HIGHER HIDA THEORY

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This is the third talk in a series of talks in higher Hida theory. We will construct the complex interpolating higher coherent cohomology and look at its application to the p-adic L-function for GSp_4 in a simple case. Most of the hard part is already done. The main reference is [LPSZ19], with more details found in [Pil20].

1. Main theorem

Let X (resp. $X_{\mathrm{Kl}}(p)$) be some toroidal compactification of the Shimura variety of GSp_4 with a fixed tame level and hyperspecial (resp. Klingen) level at p, defined over $\mathbf{Z}_p$. Over them, there are modular sheaves $\omega(k_1, k_2)$, parametrized by integers $k_1 \geq k_2$. We will recall their moduli interpretations shortly. The goal of higher Hida theory is to interpolate the coherent cohomology $H^\bullet(X_{/\mathbf{Q}_p}, \omega(k_1, k_2)(-D))$. Unlike in classical Hida theory where we can interpolate both variables, here we are only interpolating in k_1. The main theorem is the following.

Theorem 1.1. *Let $\Gamma = \mathbf{Z}_p^\times$ and $\Lambda = \mathbf{Z}_p[[\Gamma]]$. Let $k_2 \in \mathbf{Z}_{\leq 0} \cup \{2\}$. There exists a perfect complex $M^\bullet_{k_2}$ with actions of Hecke operators away from p and the U_p operator. If $k_1 + k_2 \geq 4$, then there exists a Hecke-equivariant quasi-isomorphism*

$$M^\bullet_{k_2} \otimes^{\mathbf{L}}_{\Lambda, k_1} \mathbf{Q}_p = e_{\mathrm{Kl}} R\Gamma(X_{\mathrm{Kl}}(p)_{/\mathbf{Q}_p}, \omega(k_1, k_2)(-D))$$

where the derived tensor product is induced from the map $k_1 : \Gamma \to \mathbf{Q}_p^\times$, $\gamma \mapsto \gamma^{k_1}$. Moreover, if $k_2 \leq 0$, then $M^\bullet_{k_2}$ is concentrated in degree 1. If $k_2 = 2$, then it is concentrated in degrees $[0, 1]$.

The proof has three parts. The first two parts will be integral, and at least when $k_2 = 2$, all results are known from previous talks. The final part is entirely new.

(1) We construct an Igusa tower $\pi : \mathfrak{Ig}(p^\infty) \to \mathfrak{X}_{\mathrm{Kl}}(p^\infty)^{\geq 1}$, which is a $\mathbf{Z}_p^\times$-torsor. The pushforward of the structure sheaf will carry a $\mathbf{Z}_p^\times$-action. Denoting the sections isotypic with respect to a certain universal character κ by $\mathfrak{F}(\kappa, k_2)$, our complex is essentially $e_{\mathrm{Kl}} R\Gamma(\mathfrak{X}_{\mathrm{Kl}}(p)^{\geq 1}, \mathfrak{F}(\kappa, k_2)(-D))$.
(2) There is a comparison morphism $\omega(k_1, k_2) \to \mathfrak{F}(\kappa, k_2) \otimes^{\mathbf{L}}_{\Lambda, k_1} \mathbf{Z}_p = \mathfrak{F}(k_1, k_2)$. We show that this induces a quasi-isomorphism
$$e_{\mathrm{Kl}} R\Gamma(\mathfrak{X}_{\mathrm{Kl}}(p)^{\geq 1}, \omega(k_1, k_2)(-D)) = e_{\mathrm{Kl}} R\Gamma(\mathfrak{X}_{\mathrm{Kl}}(p)^{\geq 1}, \mathfrak{F}(k_1, k_2)(-D))$$
This is "classicality along sheaf".
(3) For actual classicality, we need to stop working integrally and impose weight conditions. The key statement is that there is a quasi-isomorphism
$$e_{\leq 0} R\Gamma(\mathcal{X}_{\mathrm{Kl}}(p), \omega(k_1, k_2)(-D)) = e_{\leq 0} R\Gamma(\mathcal{X}_{\mathrm{Kl}}(p)^{\geq 1}, \omega(k_1, k_2)(-D))$$
as adic spaces. Here, $e_{\leq 0}$ is the ordinary projector constructed using slope decomposition for U_p. This uses results from higher Coleman theory, which we will explain.

1.1. Igusa tower.

This part is entirely analogous to the constructions we have seen from the previous part of the seminar. We use this opportunity to fix notations and recall facts from the previous talks.

Let $\mathfrak{X}^{\geq 1}_{\mathrm{Kl}}(p^m)$ be the formal scheme classifying semiabelian surfaces A with a subgroup $H_m \subseteq A[p^m]$ étale-locally isomorphic to μ_{p^m} (and other data coming from PEL structure). Note that when $m = 1$, this is strictly smaller than $(\mathfrak{X}_{\mathrm{Kl}}(p))^{\leq 1}$, which just requires the multiplicative rank of $A[p]$ to be 1. The transition map from p^m to p^{m-1} are étale and affine but not finite (Lemma 9.1.1.1 and Remark 9.1.1.2 [Pil20]). The affineness allows us to form the formal schemes $\mathfrak{X}^{\geq 1}_{\mathrm{Kl}}(p^\infty)$. Over it, we have a $\mathbf{Z}_p^\times$-torsor $\mathfrak{Ig}(p^\infty)$ classifying isomorphisms $\mu_{p^\infty} \to H_\infty$. Let π be the map $\mathfrak{Ig}(p^\infty) \to \mathfrak{X}^{\geq 1}_{\mathrm{Kl}}(p)$.

