COLEMAN FAMILY AND THE EIGENCURVE

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This is the notes for a talk given at the student seminar on p-adic automorphic forms in Fall 2020. We will first construct Coleman families following [Pil13] and [AIS14]. Then we will survey Buzzard's axiomatic construction of eigenvarieties [Buz07] and apply it to the case at hand.

Fix a prime p throughout. For notational simplicity, we assume $p \neq 2$. Further fix a positive integer N coprime to p, which will be the tame level.

1. Coleman family

1.1. **Review from last time.** Let $X = X_1(N)$ be the (compactified) modular curve of level $\Gamma_1(M)$, viewed as an algebraic curve defined over $\mathbf{Z}_p$. It classifies pairs $\{(E, \varphi_N : \mu_N \hookrightarrow E[N])\}$. We now summarize some key points from the talk last time.

– There is a line bundle ω on X such that classical modular forms of level $\Gamma_1(N)$ and weight k are elements of $H^0(X, \omega^{\otimes k})$.
– If $p \geq 5$, then there is a modular form of level 1 and weight $p-1$ which lifts the Hasse invariant. It defines a section $\mathcal{E} \in H^0(X, \omega^{\otimes(p-1)})$.
– We have the following diagram

$$\begin{array}{ccccccccc}
 & & \mathrm{Ig}_{1,\infty} & \hookrightarrow & \mathrm{Ig}_{m,\infty} & \hookrightarrow & \dots & \rightsquigarrow & \mathfrak{Ig}_\infty \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & {\scriptstyle 1+p\mathbf{Z}_p} & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & \mathrm{Ig}_{1,1} & \hookrightarrow & \mathrm{Ig}_{m,1} & \hookrightarrow & \dots & \rightsquigarrow & \mathfrak{Ig}_1 \\
 & & {\scriptstyle (\mathbf{Z}/p\mathbf{Z})^\times}\downarrow & & \downarrow & & & & \downarrow \\
X & & X[\mathcal{E}^{-1}]_{/p} & \hookrightarrow & X[\mathcal{E}^{-1}]_{/p^m} & \hookrightarrow & \dots & \rightsquigarrow & \mathfrak{X}^{\mathrm{ord}} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\operatorname{Spec}\mathbf{Z}_p & & \operatorname{Spec}\mathbf{Z}/p\mathbf{Z} & \hookrightarrow & \operatorname{Spec}\mathbf{Z}/p^m\mathbf{Z} & \hookrightarrow & \dots & \rightsquigarrow & \operatorname{Spf}\mathbf{Z}_p
\end{array}$$

where $\mathrm{Ig}_{m,n}$ is the affine scheme over $\mathbf{Z}/p^m\mathbf{Z}$ classifying triples

$$(E, \varphi_p, \varphi_N),\ \varphi_p : \mu_{p^n} \hookrightarrow E[p^n],\ \varphi_N : \mu_N \hookrightarrow E$$

It is a $(\mathbf{Z}/p^n\mathbf{Z})^\times$-torsor over $X[\mathcal{E}^{-1}]_{/p^m}$.
– The space of p-adic modular forms is defined by

$$V := \varprojlim_m \varinjlim_n H^0(\mathrm{Ig}_{m,n}, \mathcal{O}_{\mathrm{Ig}_{m,n}}) = H^0(\mathfrak{Ig}_\infty, \mathcal{O}_{\mathfrak{Ig}_\infty})$$

It carries a natural action of $\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$, coming from the torsor structure.
– If $k \geq 2$ is an integer, then we have an embedding

$$H^0(X, \omega^{\otimes k}) \hookrightarrow V[k]$$

where $V[k]$ is the subspace where Λ acts by the character $z \mapsto z^k$.
– The Hecke algebra acts on V, and the above embedding is Hecke-equivariant. The part of V where U_p acts by a p-adic unit is particularly nice, described by Hida theory.

We now reformulate parts of this slightly. Working with formal schemes directly, let $\pi : \mathfrak{Ig}_\infty \to \mathfrak{X}^{\mathrm{ord}}$ be the projection and $\Omega = \pi_* \mathcal{O}_{\mathfrak{Ig}_\infty}$, then

$$V = H^0(\mathfrak{Ig}_\infty, \mathcal{O}_{\mathfrak{Ig}_\infty}) = H^0(\mathfrak{X}^{\mathrm{ord}}, \Omega)$$

There is a natural action of Λ on the sheaf Ω. Given an algebra homomorphism $\kappa : \Lambda \to \mathbf{C}_p$, we can take the κ-isotypic component $\Omega[\kappa]$, then

- $M_\kappa := H^0(\mathfrak{X}^{\mathrm{ord}}, \Omega[\kappa])$ is the space of p-adic modular forms of "weight" κ.
- If $k \in \mathbf{Z}$ and κ_k is induced from $z \mapsto z^k$, then $\Omega[\kappa_k] \simeq \omega^{\otimes k}$.
- The restriction $H^0(\mathfrak{X}, \omega^{\otimes k}) \to H^0(\mathfrak{X}^{\mathrm{ord}}, \omega^{\otimes k}) \simeq M_k$ is the injection of classical modular forms of weight k into p-adic modular forms.

