COMPLEX GEOMETRY IN A NUTSHELL

SAAD SLAOUI

Contents

1. Generalizations of results from complex analysis in one variable

Throughout, U refers to some open subset of $\mathbb{C}^n$, $n \geq 2$, and $\mathcal{O}(U)$ is the ring of holomorphic functions on U.

Many results from complex analysis in one variable generalize well to several complex variables. These include the identity theorem (if $f, g \in \mathcal{O}(U)$ agree on some open $V \subset U$ then $f = g$), the open mapping theorem, the Cauchy integral formula (simply induct on the dimension and apply the 1 dimensional result), the existence of (generalized) power series expansions near a point for holomorphic functions. One important result that does not generalize is the Riemann mapping theorem: for $n \geq 2$, it is not true that all simply connected regions are biholomorphic to the open unit ball $B_1^n(0)$.

By Cauchy's integral formula, holomorphicity of a continuous function can be checked one variable at a time. A useful lemma for proving many of these results is the fact that if f is a holomorphic function of $(n-1)$-variables, then we can obtain a holomorphic function in n-variables via an integral formula (here C_ϵ is an ϵ-circle in the z_1-complex line):

$$g(z_1, ..., z_n) := \frac{1}{2\pi i} \int_{C_\epsilon} f(w, z_2, ..., z_n) dw.$$

Thinking of isolated points as zero sets of holomorphic functions in one variable, the Riemann removable singularities theorem generalizes to an extension result for holomorphic functions $g \in \mathcal{O}(U - Z(f))$, for $Z(f)$ the vanishing set of a holomorphic function f and g bounded near $Z(f)$. We also have Hartog's theorem for $n \geq 2$, stating that for compact $K \subset U$ such that $U - K$ is connected, the restriction map

$$\text{res} : \mathcal{O}(U) \to \mathcal{O}(U - K)$$

is surjective (hence bijective by the identity theorem).

2. Local theory and analytic germs

One can fruitfully study analytic sets locally by thinking about the stalk of holomorphic functions $\mathcal{O}_{\mathbb{C}^n,0}$ at zero (other stalks $\mathcal{O}_{\mathbb{C}^n,x}$ are isomorphic by translation). Two key theorems are used throughout in working with this stalk: the Weierstrass preparation theorem (WPT), which states that any $f \in \mathfrak{m} \subset \mathcal{O}_{\mathbb{C}^n,0}$ (where $\mathfrak{m} = \ker \text{ev}_0$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ as a local ring) can be expressed in the form $f_w = hg_w$, where $h \in \mathcal{O}_{\mathbb{C}^n,0} - \mathfrak{m}$ is a unit, $f_w(z_1) := f(z_1, w)$, and $g_w \in \mathcal{O}_{\mathbb{C}^{n-1},0}[z_1]$ is a Weierstrass polynomial, i.e. a function of the form

$$g_w(z_1) = z_1^d + a_1(w)z_1^{d-1} + ... + a_d(w), \text{ each } a_i(w) \in \mathcal{O}_{\mathbb{C}^{n-1},0}$$

which can be thought of as a holomorphic deformation of the trivial degree d polynomial z_1^d characterizing the behavior of f near zero; and the Weierstrass division theorem (WDT), which states that $\mathcal{O}_{\mathbb{C}^n,0}$ admits Euclidian division in terms of Weierstrass polynomials (i.e. for given $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and Weierstrass g_w, can uniquely write $f_w = hg_w + r_w$, with r_w Weierstrass and $\deg r_w < \deg g_w$). Using these results, one can prove that $\mathcal{O}_{\mathbb{C}^n,0}$ is a UFD and a Noetherian ring.

Locally, we model analytic sets as germs of sets ("infinitesimal patches near zero") $(X, 0)$ equivalent to $(Z(S), 0)$ for some collection of holomorphic germs $S \subset \mathcal{O}_{\mathbb{C}^n,0}$, where $(X, 0) \sim (Y, 0)$ are considered equivalent whenever $X \cap U = Y \cap X$ for some open $U \ni 0$. The basic theory of analytic germs is very similar to the classical theory of affine algebraic sets: we have assignments Z, $\mathcal{I}$ between analytic germs and ideals of $\mathcal{O}_{\mathbb{C}^n,0}$, contravariant under inclusion, and Hilbert's nullstellensatz states that for any ideal $I \trianglelefteq \mathcal{O}_{\mathbb{C}^n,0}$, we have that

$$\mathcal{I}(Z(I)) = \sqrt{I}$$

outputs the radical of I, resulting in a 1-1 correspondence between analytic germs in $\mathbb{C}^n$ and radical ideals in $\mathcal{O}_{\mathbb{C}^n,0}$. One can also exhibit some specializations of this correspondence, such as the fact that prime ideals correspond to irreducible analytic germs.

