perform the modification procedure as described in [Hsi20] to each of them and denote the results by $\boldsymbol{f}_i^\circ$. Moreover, let $\boldsymbol{f}_1^{\circ,*}$ be the Atkin–Lehner involution applied to $\boldsymbol{f}_1$, and let $\boldsymbol{f}_2^{\circ,[p]}$ be the $p$-depletion of $\boldsymbol{f}_2^\circ$. Let $\boldsymbol{s} = \frac{1}{2}(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3)$, where we again assume $p > 2$ for simplicity.

Suppose $\boldsymbol{f}_1$ has slope at most $h$, then we will define

$$\mathcal{L}_p = \mathrm{pr}_{\boldsymbol{f}_1^{\circ,*}}(H^{\dagger, \leq h}(\nabla^{\boldsymbol{s}} \boldsymbol{f}_2^{\circ,[p]} \times \boldsymbol{f}_3^\circ)) \in \mathrm{Frac}(A_1) \hat{\otimes} A_2 \hat{\otimes} A_3$$

where $A_i = \mathcal{O}(\mathcal{U}_i)$ and we will briefly talk about $\mathrm{pr}_{\boldsymbol{f}_1^\circ}$ later. Recall that this is only defined if $\boldsymbol{\kappa}_2$ and $\boldsymbol{s}$ are 1-close to classical weights, so this can only interpolate near the centre of the weight space. There are two types of poles in $\mathcal{U}_1$: the overconvergent projector gives simple poles at $0 \leq i \leq N_h$ for some $N_h$ depending on $h$, and the projector gives poles at certain ramification points. It would be interesting to understand if the poles of the first type, corresponding to when $k_1$ is not much bigger than $k_2 + k_3$, actually exists.

The actual interpolation formula states that if $(k_1, k_2, k_3)$ are classical weights such that the above hypotheses are satisfied, and moreover $k_1 - (k_2 + k_3)$ is positive and even, then

$$\mathcal{L}_p(k_1, k_2, k_3)^2 = (*)\mathcal{E} \cdot \frac{L(f_{k_1} \times f_{k_2} \times f_{k_3}, \frac{k_1+k_2+k_3-2}{2})}{\langle f_{k_1}, f_{k_1} \rangle^2}$$

where $(*)$ denotes some local terms which is non-zero, and $\mathcal{E}$ is some Euler factor coming from the $p$-depletion procedure on $\boldsymbol{f}_2^\circ$. One should be able to write the local terms explicitly using the computations of [Hsi20]. The Euler factor is written down explicitly in [AI19]. It is different from the Euler factor from [DR14] and [Hsi20] in the ordinary case, and we do not have a good framework to predict what the right form should be.

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