The goal of this talk is to refine the picture by introducing overconvergence.

Remark 1.1. As stated, the theory already sounds perfectly reasonable. The issue however is that V is too large for a rich structural theory. One aspect is that for any weight k, the spectrum of U_p acting on $V[k] \otimes_{\mathbf{Z}_p} \mathbf{C}_p$ is all of $\mathbf{C}_p$, so we don't have the classical eigenform theory, and we cannot expect a nice classicality result. Introducing overconvergence is also necessary to relate it to p-adic cohomology theories.

Disclaimer. For the rest of the talk, we will work entirely in the framework of rigid geometry. This has the effect of replacing various global sections by their rational versions, viewed as p-adic Banach spaces. However, it is necessary to introduce formal models at various stages. We will sweep it (and a few other things) under the rug and refer to [Pil13] and [AIP18] for the details.

1.2. **Overconvergence.**

We start with a model example.

Example 1.2. Let $D = \operatorname{Sp} \mathbf{Q}_p\langle T\rangle$ be the rigid unit disc, then the space of overconvergent functions on D is

$$\mathbf{Q}_p\langle T\rangle^\dagger := \bigcup_{r>1} \left\{ \sum_{n\geq 0} a_n T^n \,\middle|\, |a_n|_p r^n \to 0 \right\}$$

In other words it consists of power series which converge on a disc of radius $1+\epsilon$ for some $\epsilon > 0$. This is no longer a Banach space. Instead, it is a direct limit of Banach spaces, with compact transition maps.

Similarly to how rigid spaces are built from D, there is a general theory of "dagger spaces" built from $D^\dagger$, with the associated integral theory of "weak formal schemes". We will define what's needed by hand.

In the modular curve case, we have a rigid space X_{rig} and an affinoid open subspace X^{ord} obtained by removing an open disc of radius 1 around each supersingular point. More precisely, taking the valuation of (a local lift of) the Hasse invariant determines a function $H : X_{\mathrm{rig}} \to [0,1]$, and the ordinary locus is exactly $H^{-1}(\{0\})$. For $v < 1$, we can enlarge X^{ord} by setting

$$X(v) = H^{-1}([0,v])$$

If v is rational, then $X(v)$ is affinoid. It has an invertible sheaf ω.

Definition 1.3. Given an integer k, the space of *overconvergent modular forms* of weight k is

$$M_k^\dagger := H^0(X^{\mathrm{ord},\dagger}, \omega^{\otimes k}) = \lim_{v\to 0} H^0(X(v), \omega^{\otimes k})$$

As in the model example, the transition maps in the inductive limit are very well-behaved. Recall from functional analysis that a map between topological vector spaces is *completely continuous* if it is a strong limit of finite-rank maps.

Proposition 1.4. *Let $\mathcal{F}$ be a coherent sheaf on X. If $v < v'$, then the restriction map*

$$H^0(X(v'), \mathcal{F}) \to H^0(X(v), \mathcal{F})$$

is completely continuous.

Proof. See section 6.4 of [Bos14]. □

This gives a good theory when k is an integral weight. To construct p-adic families, Coleman used Eisenstein series in [Col97]. In this talk, we will follow Pilloni [Pil13] and directly construct a sheaf ω^κ on $X(v)$ for v sufficiently small, depending on κ.

1.3. **Partial Igusa tower.** Let E be an elliptic curve over a p-adic complete local ring, then the p-divisible group associated to E has the étale-connected exact sequence

$$0 \to E[p^\infty]^\circ \to E[p^\infty] \to E[p^\infty]^{\text{ét}} \to 0$$

Moreover, if E is ordinary, then $\mu_{p^\infty} \simeq E[p^\infty]^\circ$. This determines a subgroup $H_{\infty,E} \subseteq E[p^\infty]$. The Igusa curve Ig_∞ can be described as follows: for K a finite extension of $\mathbf{Q}_p$, over a point $x \in X^{\text{ord}}(K)$ corresponding to a pair (E, φ_N),

$$\mathrm{Ig}_\infty|_x(\mathbf{C}_p) = \{y : H_{\infty,E} \xrightarrow{\sim} \mu_{p^\infty}\}$$

If E is not ordinary, then the connected component of its reduction is everything, so we can no longer single out a subgroup of $E[p^\infty]$. However, if it is not too supersingular, the theory of *canonical subgroup* gives us a good extension at finite level.