There is also a nice Noether normalization-type theorem, stating that for any irreducible analytic germ X, it is possible to exhibit a surjection of germs $X \twoheadrightarrow V$ with finite fibers, for some open subset $V \subset \mathbb{C}^d$, with d independent of the choice of surjection - we define $\dim X := d$. Thus in some sense one can think of irreducible analytic germs as branched coverings of some open $V \subset \mathbb{C}^d$.

3. Defining complex manifolds

The definition of complex manifolds is very similar to that of smooth manifolds, except that we require charts to map into some open of $\mathbb{C}^n$ and transition maps to be biholomorphic. The inverse function theorem and implicit function theorems readily generalize to the complex case, allowing us to exhibit analytic sets $Z(f_1, ..., f_r)$, for given $f_i \in \mathcal{O}(U)$ with 0 as a common regular value as submanifolds of $\mathbb{C}^n$.

When doing calculus in local coordinates, it is often useful to work with the Wirtinger basis, which is related to the standard basis via the formulas:

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}).$$

For holomorphic $f : U \to V$, the complexification of the real Jacobian (after change of basis) takes the form (using the relationship $\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}$):

$$\begin{pmatrix} J_{\mathbb{C}} f & 0 \\ 0 & \overline{J_{\mathbb{C}} f)} \end{pmatrix},$$

where $J_{\mathbb{C}} f$ denotes the complex Jacobian. In particular, given any biholomorphic $f : U \to V$, it follows that $\det(J_{\mathbb{R}} f) = |\det(J_{\mathbb{C}} f)|^2 > 0$, implying that any complex n manifold viewed as a smooth $2n$-manifold comes equipped with a canonical orientation.

Complex manifolds come with a sheaf of holomorphic functions $\mathcal{O}_X$, and it has no non-constant global sections in the case of compact complex manifolds by the maximum principle - differing from the behavior of smooth manifolds on which one can always construct bump functions. It is also interesting to study the sheaf of meromorphic functions on a complex manifold X, denoted by $\mathcal{K}_X$. For compact complex connected manifolds, Siegel's theorem states that

$$\text{trdeg}_{\mathbb{C}} \mathcal{K}_X(X) \leq \dim_{\mathbb{C}} X.$$

We call $a(X) := \text{trdeg}_{\mathbb{C}} \mathcal{K}_X(X)$ the algebraic dimension of X.

It turns out that every nonsingular algebraic variety over $\mathbb{C}$ satisfies $a(X) = \dim_{\mathbb{C}} X$ by dimension theory, but general complex manifolds can have arbitrary algebraic dimension $0 \leq a(X) \leq \dim_{\mathbb{C}} X$ - so there are many complex manifolds not of algebraic origin. Nonetheless, Chow's theorem states that any projective complex manifold $X \subset \mathbb{P}^N$ can actually be obtained as a complex algebraic variety.

One can also speak of differential forms over a complex manifold. The situation differs from smooth manifolds in the sense that under the splitting $T_{\mathbb{C}} X = TX \oplus \overline{TX}$ of the complexified tangent bundle, one is naturally led to studying (p,q)-forms on X, which are defined to be sections of the bundle

$$\mathcal{A}^{p,q} := \Lambda^p(T^*X) \otimes \Lambda^q(\overline{T^*X}).$$

Locally over $U \subset X$, with respect to the Wirtinger basis, sections of this bundle can be expressed as $\mathcal{O}(U)$-linear combinations in the basis

$$dz_{i_1} \wedge ... \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q} \text{ for some } i_1 < ... < i_p, j_1 < ... < j_q.$$

4. Examples of complex manifolds

Quotients of complex manifolds by topological group actions lead to a rich class of examples, often relying on the fact that if the action of a topological group G acts freely and via a proper map $G \times X \to X$ on a complex manifold X, then X/G with the quotient topology inherits the structure of a complex manifold (quotients under non-free actions are harder to study, but can sometimes be approached via GIT). In what follows, we go over some (but not all) important examples.

One can define complex projective space of dimension n as the quotient

$$\mathbb{P}^n := (\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$$

under scalar multiplication by $\mathbb{C}^*$, with charts defined on opens $U_i := \{[z_0 : ... : z_n] \mid z_i \neq 0\} \xrightarrow{\varphi_i} \mathbb{C}^n$ and transition maps for $i < j$ (wlog) given by:

$$\varphi_j \circ \varphi_i^{-1} : (z_1, ..., z_n) \mapsto (\frac{z_1}{z_j}, ..., \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, ..., \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, ..., \frac{z_n}{z_j}).$$

The space $\mathbb{P}^n$ is a compact complex manifold, and every homogeneous polynomial $f \in \mathbb{C}[z_0, ..., z_n]$ such that 0 is a regular value for $f : \mathbb{C}^{n+1} \to \mathbb{C}$ can be shown to define a complex hypersurface $V(f) \subset \mathbb{P}^n$ via the implicit function theorem. One can speak more generally of projective submanifolds $V(f_1, ..., f_r) \subset \mathbb{P}^n$ associated to several homogeneous polynomials (possibly of different degrees) with common regular value at 0. Chow's theorem stated above is then the statement that any complex submanifold of $\mathbb{P}^n$ can be locally expressed in this form.