Before defining the interpolation sheaf, we recall the definition of classical modular sheaves, using the labelling of [LPSZ19]. Let $\mathcal{A} \to X$ be the universal semiabelian scheme over X. Let $\omega_{\mathcal{A}}$ be the conormal bundle of $\mathcal{A}$, which has rank 2. Define

$$\omega(k_1, k_2) = \mathrm{Sym}^{k_1 - k_2}(\omega_{\mathcal{A}}) \otimes \det^{k_2} \omega_{\mathcal{A}}$$

We will interpolate this sheaf as k_1 vary p-adically.

Definition 1.2. Let $\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ and $\kappa : \mathbf{Z}_p^\times \to \Lambda$ be the universal character. For an integer k_2, define

$$\mathfrak{F}(\kappa, k_2) = \pi_* \mathcal{O}_{\mathfrak{Ig}(p^\infty)}[\kappa - k_2] \otimes \det^{k_2} \omega_{\mathcal{A}/\mathfrak{X}_{\mathrm{Kl}}(p)}$$

If k_1 is an integer, define $\mathfrak{F}(k_1, k_2) = \mathfrak{F}(\kappa, k_2) \otimes_{\Lambda, k_1} \mathbf{Z}_p$.

We need to construct a natural map $\omega(k_1, k_2) \to \mathfrak{F}(k_1, k_2)$. Recall that there is a Hodge–Tate map $\mathrm{HT} : H_\infty^D \to \omega_{H_\infty}$ sending $f : H_\infty \to \mu_{p^\infty}$ to $f^* \frac{dT}{T}$. Unlike the Coleman theory talks, ω_{H_∞} is locally free of rank 1, so this induces an isomorphism

$$\mathrm{HT} : H_\infty^D \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p^\infty)} \xrightarrow{\sim} \omega_{H_\infty}$$

If k is an integer, we get an isomorphism $\mathfrak{F}(k, 0) \xrightarrow{\sim} \omega_{H_\infty}^{\otimes k}$. Moreover, the inclusion $H_\infty \hookrightarrow \mathcal{A}[p^\infty]$ induces a surjection $\omega_{\mathcal{A}} \to \omega_{H_\infty}$. By considering projection onto the highest weight vector, we get a surjection

$$\omega(k, 0) = \mathrm{Sym}^k \omega_{\mathcal{A}} \to \omega_{\mathcal{A}}^{\otimes k} \to \omega_{H_\infty}^{\otimes k} \xrightarrow{\sim} \mathfrak{F}(k, 0)$$

Twisting this by $\det \omega_{\mathcal{A}}^{k_2}$ gives the comparison map with a short exact sequence

$$0 \to K\omega(k_1, k_2) \to \omega(k_1, k_2) \to \mathfrak{F}(k_1, k_2) \to 0$$

where $K\omega(k_1, k_2)$ is defined to be the kernel.

Remark 1.3. In classical Hida theory (see for example [Pil12] or [Hid04]), the Igusa tower over the ordinary locus $\mathfrak{X}_{\mathrm{Kl}}^{\geq 2}(p^\infty)$ is a $\mathrm{GL}_2(\mathbf{Z}_p)$-torsor. We can then construct a 2-variable universal sheaf $\mathfrak{E}(\kappa_1, \kappa_2)$ over $\mathfrak{X}_{\mathrm{Kl}}^{\geq 2}(p)$. For classical weights (k_1, k_2), there is an *injection* $\omega(k_1, k_2) \to \mathfrak{E}(k_1, k_2)$. Roughly, when working étale-locally, $\omega(k_1, k_2)$ is the algebraic induction and $\mathfrak{E}(k_1, k_2)$ is the continuous induction. In our case, the Igusa tower is not enough to trivialize $\omega(k_1, k_2)$, so we need to lose some information first, which is the projection to the highest weight subspace.

Definition 1.4. $M^\bullet_{\kappa, k_2} = R\Gamma(\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p), \mathfrak{F}(\kappa, k_2)(-D))$

1.2. **Classicality on sheaf.**

Theorem 1.5 ([Pil20, Theorem 11.3.1], [LPSZ19, Lemma 3.8])**.** *The U_p-operator acts locally finitely on $R\Gamma(\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1, k_2)(-D))$ and $M^\bullet_{\kappa, k_2}$. Let $e_{\mathrm{Kl}} = \lim_{n \to \infty} U_p^{n!}$ be the ordinary projector, then we have a quasi-isomorphism*

$$e_{\mathrm{Kl}} R\Gamma(\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1, k_2)(-D)) = e_{\mathrm{Kl}} R\Gamma(\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p), \mathfrak{F}(k_1, k_2)(-D))$$

Proof. We do everything modulo p. The general case follows from Nakayama lemma, as explained in Section 2 of [Pil20]. In Gyujin's talk, he proved the finiteness statement for classical cohomology under the additional hypothesis that $k_2 \geq 2$ and $k_1 - k_2 \gg 0$. Recall the structure of the proof: first show that $R\Gamma(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \omega(k_1, k_2))$ can be computed by the complex

$$[H^0(X_{\mathrm{Kl}}^{=2}(p)_{/\mathbf{F}_p}, \omega(k_1, k_2)(-D)) \to \varinjlim H^0(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \omega(k_1 + \ell(p-1), k_2 + \ell(p-1))(-D))/\mathrm{Ha}^\ell]$$

On each term, relate the U_p-operator to T_p-operator acting on cohomology of $X_{/\mathbf{F}_p}$. Finally, use nilpotency of T_p on the supersingular locus to show that each term is the cohomology of a closed subscheme of $X_{/\mathbf{F}_p}$. The final step uses the fact that T_p commutes with the Hasse invariants, which requires the weight conditions.