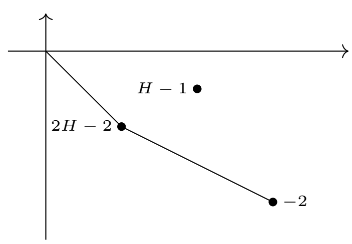
Example 1.5 (Buzzard)**.** Let $p = 2$. For $a \in \mathbf{C}_2$, let E_a be the elliptic curve

$$y^2 + y + axy = x^3 + x^2$$

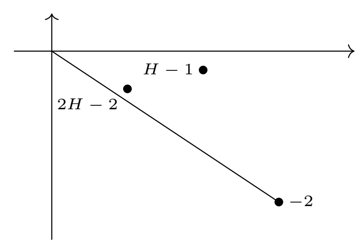
This has good reduction at 2. With the differential $\omega = \frac{dx}{ax+1}$, it has Hasse invariant a. The x-coordinates of the three non-trivial 2-torsion points are roots of

$$x^3 + \left(\frac{a^2}{4} + 1\right)x^2 + \frac{a}{2}x + \frac{1}{4}$$

Let $H = v_p(a)$, then we can draw its Newton polygon.



$H < \frac{2}{3}$



$H \geq \frac{2}{3}$

If H is small, then one root is singled out by its valuation, so we have a canonical $\mathbf{Z}/2\mathbf{Z} \subseteq E_a[2]$. As $H \to 0$, this subgroup reduces to $E_a[2]^\circ$ modulo 2. It can be seen as a deformation of the connected part as E_a becomes slightly supersingular.

Theorem 1.6 (Katz–Lubin, see [Kat73, Chapter 3])**.** *Let E be an elliptic curve over a p-adic ring. Suppose*

$$H(E_x) < \frac{1}{p^{n-2}(p+1)}$$

for every geometric point x, then E has a canonical subgroup $H_{n,E} \subseteq E[p^n]$, in the sense that

- $H_{n,E}(\mathbf{C}_p) \simeq \mathbf{Z}/p^n\mathbf{Z}$.
- *The construction is compatible with base change.*
- *If x is a point of characteristic p and E_x is ordinary, then H_{n,E_x} is the connected component of $E_x[p^n]$.*

Fix $n \geq 1$ and $v < \frac{1}{p^{n-2}(p+1)}$, then the universal elliptic curve E_{univ} over $X(v)$ satisfies the condition of the theorem, so it has a canonical subgroup $H_{n,E_{\text{univ}}} \subseteq E_{\text{univ}}[p^n]$. Using this subgroup, we can extend the partial Igusa tower $\mathrm{Ig}_n \to X^{\text{ord}}$ to $X(v)$, characterized by the property that give a finite extension K of $\mathbf{Q}_p$ and a point $x \in X(v)(K)$ corresponding to a pair (E, φ_N),

$$\mathrm{Ig}_n|_x(\mathbf{C}_p) = \{y : H_{n,E} \xrightarrow{\sim} \mu_{p^n}\}$$

The restriction of this to X^{ord} is exactly the Igusa tower of level n we defined before.

1.4. **Construction of the sheaf.** Recall that the Igusa tower Ig_∞ is a $\mathbf{Z}_p^\times$-torsor over X^{ord}, and this $\mathbf{Z}_p^\times$-action is used to define weight. The partial Igusa tower we constructed is only a $(\mathbf{Z}/p^n\mathbf{Z})^\times$-torsor. We will construct a further covering of the partial Igusa tower with an action by $1+p^M\mathbf{Z}_p$ for some M. This space will be the replacement of the full Igusa tower in the overconvergent setting.

Suppose we are given a morphism of group schemes $y : H_{n,E} \to \mu_{p^n}$, with everything defined over some formal scheme. On μ_{p^n}, there is the canonical differential $\frac{dT}{T}$. Its pullback under y defines a differential in $\omega_{H_{n,E}}$, which we denote by $\mathrm{HT}(y)$.