One can interpret $\mathbb{P}^n = \mathrm{Gr}(1, n)$ as a special of a more general class of complex manifolds known as complex Grassmannians, where $\mathrm{Gr}(k, n)$ is taken to be the complex manifold of dimension $k(n-k)$ consisting of all k-dimensional complex linear subspaces of $\mathbb{C}^n$. Each $\mathrm{Gr}(k, n)$ can be shown to be a projective manifold, i.e. a complex submanifold of $\mathbb{P}^{\binom{n}{k}}$ via the Plücker embedding sending a k-dimensional subspace to its top exterior power as an element of the projectivization $\mathbb{P}(\Lambda^k \mathbb{C}^n) \simeq \mathbb{P}^{\binom{n}{k}}$:

$$L \mapsto [\Lambda^k L].$$

It follows by Chow's theorem that complex Grassmannians are algebraic. This class of manifolds generalizes further to complex flag varieties parametrizing ascending sequences of complex linear subspaces $V_1 \subset ... \subset V_r, \dim V_i = k_i$, for some choice of sequence of integers $1 < k_1 < ... < k_r < n$.

Another fruitful group quotient construction comes from starting with a $\mathbb{Z}$-lattice $\Gamma \subset \mathbb{C}^n$ of maximal rank, which acts freely and discretely on $\mathbb{C}^n$ by translation, and to take the quotient $\mathbb{C}^n/\Gamma$, a compact complex manifold called a complex torus. Under the abelian group structure induced on $\mathbb{C}^n/\Gamma$ by $\mathbb{C}^n$, this is in fact a complex Lie group (a class which includes some matrix groups such as $SL_n(\mathbb{C})$ and $\mathrm{Sp}(n)$), and it can be shown that every connected compact complex Lie group is of this form (and in particular that it is automatically abelian). Of particular interest is the case $n = 1$, and we call complex tori of dimension 1 elliptic curves. Up to scalar multiplication, we can always ensure that a given elliptic curve is of the form

$$E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \text{ for some } \tau \in \mathbb{H}.$$

There is an action of $\mathrm{SL}_2(\mathbb{Z})$ on the Poincar half-plane $\mathbb{H}$ by Möbius transformations (i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau+b}{c\tau+d}$), and it can be shown that $E_\tau \simeq E_{\tau'}$ are biholomorphic as complex manifolds if and only if τ and τ' are in the same $\mathrm{SL}_2(\mathbb{Z})$ orbit. Thus there is a 1-1 correspondence:

$$\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \simeq \big\{\text{elliptic curves up to bioholomorphism}\big\}$$

so that one can think of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ as a moduli space of elliptic curves.

Finally, letting $\mathrm{SU}(1, n)$ denote the isometry subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$ associated to the hermitian form

$$< z, w >= z_0\bar{w_0} - \sum_{i=1}^{n} z_i\bar{w_i},$$

which is in 1-1 correspondence with $\mathrm{Aut}(B_1^n(0))$, one can construct so-called ball quotients $B_1^n(0)/\Gamma$ for some choice of discrete cocompact subgroup $\Gamma \subset \mathrm{SU}(1,n)$ acting freely on $B_1^n(0)$ (here we view $B_1^n(0)$ as a subspace of $U_0 \subset \mathbb{P}^n$ to make sense of the $\mathrm{SU}(1,n)$ action). By covering space theory, one can in particular recover the class of compact Riemann surfaces of genus g as the quotient D^1/Γ_g by the genus g surface group $\Gamma_g = \pi_1(\Sigma_g) =< a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 >$.

5. Vector bundles and sheaf theory

Vector bundles offer a way to study a given complex manifold X by means of "parametrized linear algebra". Most linear algebra constructions carry over to the settings of vector bundles: given vector bundles E, F over X, one can define $E \oplus F$, $E \otimes F$, $\mathrm{Hom}(E, F)$, $E^\vee$, $\Lambda^p E$,... Most relevant for us are the tangent bundle T_X, its dual Ω_X^1: the bundle of $\mathbb{C}$-valued differential 1-forms and the associated bundles of k-forms $\Omega_X^k = \Lambda^k \Omega_X^1$. Moreover, we have a natural splitting of complex vector bundles $\Omega_X^1 = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$ into holomorphic (spanned by dz_i's) and anti-holomorphic (spanned by $d\bar{z}_i$'s) parts, and exterior powers of these give us subbundles of forms of type (p,q): $\Omega_X^{p,q} \subseteq \Omega_X^k$ for $p+q=k$. We denote global sections of Ω_X^k resp. $\Omega_X^{p,q}$ by $\mathcal{A}^k$ resp. $\mathcal{A}^{p,q}$.[1] Exterior differentiation decomposes as $d = \partial + \bar{\partial}$, where $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$.