To deal with arbitrary $k_2 \in \mathbf{Z}$, we consider the short exact sequence

$$0 \to \omega(k_1, k_2) \to \omega(k_1 + p - 1, k_2 + p - 1) \to \omega(k_1 + p - 1, k_2 + p - 1)/\mathrm{Ha} \to 0$$

obtained by multiplication by Ha and use downwards induction on k_2. Crucially, this is U_p-equivariant since U_p commutes with the two Hasse invariants without any weight condition [Pil20, Lemma 10.5.2.1, Lemma 10.5.3.1]. The cohomology of the middle term is locally finite by induction hypothesis. The final term is supported on $X_{\mathrm{Kl}}^{=1}(p)_{/\mathbf{F}_p}$. On that stratum, we use the second Hasse invariant to boost the weight up further. The details can be found in [LPSZ19].

We want to apply the same argument to $\mathfrak{F}(k_1, k_2)$. There is one issue: to reduce modulo p, we also need to truncate the definition of $\mathfrak{F}(k_1, k_2)$ at a finite level, since we needed the Hodge–Tate map to be an isomorphism. We also cannot work directly at the formal scheme level since some algebraic results require actual nilpotency. To boost the level from 1 to m, we show using the moduli interpretation of U_p^m that there is a commutative diagram

$$\begin{array}{ccc} R\Gamma(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \mathcal{F}_{m,1}(k_1, k_2)(-D)) & \xrightarrow{U_p^m} & R\Gamma(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \mathcal{F}_{m,1}(k_1, k_2)(-D)) \\ \uparrow & \searrow & \uparrow \\ R\Gamma(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \mathcal{F}_{1,1}(k_1, k_2)(-D)) & \xrightarrow{U_p^m} & R\Gamma(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \mathcal{F}_{1,1}(k_1, k_2)(-D)) \end{array}$$

where $\mathcal{F}_{m,n}(k_1, k_2)$ are the truncated versions of $\mathfrak{F}(k_1, k_2)$. They satisfy $\mathfrak{F}(k_1, k_2) = \varprojlim_m \varinjlim_n \mathcal{F}_{m,n}(k_1, k_2)$. The above diagram then gives a quasi-isomorphism

$$e_{\mathrm{Kl}} R\Gamma(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \mathcal{F}_{1,1}(k_1, k_2)(-D)) = e_{\mathrm{Kl}} R\Gamma(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \varinjlim_m \mathcal{F}_{m,1}(k_1, k_2)(-D))$$

so everything is now reduced to a level-1 Igusa tower over $\mathbf{F}_p$.

The construction of the short exact sequence $0 \to K\omega(k_1, k_2) \to \omega(k_1, k_2) \to \mathfrak{F}(k_1, k_2) \to 0$ has a truncated version with $\mathfrak{F}$ replaced by $\mathcal{F}_{1,1}$. We denote the kernel by the same notation. Now, each term has a 2-step resolution analogous to the classical one. Still assuming $k_1 - k_2 \gg 0$, the middle term is computed by a perfect complex with locally finite U_p-action on each term, which implies the same for the other two terms by [Pil20, Lemma 2.1.1]. We will show next that U_p is divisible by p on $K\omega(k_1, k_2)$, which is enough to produce the quasi-isomorphism in the theorem.

In the construction of U_p, we used the correspondence $\mathfrak{C}^1(p^\infty)$ parametrizing pairs $(A \to A', H_\infty \subseteq A)$, where $A \to A'$ is an isogeny of degree p^3, H_∞ is étale-locally isomorphic to μ_{p^∞}, and $\ker(A \to A') \cap H_\infty = \{0\}$. In the process, we need to differentiate the universal isogeny. But recall that the comparison map contains $\omega_{\mathcal{A}} \to \omega_{H_\infty}$, so $K\omega(k_1, k_2)$ contains contributions from $\ker(\omega_{\mathcal{A}} \to \omega_{H_\infty})$. The differential of the universal isogeny is divisible by p, but it is an isomorphism on ω_{H_∞} since H_∞ is not in its kernel. It follows that U_p is divisible by p on the kernel.

Finally, we need to remove the assumption that $k_1 - k_2 \gg 0$. First, the Nakayama lemma argument from specializing at a classical weight shows that U_p acts locally finitely on $M^\bullet_{\kappa, k_2}$. Now we can specialize at any classical weight and use the argument from the previous paragraph in reverse to show local finiteness and the quasi-isomorphism for $R\Gamma(\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1, k_2)(-D))$. □

1.3. **Full classicality.** In Gyujin's talk, he proved integral classicality results on ordinary classes under certain weight conditions by relating them to the hyperspecial level. More precisely, given $k_2 \geq 2$ it can be shown that there exists a constant C independent of the tame level such that for all $k_1 \geq C$, we have a quasi-isomorphism [Pil20, Theorem 11.2.1]

$$e_{\mathrm{Kl}} R\Gamma(\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1, k_2)(-D)) = e_T R\Gamma(\mathfrak{X}, \omega(k_1, k_2)(-D))$$

where e_T is the ordinary projector at the hyperspecial level. There are also partial statements for $k_2 = 2$. A similar statement holds for $k_2 \leq 0$ (see Theorem 1.8), but it is difficult to remove the condition on k_1. We will prove the classicality theorem for $k_2 \leq 0$ after inverting p. The proof goes via higher Coleman theory, which we will outline in the next section.