Theorem 1.7 (Théorème 3.1 of [Pil13])**.** *For each $n \geq 1$ and $v < \frac{1}{p^{n-2}(p+1)}$, there is a diagram of rigid spaces $F_n \to \mathrm{Ig}_n \to X(v)$. It is characterized by the following property: given a finite extension K of $\mathbf{Q}_p$ and a point $x \in X(v)(K)$ corresponding to a pair $(\mathcal{E}, \varphi_N)$ over $\mathcal{O}_K$,*

$$F_n|_x(\mathbf{C}_p) = \{(y,\omega) \,|\, y : H_{n,\mathcal{E}/K} \xrightarrow{\sim} \mu_{p^n},\ \omega \in \omega_{\mathcal{E}},\ \mathrm{HT}(y) = \omega|_{\omega_{H_{n,\mathcal{E}}}}\}$$

To understand this definition, fix an elliptic curve over $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_K$ with a level n canonical subgroup $H_{n,\mathcal{E}} \subseteq \mathcal{E}$. We now have a morphism of coherent sheaves over $\mathcal{O}_K$

$$\mathcal{O}_K \simeq \omega_{\mathcal{E}} \to \omega_{H_{n,\mathcal{E}}}$$

defined by pullback. Over the generic fibre, $H_{n,\mathcal{E}}$ is étale, so $\omega_{H_{n,\mathcal{E}}}|_K = 0$. Therefore, the final equation in the definition of F_n is a congruence condition, and F_n replaces a point of Ig_n by a ball. The map $q_n : F_n \to X(v)$ has a $\mathbf{Z}_p^\times$-action given by $(y,\omega) \mapsto (uy, u\omega)$.

Definition 1.8. Let $\kappa : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ be a character. Define the modular sheaf ω^κ on $X(v)$ by

$$\omega^\kappa := ((q_n)_* \mathcal{O}_{F_n})[\kappa]$$

For this to be a reasonable definition, we expect that (1) ω^κ is an invertible sheaf; (2) $\omega^{\kappa_k} = \omega^{\otimes k}$ if $k \in \mathbf{Z}$; (3) as n and v vary, the various ω^κ are compatible. We need an exact description of the fibres of π in order to verify these properties.

Theorem 1.9 (Proposition 3.1 of [Pil13])**.** *Let $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_K$ be an elliptic curve with a canonical subgroup of level n. The module $\omega_{H_{n,\mathcal{E}}}$ is isomorphic to $\mathcal{O}_K/a\mathcal{O}_K$. Its submodule generated by the image of* HT *is $c\mathcal{O}_K/a\mathcal{O}_K$, where*

$$v_p(a) = n - \frac{p^n-1}{p-1}H(\mathcal{E}), \quad v_p(c) = \frac{1}{p-1}H(\mathcal{E})$$

Therefore, $F_n \to X(v)$ is a bundle of $p^{n-1}(p-1)$ balls, with centres of valuation $\frac{H}{p-1}$ and radii $p^{-n+\frac{1-p^n}{1-p}H}$.

Example 1.10. In the example with E_a, suppose $a \in \mathcal{O}_K$, then using the local parameter $z = -x/y$ at infinity, it's easy to compute that $H_{1,E_a} \simeq \operatorname{Spec} \mathcal{O}_K[z]/(z^2 - a'z)$, where $v_p(a') = 1 - v_p(a)$. It then follows that $\omega_{H_{1,E_a}} \simeq \mathcal{O}_K/a'\mathcal{O}_K$. The non-trivial group scheme morphism $H_{1,E_a} \to \mu_2$ is given on coordinates by $z \mapsto 1 - \frac{2}{a'}z \in \mu_2$. Under HT, it is mapped to an element of valuation $v_p(a)$.

So concretely, the fibre of ω^κ above a point x consists of analytic functions on

$$\bigsqcup_{u \in (\mathbf{Z}/p^n\mathbf{Z})^\times} B(ux_0, p^{-M}), \quad \text{where } v(x_0) = \frac{H}{p-1},\ M = n - \frac{p^n-1}{p-1}H$$

which transforms by κ under the action of $\mathbf{Z}_p^\times$. Since the action is transitive on the components, it remains to look at functions on $B(x_0, p^{-M})$ satisfying the condition

$$f(ux) = \kappa(u)f(x) \text{ for all } u \in 1 + p^n\mathbf{Z}_p$$

The function $g(x) := \kappa\big(\frac{x}{x_0}\big)f(x_0)$ is analytic on $B(x_0, r)$ for some $r > 0$ depending on κ. It agrees with f on $B(x_0, \min(r, p^{-M}))$, so by analytic continuation, *if* any such f exists, it must be g. Therefore, we have shown that $\omega^\kappa|_x$ is either 0 or a 1-dimensional space, depending on the relation between κ and H. We need to make this relation precise.

Asides on weight space. The weight space $\mathcal{W}$ consists of all characters $\mathbf{Z}_p^\times \to \mathbf{C}_p^\times$. As a rigid space, we have an identification

$$\mathcal{W} \xrightarrow{\sim} (\widehat{\mathbf{Z}/p\mathbf{Z}})^\times \times B(1, 1^-)$$

where the first projection takes the tame character and the second projection is evaluation at $1+p$. For each $t > 0$, we can define a subset $\mathcal{W}_t = \{(\chi, \lambda) \in \mathcal{W} \mid v_p(\lambda - 1) \geq t\}$. The subsets $\mathcal{W}_t$ exhausts $\mathcal{W}$ as $t \downarrow 0$.

Given $w > 0$, a character $\kappa = (\chi, \lambda) \in \mathcal{W}$ extends to an analytic function on $1 + p^w \mathcal{O}_{\mathbf{C}_p}$ if and only if

$$\inf_{n\geq 1}(nw - v_p(n)) + \inf_{n \geq 1}(nv_p(\lambda - 1) - v_p(n)) > \frac{p}{p-1}$$

Therefore, there is a function $w(t)$ such that all $\kappa \in \mathcal{W}_t$ extends to $1 + p^w\mathcal{O}_{\mathbf{C}_p}$ if and only if $w > w(t)$. The function $w(t)$ is decreasing in t, i.e. near the boundary of $\mathcal{W}$, characters are only analytic on a small disc.

Theorem 1.11. *Fix parameters n, v, t satisfying*

$$v < \frac{p-1}{p^n}, \quad w(t) < n - \frac{p^n}{p-1}v$$

For all $\kappa \in \mathcal{W}(t)$, the sheaf ω^κ is invertible. If $k \in \mathbf{Z}$ and κ_k is the character $z \mapsto z^k$, then $\omega^{\kappa_k} \simeq \omega^{\otimes k}$.

Proof. The first part follows from the discussion before the aside and the definition of $w(t)$. For the second part, we have a natural restriction map $\omega^{\otimes k} \to \omega^{\kappa_k}$. Concretely, over a point x, this is induced from the disjoint union of inclusions

$$\bigsqcup_{u \in (\mathbf{Z}/p^n\mathbf{Z})^\times} B\big(ux_0, p^{-M}\big) \to \mathbf{C}_p^\times$$

using the notations from before. Under the assumption on v, this is an injection. Moreover, the weight-κ_k functions on the left hand side are translates of $z \mapsto z^k$, which extends to an analytic function on $\mathbf{C}_p^\times$. This extension property proves the isomorphism. □

Definition 1.12. Let $\kappa \in \mathcal{W}$. The space of *overconvergent modular forms* of weight κ is

$$M_\kappa^\dagger := \lim_{v \to 0} H^0(X(v), \omega^\kappa)$$

More precisely, given κ, it lies in some space $\mathcal{W}(t)$. We first choose $n > w(t) + 1$, then choose $v < \frac{p-1}{p^n}$. Now the conditions of the theorem are satisfied. It is easy to check compatibility between different n. The theorem also tells us $M_{\kappa_k}^\dagger$ agrees with the integer weight space defined before.

1.5. **Hecke operators.** We now look at the actions of Hecke operators. For $\ell \neq p$, the construction by correspondence works in very much the same way as usual, and we refer to [Pil13, Section 4.1] for more details. Instead, we will focus on the U_p-operator.

Suppose $v < \frac{p}{p+1}$, so the first canonical subgroup exists. Consider the following diagram

$$\begin{array}{ccc} & D_p & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ X(v) & & X\left(\frac{v}{p}\right) \end{array}$$

Here, D_p parametrizes the tuples (E, φ_N, D) with $(E, \varphi_N) \in X(v)$ and $D \subseteq E$ a subgroup of rank p, which is *not* the canonical subgroup. The map p_1 is the "forget D" map, which is finite of degree p. The map p_2 sends (E, φ_N, D) to $(E/D, \varphi_N/D)$. A key observation implicit in the diagram is that $H(E/D) = \frac{1}{p}H(E)$, which crucially uses $D \neq H_{1,E}$, see [Kat73, Theorem 3.10.7].

Theorem 1.13. *Suppose $n \geq 2$ and $v < \frac{p-1}{p^n}$, then there is a natural map*

$$\pi : F_n \times_{X(v), p_1} D_p \to F_{n-1} \times_{X(\frac{v}{p}), p_2} D_p$$

Proof. We show this at the level of points. Given a finite extension K of $\mathbf{Q}_p$, the K-points of the left hand side parametrizes tuples $(\mathcal{E}, \varphi_N, y, \omega, D)$, where

– $(\mathcal{E}, \varphi_N)$ is an elliptic curve over $\mathcal{O}_K$ such that $H(\mathcal{E}) < v$, so it has a level n canonical subgroup $H_{n,\mathcal{E}}$.
– $D \subseteq \mathcal{E}$ is an order p subgroup which is not the canonical subgroup.
– y is an isomorphism $H_{n,\mathcal{E}/K} \xrightarrow{\sim} \mu_{p^n}$.