Remark 1.6. We cannot expect a quasi-isomorphism for $k_2 = 2$ since there are ordinary newforms at the Klingen level, but we also cannot formulate a comparison to ordinary forms at the Klingen level since we did not define U_p on the whole formal scheme $\mathfrak{X}_{\mathrm{Kl}}(p)$ due to issues with integral models. These issues are not present if we work rationally and use slope decomposition instead of ordinary projector.

First, we sketch an outline. A feature for $k_2 \leq 0$ is that all cohomologies are concentrated in degree 1. This is classical for the cohomology of $\mathfrak{X}_{\mathrm{Kl}}(p)$. We will prove a similar statement for $\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p)$ in Theorem 1.7. Given this, it remains to show that the restriction map

$$e_{\leq 0} H^1(\mathcal{X}_{\mathrm{Kl}}(p), \omega(k_1, k_2)(-D)) \to e_{\leq 0} H^1(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1, k_2)(-D))$$

is an isomorphism. To do this, we insert an overconvergent term in the middle

$$e_{\leq 0}H^1(\mathcal{X}_{\mathrm{Kl}}(p), \omega(k_1,k_2)(-D)) \xrightarrow{i_1} e_{\leq 0}H^1(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p)^{\dagger}, \omega(k_1,k_2)(-D)) \xrightarrow{i_2} e_{\leq 0}H^1(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1,k_2)(-D))$$

The key theorem of higher Coleman theory, which we will discuss in the next section, is that i_1 is an isomorphism if $k_1+k_2 \geq 4$ (no condition on k_2). It is fairly easy to show that i_2 is always surjective using a Čech complex computation (Proposition 2.2). Finally, the integral classicality result of Theorem 1.8 shows that $i_2 \circ i_1$ is injective if $k_1 \gg 0$. By p-adic density, this implies i_2 is an isomorphism for all $k_1 \geq 4-k_2$, which finally shows that $i_2 \circ i_1$ is an isomorphism for all $k_1 \geq 4-k_2$.

We now state and prove the two results needed above which do not require overconvergent cohomology.

Theorem 1.7. *If $k_2 \leq 0$, then $e_{\mathrm{Kl}}H^0(\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1,k_2)(-D)) = 0$.*

Proof. This has an entirely different flavour from the rest of the talk. The idea is to show any section vanishes when restricted to certain modular curves in $\mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p)$, and that these modular curves are Zariski dense.

We can work over $\mathbf{F}_p$ by the Nakayama lemma. It carries a universal multiplicative group $H \subseteq \mathcal{A}$, which we previously denoted by H_1. By the arguments from classicality on sheaf, there is an injection

$$e_{\mathrm{Kl}}H^0(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \omega(k_1,k_2)(-D)) \to e_{\mathrm{Kl}}H^0(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \omega_H^{k_1-k_2} \otimes \det^{k_2} \omega_{\mathcal{A}}(-D))$$

Let $s \in H^0(X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}, \omega_H^{k_1-k_2} \otimes \det^{k_2} \omega_{\mathcal{A}}(-D))$. We need to show it is 0. Fix an ordinary elliptic curve E_0 over $\mathbf{F}_p$. Define a map

$$j : X_{\mathrm{GL}_2,\mathbf{F}_p} \to X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}$$

by sending an elliptic curve E to the abelian surface $E \times E_0$ with multiplicative subgroup $E_0[p]^{\circ}$, taking care of tame level structure and boundary appropriately. Now, j^*s is a section of the sheaf $\omega_E^{k_2}(-D)$. Since $k_2 \leq 0$, this vanishes.

By applying Hecke translates away from p, we get a collection of modular curves in $X_{\mathrm{Kl}}^{\geq 1}(p)_{/\mathbf{F}_p}$ on which s vanishes. Varying E_0 expands them into a product of ordinary parts of modular curves. It remains to show that their union is Zariski dense. Since each product is codimension 1, we just need to show that p-prime Hecke operators can produce infinitely many such products. It is easy to reduce the problem to the ordinary locus without additional level structure $X_{/\mathbf{F}_p}^{\geq 2}$. Now we just need to exhibit a sequence of abelian surfaces A_n, each obtained from a product of ordinary elliptic curves by some p-prime isogeny, with the further requirement that the degree of any such isogeny is bounded below. For this, take non-isogenous ordinary elliptic curves E, F over $\mathbf{F}_p$. Let $\{e_1, e_2\}$ (resp. $\{f_1, f_2\}$) be bases of $T_\ell E$ (resp. $T_\ell F$). Any isogeny between $A_\ell = (E \times F)/\langle e_1 + f_1 \pmod{\ell}, e_2 - f_2 \pmod{\ell}\rangle$ and a product of two elliptic curves has degree at least ℓ^2. ☐