– $\omega \in \omega_E$ is equal to $\mathrm{HT}(y)$ when restricted to $\omega_{H_{n,\mathcal{E}}}$.

Let $(\mathcal{E}', \varphi'_N) = (\mathcal{E}/D, \varphi_N/D)$. We must construct the corresponding (y', ω') on this pair.

Since D is not the canonical subgroup, the image of $H_{n,\mathcal{E}}$ in $\mathcal{E}'$ must be the level n canonical subgroup. The projection $H_{n,\mathcal{E}/K} \to H_{n,\mathcal{E}'/K}$ is an isomorphism over K, since it is surjective and both sides have rank p^n, so y determines an isomorphism $y'' : H_{n,\mathcal{E}'/K} \xrightarrow{\sim} \mu_{p^n}$, and we define y' to be the restriction of y'' to $H_{n-1,\mathcal{E}'/K}$. Pullback also gives an isomorphism $\omega_{\mathcal{E}'/K} \simeq \omega_{\mathcal{E}/K}$, which defines an $\omega' \in \omega_{\mathcal{E}'/K}$.

It remains to check that ω' is integral and satisfies the congruence condition with $\mathrm{HT}(y')$. The key point is that at an integral level, $\omega_{\mathcal{E}'} \to \omega_{\mathcal{E}}$ is multiplication by an element with valuation $\frac{1}{p}H(\mathcal{E})$. Using the explicit description of the fibres of F_n from earlier, we see that the congruence relation only holds at level $n-1$. □

Using the theorem, we can define the following composite

$$H^0\left(X\left(\frac{v}{p}\right), (q_{n-1})_*\mathcal{O}_{F_{n-1}}\right) \xrightarrow{p_2^*} H^0(D_p, p_2^*(q_{n-1})_*\mathcal{O}_{F_{n-1}})$$
$$\xrightarrow{\pi^*} H^0(D_p, p_1^*(q_n)_*\mathcal{O}_{F_n}) \xrightarrow{\frac{1}{p}(p_1)_*} H^0(X(v), (q_n)_*\mathcal{O}_{F_n})$$

Suppose $\kappa \in \mathcal{W}(t)$ for a sufficiently large t, then we can take κ-components of everything and get a map $\alpha_v : H^0\big(X\big(\frac{v}{p}\big), \omega^\kappa\big) \to H^0(X(v), \omega^\kappa)$. The U_p operator is its composition with restriction

$$U_{p,v} : H^0(X(v), \omega^\kappa) \xrightarrow{\mathrm{res}} H^0\left(X\left(\frac{v}{p}\right), \omega^\kappa\right) \xrightarrow{\alpha_v} H^0(X(v), \omega^\kappa)$$

Note that the first map is completely continuous by Proposition 1.4, so $U_{p,v}$ is a completely continuous endomorphism of $H^0(X(v), \omega^\kappa)$. It is compatible between different v.

Lemma 1.14. *Suppose $n \geq 2$ and $0 < v' < v < \frac{p-1}{p^n}$, then the following diagram commutes*

$$\begin{array}{ccc} H^0(X(\frac{v}{p}), \omega^\kappa) & \xrightarrow{\alpha_v} & H^0(X(v), \omega^\kappa) \\ \big\downarrow{\scriptstyle \mathrm{res}} & & \big\downarrow{\scriptstyle \mathrm{res}} \\ H^0(X(\frac{v'}{p}), \omega^\kappa) & \xrightarrow{\alpha_{v'}} & H^0(X(v'), \omega^\kappa) \end{array}$$

It follows that all the $U_{p,v}$ as $v \to 0$ defines an operator $U_p : M_\kappa^\dagger \to M_\kappa^\dagger$. Moreover, we also have $U_{p,\frac{v}{p}} = \mathrm{res} \circ \alpha_v$, which will be important later.