Theorem 1.8. *For a fixed $k_2 \leq 0$, there exists an absolute constant C independent of tame level such that for all $k_1 > C$, we have a commutative diagram*

$$\begin{array}{ccc} e_{T,\leq 0}H^1(\mathcal{X}, \omega(k_1,k_2)(-D)) & \xrightarrow{\sim} & e_{\leq 0}H^1(\mathcal{X}_{\mathrm{Kl}}(p), \omega(k_1,k_2)(-D)) \\ \big\downarrow & & \big\downarrow \\ e_{T,\leq 0}H^1(\mathcal{X}^{\geq 1}, \omega(k_1,k_2)(-D)) & \xrightarrow{\sim} & e_{\leq 0}H^1(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1,k_2)(-D)) \end{array}$$

The bottom isomorphism also holds integrally.

Proof. The top isomorphism is a purely automorphic computation plus estimates on Hecke eigenvalues coming from p-adic Hodge theory, see [LPSZ19, Lemma 3.22]. The left injection holds since $\mathcal{X}$ is smooth and the complement $\mathcal{X}^{=0}$ has codimension 2. The bottom isomorphism was proven if $k_2 \geq 2$ in Gyujin's talk by studying the behaviour of U_p and T_p on each stratum. The same analysis is needed here, see the discussions before [LPSZ19, Lemma 3.21]. ☐

2. Higher Coleman theory

We now want to do Coleman theory based on the approach in [AIP15] using the same spaces. This is again very technical and should be the subject of at least another talk, so we will just give some flavour here. The recent preprint of Boxer–Pilloni on higher Coleman theory for other Shimura varieties also seems to clarify certain arguments here.

Theorem 2.1. *Let κ be the universal character on an affinoid subspace of the 1-dimensional weight space. There exists overconvergent cohomology groups $H^i(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p)^\dagger, \omega(\kappa, k_2)(-D))$. It carries a completely continuous action of a U_p-operator. Moreover, there exists an explicit slope bound $h = h(k_1, k_2)$ such that*

$$H^i(\mathcal{X}_{\mathrm{Kl}}(p), \omega(k_1, k_2)(-D))^{<h} = H^i(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p)^\dagger, \omega(k_1, k_2)(-D))^{<h}$$

The same statements hold with $(-D)$ removed.

About the Proof. The construction is similar to what [AIP15] did for the ordinary locus. They use Iwahori level structures, but we use Klingen level structures. The corresponding diagram of formal schemes is

$$\pi : \mathfrak{Fl}_w^+ \xrightarrow{\pi_1} \mathfrak{Fl}_w \xrightarrow{\pi_2} \mathfrak{X}(p^n)_\epsilon \xrightarrow{\pi_3} \mathfrak{X}_{\mathrm{Kl}}(p^n)_\epsilon \xrightarrow{\pi_4} \mathfrak{X}_{\mathrm{Kl}}(p)_\epsilon$$

where the formal models of Shimura varieties are defined by normalization, and the flag variety here is the $\mathbf{P}^1$-bundle parametrizing free direct summands of a modification of $\omega_{\mathcal{A}}$. The map π is affine, and it carries an action of a formal group $\mathfrak{T}_w = \mathbf{Z}_p^\times(1 + p^w \mathbf{G}_a)$ and a finite group $\mathrm{Kl}(\mathbf{Z}/p^n\mathbf{Z})$. Suppose w is large enough so that κ extends to an analytic function on $\mathfrak{T}_w$, then we can define

$$\mathfrak{F}^{\kappa, w} = (\pi_* \mathcal{O}_{\mathfrak{Fl}_w^+} \hat{\otimes} R)^{\mathfrak{T}_w \times_{\mathfrak{T}_w / \mathfrak{T}_w^\circ} \mathrm{Kl}(\mathbf{Z}/p^n\mathbf{Z})}$$

where the action of the group on R is entirely via the first factor, and we view the group as a thickening of $\mathrm{Kl}(\mathbf{Z}/p^n\mathbf{Z})$ via its Levi subgroup. It is not difficult to prove compatibilities between $\mathfrak{F}^{\kappa,w}$ as w varies, and to construct the rational version $\mathscr{F}^{\kappa,w}$ on the adic generic fibre. We define

$$H^i(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p)^\dagger, \omega(\kappa, k_2)(-D)) = \lim_{\epsilon \to 0, n, w \to \infty} H^i(\mathcal{X}_{\mathrm{Kl}}(p^n, \epsilon), \mathscr{F}^{\kappa - k_2, w} \otimes \det^{k_2} \omega_{\mathcal{A}}(-D))$$

and similarly without $(-D)$. Note that in [Pil20], two *different labellings* are used, with the result that $\mathfrak{X}_{\mathrm{Kl}}(p, \epsilon) \downarrow \mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p)$ as $\epsilon \to \infty$, and $\mathfrak{X}_{\mathrm{Kl}}(p)_\epsilon \downarrow \mathfrak{X}_{\mathrm{Kl}}^{\geq 1}(p)$ as $\epsilon \to 1$.

The definition of U_p-operator is easier since we work only with the generic fibre. We can prove that U_p improves overconvergence, and therefore it is completely continuous. The standard spectral theory then gives overconvergent families and finite slope projectors. There are two harder facts we need to work towards: vanishing theorem for $i \geq 2$ and the classicality theorem referred to before.