1.6. **Coleman family.** We now let the characters vary in a family. Fix numbers $t, v \in \mathbf{Q}_{>0}$ and $n \geq 2$ such that $w(t) < n - 1 - \frac{p^n}{p-1}v$. After multiplying $\pi : F_n \to X(v)$ by $\mathcal{W}(t)$, we have a sheaf

$$\Omega = (\pi \times 1)_*\mathcal{O}_{F_n \times \mathcal{W}(t)}$$

on $X(v) \times \mathcal{W}(t)$. On $\mathcal{W}(t)$, there is a universal character $\kappa_{\mathrm{univ}} : \mathbf{Z}_p^\times \to \mathcal{O}(\mathcal{W}(t))^\times$. Let ω_{univ} be the κ_{univ}-component of Ω. This interpolates ω^κ in the sense that if κ is associated to the point $x \in \mathcal{W}(t)$, then

$$x^*\omega_{\mathrm{univ}} = \omega^\kappa$$

By Theorem 1.11, this is an invertible sheaf independent of n. Since the space $X(v) \times \mathcal{W}(t)$ is affinoid, ω_{univ} is associated to a Banach module $M_{v,t}$. Its elements are Coleman families.

2. Eigenvarieties

We will now describe Buzzard's eigenvariety machine [Buz07], in particular explaining what eigenvarieties are. In the first section, we will describe the input and verify that the work in the previous section produces the right data. Afterwards, we can forget about all the geometric construction, as the problem now lies entirely in p-adic functional analysis.

2.1. **Input and output.** The eigenvariety machine can be summarized as follows.
Input:

– Weight space: a reduced affinoid rigid space $\mathrm{Sp}\,R$.
– Overconvergent automorphic forms: a *projective* Banach R-module M.
– Hecke algebra: an R-algebra $\mathbf{T}$ with a R-linear map $R \to \mathrm{End}_R(M)$.
– U_p-operator: an element $u \in \mathbf{T}$ whose induced map $u : M \to M$ is completely continuous.

Output:

– Rigid spaces $C \to Z \to \mathrm{Sp}\,R$.
– Points of C correspond to *finite slope* eigensystems of $\mathbf{T}$ appearing in M.
– Points of Z correspond to reciprocals of non-zero eigenvalues of u.
– The map $C \to Z$ is finite.
– The map $C \to \mathrm{Sp}\,R$ is locally finite on source, and each irreducible component has Zariski-dense image.

The space C is the *eigenvariety*, and Z is the spectral variety.

We need to explain the words “projective” and “finite slope”. The first is a technical hypothesis. The second is an essential restriction on the automorphic forms that we can interpolate this way.

Definition 2.1. Let R be a Banach algebra. Let M be a Banach R-module, then M is projective if for any surjection of Banach R-modules $\pi : P \to Q$ and continuous map $f : M \to Q$, we can find a continuous map $g : M \to P$ such that $f = \pi \circ g$.

In the usual way one proves projective modules are direct summands, one can prove the following equivalent characterization, though the open mapping theorem is required.

Lemma 2.2. *In the above setting, M is projective if and only if there exists an R-module N such that $M \oplus N$ is isomorphic to $c_0(I, R)$ with the supremum norm, where I is some set.*

Definition 2.3. Let λ be an eigensystem (i.e. an algebra homomorphism $\mathbf{T} \to L$ for some finite extension L of $\mathbf{Q}_p$). Its *slope* is $v_p(\lambda(u))$. It has *finite slope* if $\lambda(u) \neq 0$.

Remark 2.4. In the case of modular forms, the finite slope condition is that $a_p(f) \neq 0$. From a representation theoretic point of view, this says the component of f at p is supercuspidal. They are somehow more rigid and do not form nice families.

2.2. **The eigencurve.** In our case, the weight space is $\mathcal{W}(t)$ for some $t \in \mathbf{Q}_{>0}$. Choose v sufficiently small, then from the previous section, we have a Banach module $M_{v,t}$. The Hecke algebra $\mathbf{T}$ will be the polynomial ring over $\mathbf{Z}_p$ generated by T_ℓ for $\ell \neq p$ and U_p. In our construction of U_p, we showed that it is completely continuous. It remains to check that $M_{v,t}$ is projective. This is [Pil13, Corollaire 5.1]. The point is to take a covering by finitely many étale open subsets which trivializes the fibration $F_n \to X(v)$ and resolve $M_{v,t}$ using the associated Čech complex. The machine produces an eigenvariety $\mathcal{C}_{v,t} \to \mathcal{W}(t)$. Since $\mathcal{W}(t)$ has dimension one, $\mathcal{C}_{v,t}$ is a curve.

To construct the eigencurve over $\mathcal{W}$, we need to glue together $\mathcal{C}_{v,t}$ as $t \downarrow 0$. This requires knowing that $\mathcal{C}_{v,t}$ does not depend on v. The key fact is [Buz07, Lemma 5.6], which we now state.