First we look at the vanishing theorem for cuspidal cohomology [Pil20, Section 12.9]. The space $\mathfrak{X}(p^n)_\epsilon$ can be covered by two open formal subschemes. Consider the minimal compactification $\mathfrak{X}(p^n)_\epsilon^\star$. Using perfectoid the Shimura variety, we can define a new integral structure $\mathfrak{X}(p^n)_\epsilon^{\star - HT}$ with a projective map $\mathfrak{X}(p^n)_\epsilon^\star \to \mathfrak{X}(p^n)_\epsilon^{\star - HT}$. The two pieces are affine in this new integral structure, and general results show that one can compute the cohomology of $\mathfrak{X}(p^n)_\epsilon^\star$ using a Čech cover with those two pieces. Finally, we use the fact that the higher direct image of $\mathcal{O}_{\mathfrak{X}(p^n)_\epsilon}(-D)$ under $\mathfrak{X}(p^n)_\epsilon \to \mathfrak{X}(p^n)_\epsilon^\star$ is trivial to conclude [Pil20, Proposition 12.9.2.1]. Note that this proof only applies to the cuspidal sheaves.

The proof of classicality theorem is again divided into classicality along sheaf and analytic continuation. Both steps are done using the same ideas from [AIP15], see also Jack's notes. The technical details are explained in Section 13.3 and Chapter 14 of [Pil20]. □

Finally, we need the following easy comparison theorem between higher Coleman theory and Hida theory.

Proposition 2.2. *If $k \leq 0$, then the restriction map*

$$e_{\leq 0} H^1(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p)^\dagger, \omega(k_1, k_2)(-D)) \to e_{\leq 0} H^1(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p), \omega(k_1, k_2)(-D))$$

is surjective.

Proof. In the minimal compactification, both sides can be computed by two affinoids U_1 and U_2, so the map can be represented by the complex

$$\begin{array}{ccc} H^0(U_1, \mathcal{F}) \oplus H^1(U_2, \mathcal{F}) & \longrightarrow & H^0(U_1 \cap U_2, \mathcal{F}) \\ \uparrow & & \uparrow \\ H^0(U_1^\dagger, \mathcal{F}) \oplus H^1(U_2^\dagger, \mathcal{F}) & \longrightarrow & H^0(U_1^\dagger \cap U_2^\dagger, \mathcal{F}) \end{array}$$

where $\mathcal{F}$ is the direct image of $\omega(k_1, k_2)(-D)$ in the minimal compactification. The two vertical maps are injective with dense image. Since restriction is continuous, and $e_{\leq 0} H^1(\mathcal{X}_{\mathrm{Kl}}^{\geq 1}(p)^\dagger, \omega(k_1, k_2)(-D))$ is finite dimensional, the proposition is proven. □

3. p-ADIC L-FUNCTION

Finally, we talk about the main point of this seminar, which is to construct a p-adic L-function for GSp_4. Let Π be a stable globally generic cuspidal representation of GSp_4 such that Π_∞ has infinitesimal character (r_1+2, r_2+1). Moreover, suppose Π is unramified and Klingen-ordinary at p. Our goal is to construct a p-adic L-function $\mathcal{L}_p(\Pi, s)$ such that if s specializes to finite order character χ, then

$$\mathcal{L}_p(\Pi,\chi) = \mathcal{E}_p(\Pi,\chi) L\Big(\Pi\otimes\chi, \frac{1+d}{2}\Big)/\Omega_\Pi$$

where $d = r_1 - r_2$, Ω_Π is some archimedean period, and $\mathcal{E}_p(\Pi,\chi)$ is the Euler factor predicted by Coates–Perrin-Riou [CPR89]. We will actually interpolate the values at $\frac{1+d}{2} - a$ for $a \le d$. The same method allows one to construct a p-adic L-function for $\Pi\times\sigma$, where σ is an automorphic form on GL_2, but this involves introducing the additional technical gadget of nearly overconvergent sheaves.

As explained in Prof. Skinner's talk, the construction is based on an integral representation of the form

$$\int_{[\mathrm{GL}_2\times\mathrm{GL}_2]} f(h_1,h_2)E(h_1,\chi\chi_\pi;s_1)E(h_2,\chi^{-1};s_2)dh_1dh_2 \doteq L\Big(\Pi, s_1+s_2-\frac{1}{2}\Big)L\Big(\Pi\otimes\chi, s_1-s_2+\frac{1}{2}\Big)$$

where $E(h,\chi;s)$ is a certain Eisenstein series, and f is a vector in Π. To f, we will associate a class $\eta \in H^2(X, \omega(r_2+2, -r_1)(-D))[\Pi_f]$. The Eisenstein series at holomorphic points will define a class in the H^0 of a product of two modular curves. Its pushforward will lie in H^1, and the integral will be interpreted as the cup product of this class with η. For p-adic interpolation, we deform each Eistenstein series to an Eisenstein measure and use pushforward at the level of Igusa tower.

3.1. **Pushforward and cup product.** Let $G = \mathrm{GSp}_4$ and $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \subseteq G$ be the embedding

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}\right) \mapsto \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d\end{pmatrix}$$

Let $X_{G,\mathbf{Q}}$, $X_{H,\mathbf{Q}}$ be Shimura varieties for G and H respectively, both defined over $\mathbf{Q}$ and with some compatible level structures. The embedding induces a morphism $\iota : X_{G,\mathbf{Q}} \to X_{H,\mathbf{Q}}$. Let $\omega_G(k_1,k_2)$ be the automorphic vector bundle on $X_{G,\mathbf{Q}}$ we have been working with. Let $\omega_H(k_1,k_2)$ be the automorphic vector bundle on $X_{H,\mathbf{Q}}$ whose global section corresponds to pairs of modular forms of weights (k_1,k_2). Suppose $\ell_1+\ell_2 = k_1+k_2-2$, then there is a pushforward map

$$\iota_* : H^0(X_{H,\mathbf{Q}}, \omega_H(\ell_1,\ell_2)) \to H^1(X_{G,\mathbf{Q}}, \omega_G(k_1,k_2))$$

This comes from a computation of $\iota^!$ using Kodaira–Spencer isomorphisms [LPSZ19, Section 2.3.3]. If $(k_1,k_2) = (r_1+3, 1-r_2)$, then the right hand side pairs with $\eta \in H^2(X_{G,\mathbf{Q}}, \omega(r_2+2,-r_1)(-D))$ to give an element in the field of definition of Π, which is the algebraic analogue of the period integral by [LPSZ19, Theorem 5.4]. We have ignored the normalization of central characters.

Before p-adic interpolation, we make precise the level structure we will use. Fix $K^p \subseteq G(\mathbb{A}^{p,\infty})$. Let $X_{G,\mathrm{Kl}}(p^m)_{/\mathbf{Q}}$ be the Shimura variety for G with deep Klingen level $K^p\mathrm{Kl}(p^m)$. For H, the tame level will just be $K^p \cap H(\mathbb{A}^{p,\infty})$. The level at p most convenient for us is

$$K_{H,\Delta} := \left\{ h \mid h \equiv \left(\begin{pmatrix} x & * \\ & * \end{pmatrix}, \begin{pmatrix} x & * \\ & * \end{pmatrix}\right) \pmod{p^m}\right\}$$

Let $X_{H,\Delta}(p^m)_{/\mathbf{Q}}$ be the Shimura variety for H with this level structure. As a moduli problem, this classifies pairs of elliptic curves with an isomorphism $E_1[p^m]^\circ \simeq E_2[p^m]^\circ$. An easy group-theoretic calculation gives us a finite morphism

$$\iota_m : X_{H,\Delta}(p^m)_{/\mathbf{Q}} \to X_{G,\mathrm{Kl}}(p^m)_{/\mathbf{Q}}$$

which is a twist of ι. Implicitly, we have fixed compatible toroidal compactifications for everything.

The ordinary classes in $H^1(X_{G,\mathrm{Kl}}(p^m)_{/\mathbf{Q}}, \omega(k_1,k_2)(-D))$ are interpolated by $e_{\mathrm{Kl}}H^1(\mathfrak{X}^{\ge 1}_{G,\mathrm{Kl}}(p), \mathfrak{F}_G(\kappa,k_2)(-D))$, where $\mathfrak{F}_G$ was denoted by $\mathfrak{F}$. Similarly, by classical Hida theory, there is a sheaf $\mathfrak{F}_H(\kappa_1,\kappa_2)(-D)$ on $\mathfrak{X}^{\mathrm{ord}}_{H,\Delta}(p)$ whose global sections interpolate pairs of cusp forms.

Proposition 3.1 ([LPSZ19, Section 4.8])**.** *Suppose $\ell_1 + \ell_2 = k_1 + k_2 - 2$, then there exists a map ι_* making the following diagram commute*

$$\begin{array}{ccc} H^0(X_{H,\Delta}(p^m)_{/\mathbf{Q}_p}, \omega_H(\ell_1,\ell_2)(-D)) & \xrightarrow{\iota_{m,*}} & H^1(X_{G,\mathrm{Kl}}(p^m)_{/\mathbf{Q}_p}, \omega_G(k_1,k_2)(-D)) \\ \downarrow & & \downarrow \\ H^0(\mathfrak{X}^{\mathrm{ord}}_{H,\Delta}(p), \mathfrak{F}_H(\ell_1,\ell_2)(-D)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p & \xrightarrow{\iota_*} & H^1(\mathfrak{X}^{\geq 1}_{H,\mathrm{Kl}}(p^m), \mathfrak{F}_G(k_1,k_2)(-D)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \end{array}$$

Proof. We first define a morphism on the Igusa towers $\tilde{\iota}_m : \mathfrak{Ig}_H(p^m) \to \mathfrak{Ig}_G(p^m)$. It sends a pair of ordinary elliptic curves with level p^m structure $(E_1, \alpha_1; E_2, \alpha_2)$ to $\alpha_1 + \alpha_2 : \mu_{p^m} \to (E_1 \times E_2)[p^m]$. It is not hard to show through the moduli interpretation that this is the pullback of $\iota_m : X_{H,\Delta}(p) \to X_{G,\mathrm{Kl}}(p)$, which can be used to construct an inclusion $\mathfrak{F}_H(\ell_1,\ell_2) \hookrightarrow \iota_1^* \mathfrak{F}_G(k_1,k_2)$. The reason this is an inclusion is that the construction $\mathfrak{F}_H$ involves taking a $(\mathbf{Z}_p^\times)^2$-component, but the construction of $\mathfrak{F}_G$ only involves taking a $\mathbf{Z}_p^\times$-component. Another way of seeing this is via the branching law for the corresponding representations.