Lemma 2.5. *Let $(R, M, \mathbf{T}, u)$ and $(R, M', \mathbf{T}, u')$ be two input data. Suppose there is are continuous R-linear maps $\alpha : M \to M'$ and $r : M' \to M$ such that r is completely continuous, $r \circ \alpha = u'$, and $\alpha \circ r = u$, then the eigenvarieties associated to the two data are canonically isomorphic.*

The discussion following Lemma 1.14 exactly tells us that the restriction $M_{v,t} \to M_{\frac{v}{p},t}$ and the α map going the other way satisfies the conditions of the lemma. This gives us the compatibility required, which leads to the eigencurve.

Theorem 2.6 (Coleman–Mazur, Buzzard, Pilloni, etc.)**.** *There exists a rigid analytic curve $\mathcal{C}$ with a locally finite map $w : \mathcal{C} \to \mathcal{W}$ such that for any $\kappa \in \mathcal{W}$, the fibre of w corresponds to finite slope overconvergent eigenforms of weight κ.*

2.3. **Construction.** Let $(R, M, \mathbf{T}, u)$ be an input tuple. As a motivating example, if M is finite over R and u is an isomorphism, then the diagram $C \to Z \to \mathrm{Sp}\,R$ is associated to the following morphisms of R-algebras

$$R \to R[T]/(F(T)) \to \mathbf{T}_M$$

where $F(T) = \det(1 - Tu|M)$, and $\mathbf{T}_M$ is the image of $\mathbf{T}$ in $\mathrm{End}_R(M)$. The second map sends T to u^{-1}. In our case, M is not finite over R, which is why we need some p-adic functional analysis.

First, we construct the spectral variety. The map $u : M \to M$ is completely continuous, so we can form its characteristic power series

$$F(T) = \det(1 - Tu|M) \in R[[T]]$$

There are two ways of making this precise: using approximation by finite dimensional subspaces as in [Col97] or using the formal identity $F(T) = \sum_{n=0}^{\infty} \mathrm{Tr}(\wedge^n u)(-T)^n$ and build on a theory of traces as in [Sch02]. In either case, it satisfies the desired properties, and moreover lies in the subring $R\{\{T\}\}$ of those series $\sum_{i=0}^{\infty} a_i T^i$ satisfying $\|a_n\|_R\, r^n \to 0$ for all $r > 0$.

We now define $Z \subseteq \mathrm{Sp}\, R \times \mathbb{A}^1_{\mathrm{rig}}$ to be the the vanishing locus of $F(T)$. Above each point of $\mathrm{Sp}\, R$, the space Z encodes the reciprocals of the infinitely many non-zero eigenvalues of u. The eigenvariety introduces a finite cover of this to take care of the other Hecke operators. Classical Fredholm theory tells us to expect each individual u-eigenspace to be finite dimensional, so we may hope the construction in the motivating example works over Z.

Let $\pi : Z \to \mathrm{Sp}\, R$ be the natural projection. First suppose Y is an affinoid subdomain of Z such that $\pi|_Y$ is an isomorphism. In this case, $\mathcal{O}(Y) = R[T]/(1 - aT)$ for some $a \in R^\times$, so a is the inverse of an eigenvalue of u. By the definition of Z, we have a factorization $F(T) = (1 - aT)S(T)$, where $S(t) \in R\{\{T\}\}$ is not divisible by $1 - aT$. Now consider the operator $u - a$ acting on M. Fredholm theory gives us a decomposition $M = N \oplus F$, where $N = \ker(u - a)$ and $F = \mathrm{Im}(u - a)$. The two projectors are in the closure of $R[f]$ in $\mathrm{End}_R(M)$. Moreover, N is finite over R, and $u - a$ is invertible on F. Since $\mathbf{T}$ is commutative, it preserves this decomposition. Let $\mathbf{T}_N$ be the image of $\mathbf{T}$ in $\mathrm{End}_R(N)$, then we have a diagram

$$R \to R[T]/(1 - aT) \to \mathbf{T}_N$$

as in the introductory example. This gives a finite morphism $\mathrm{Sp}\, \mathbf{T}_N \to Y$. Any eigensystem of $\mathbf{T}$ with u acting as a cannot occur in F, so $\mathrm{Sp}\, \mathbf{T}_N$ is the piece of the eigenvariety above Y.

Similar constructions work if Y is only required to be finite above $\mathrm{Sp}\, R$. We can also drop the surjectivity requirement and only assume that $\pi|_Y$ is finite above an affinoid subdomain of $\mathrm{Sp}\, R$. The key fact, proven in [Buz07, Section 4], is that the collection of all such Y form an admissible covering of Z. The proof is based on a theorem of Conrad which deduces finiteness from quasi-finiteness under certain assumptions. Given this, it is not hard to glue the pieces above Y together for different Y.

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