Once we have this inclusion, we can apply the relation between ι^* and $\iota^!$ to construct the adjunct of ι_* required in the statement. The commutativity comes from showing that the above moduli interpretation really corresponds to the group-theoretic definition of ι_m. □

Now let R be a flat p-adic $\mathbf{Z}_p$-algebra with two characters $\tau_1, \tau_2 : \mathbf{Z}_p^\times \to R^\times$. In our application, R will be the two variable Iwasawa algebra $\Lambda_2 = \mathbf{Z}_p[[\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times]]$, and the two characters are the canonical characters in each variable. Suppose

$$\mathcal{E} \in H^0(\mathfrak{X}^{\mathrm{ord}}_{H,\Delta}(p), \mathfrak{F}_H(\tau_1,\tau_2)(-D))$$

is a pair of p-adic modular forms. If $\tau_1 + \tau_2 = r_1 - r_2 + 2$, then we can take its pushforward

$$\iota_* \mathcal{E} \in H^1(\mathfrak{X}^{\geq 1}_{G,\mathrm{Kl}}(p), \mathfrak{F}_G(r_1 + 3, 1 - r_2)(-D))$$

After applying the ordinary projector e_{Kl}, by classicality, the result is a class in

$$e_{\leq 0} H^1(X_{G,\mathrm{Kl}}(p)_{/\mathbf{Q}_p}, \omega_G(r_1+3, 1-r_2)(-D)) \otimes_{\mathbf{Q}_p} R[1/p]$$

which can be paired with the fixed element η to give an element in $R[1/p]$, which we denote by $\langle \iota_* \mathcal{E}, \eta \rangle$.

Proposition 3.2. *If $\varphi : R \to \mathbf{Q}_p$ is such that the specialization $\mathcal{E}_\varphi$ is a classical modular form of level p^m, then $\varphi(\langle \iota_* \mathcal{E}, \eta \rangle) = \langle \iota_{m,*} \mathcal{E}_\varphi, \eta \rangle$.*

Proof. This is more subtle than it appears. Indeed, consider the following diagram

$$\begin{array}{ccc} H^1(X_{G,\mathrm{Kl}}(p^m)_{/\mathbf{Q}_p}, \omega_G(k_1,k_2)(-D)) & \longrightarrow & H^1(X_{G,\mathrm{Kl}}(p^m)_{/\mathbf{Q}_p}, \omega_G(k_1,k_2)) \\ \downarrow & & \downarrow \\ H^1(\mathcal{X}^{\geq 1}_{G,\mathrm{Kl}}(p), \mathscr{F}_G(k_1,k_2)(-D)) & \longrightarrow & H^1(\mathcal{X}^{\geq 1}_{G,\mathrm{Kl}}(p), \mathscr{F}_G(k_1,k_2)) \end{array}$$

After taking ordinary parts, the left arrow is an isomorphism by classicality, but the right arrow is *not* injective, so we don't know that the class we obtained from classicality is really $\iota_{m,*}\mathcal{E}_\varphi$, only that they agree when restricted to $\mathcal{X}^{\geq 1}_{G,\mathrm{Kl}}(p)$.

The work-around is to insert a middle layer of overconvergent cohomology. The argument above shows that the two classes agree on a strict neighbourhood of $\mathcal{X}^{\geq 1}_{G,\mathrm{Kl}}(p)$. However, higher Coleman theory works for both $\omega(k_1,k_2)$ and $\omega(k_1,k_2)(-D)$, so from this we can deduce the equality of the two classes. □

3.2. **Eisenstein measure.** We now need to construct a suitable class $\mathcal{E}$ from above. This will be the box product of two Eisenstein measures, already constructed by Katz.

Theorem 3.3. *Let $\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$ with canonical character κ. There exists an Eisenstein measure $\mathcal{E}(\kappa)$, which is a p-adic cusp form of weight $\kappa + 1$ which specializes to Eisenstein series at classical $k \geq 0$.*

3.3. **Interpolation property.** Using the Eisenstein measure, define

$$\mathcal{H} = \mathcal{E}(\kappa) \boxtimes \mathcal{E}(d-\kappa)$$

We can now define an element $\mathcal{L}_p \in \Lambda_2[1/p]$ to be $\langle \iota_* \mathcal{H}, \eta \rangle$. By what we have done before, there is the following vague interpolation property: suppose $x = k + \chi$ is an algebraic weight, where $0 \leq k \leq d$ and χ is a finite order character, then

$$\mathcal{L}_p(x) \doteq L\big(\Pi, \frac{1+d}{2}\big) L\big(\Pi \otimes \chi, \frac{1-d}{2} + k\big)$$

The Euler factor comes out of analyzing the local zeta integral at p, and there is an archimedean period, which can be defined using Whittaker models at infinity.

We would be done if we can show that $L\big(\Pi, \frac{1+d}{2}\big) \neq 0$. If $d \geq 2$, then this is in the region of absolute convergence for the Euler product. If $d = 1$, then this is on the line $\mathrm{Re}(s) = 1$, and we can apply [JS77]. If $d = 0$, then this is the central critical value, and the non-vanishing is not-guaranteed. It is expected that it is non-vanishing for the twist of Π by "most" characters, but unlike in the GL_2-case, this is not known.

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