Vector bundles admit an explicit description amenable to computations in terms of cocycle data of the form $\{U_i, \psi_{ij}\}$, consisting of an open cover of X ("local trivializations") together with holomorphic transition functions $\psi_{ij} : U_{ij} \to \mathrm{GL}_k(\mathbb{C})$, where $k = \mathrm{rk}(E)$. In fact, isomorphism classes of vector bundles are in 1-1 correspondence with equivalence classes of cocycle data (under an appropriate relation). For line bundles, the cocycle data is visibly a Čech 1-cochain with coefficients in the sheaf $\mathcal{O}_X^*$, so that we obtain a natural map

$$\mathrm{Pic}(X) \to \check{H}^1(X, \mathcal{O}_X^*) \simeq H^1(X, \mathcal{O}_X^*)$$

which is in fact a group isomorphism. The cocycle viewpoint gives a quick proof of the adjunction formula, which states that for a complex submanifold $Y \subseteq X$, if $K_X = \det(\Omega_X^1)$ resp. K_Y are the canonical line bundles and $\mathcal{N}_{Y/X}$ the associated normal bundle, then $K_Y \simeq K_X|_Y \otimes \det(\mathcal{N}_{Y/X})$.

Holomorphic vector bundles are a particular kind of sheaves: they are in 1-1 correspondence with locally free sheaves of $\mathcal{O}_X$-modules of finite rank. Under this correspondence, a vector bundle E is sent to its sheaf of sections $\mathcal{E}$. Sheaf theory offers a robust framework for defining cohomology groups on the space X with coefficients in a given sheaf: via the general framework of derived functors in abelian categories, because the global sections functor Γ is left exact, we may define $H^q(X, \mathcal{F}) := R^q\Gamma(X, -)(\mathcal{F})$. In particular, one recovers singular cohomology on constant sheaves (hence in particular de Rham cohomology on $\mathbb{R}_X$). The Dolbeault cohomology groups $H^{p,q} = H^q(X, \Omega_X^p)$ are of particular interest for Hodge theory.

Computing sheaf cohomology groups can be done in various ways. One can look for Γ-acyclic resolutions of a given sheaf $\mathcal{F}$ and take cohomology of the complex

[1] **Notation**: We sometimes write $\mathcal{A}^{p,q}, H^{p,q}, ...$ instead of $\mathcal{A}^{p,q}(X), H^{p,q}(X), ...$ when X is clear from context. Unless otherwise indicated, X is always a manifold.

of global sections. This can be done by resolving $\mathcal{F}$ by injective sheaves (this is the way in which right derived functors are usually first defined), flasque sheaves (a canonical such resolution called the Godement resolution can always be given; these include singular cochains with coefficients in any commutative ring), or soft sheaves (these include sheaves of differential forms). In particular, noticing that both flasque and soft sheaves can be used to resolve $\mathbb{R}_X$ gives a sheaf-theoretic proof of de Rham's theorem that $H^*_{\text{sing}}(X,\mathbb{R}) \simeq H^*_{\text{dR}}(X)$. There is also a more combinatorial approach via Čech cohomology, which gives for instance the cohomology groups $H^*(\mathbb{P}^n, \mathcal{O}(m))$ of twisting sheaves on projective space (defined below).

To first approximation, sheaf cohomology corrects the failure of the global sections functor to be right exact: any short exact sequence of sheaves induces a long exact sequence (LES) in cohomology. An instructive example comes from considering the exponential sequence on $X = S^1$:

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{\text{cstt}} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

which is exact because of the existence of logarithms locally, but not exact on global sections because S^1 is not simply connected - and $H^1(X,\mathbb{Z}) = \mathbb{Z}$. This sequence is also interesting for X compact Kähler (defined later), in which case the RHS map above is surjective on global sections, and the resulting LES in cohomology gives:

$$\begin{array}{cccccccccc}
0 \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X^*) & \xrightarrow{\partial} & H^2(X,\mathbb{Z}) & \longrightarrow ... \\
& \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & \\
& \mathbb{Z}^{b_1(X)} & \xrightarrow{\text{lattice incl.}} & \mathbb{C}^{b_1(X)/2} & \longrightarrow & \text{Pic}(X) & \xrightarrow{\mathcal{L}\mapsto c_1(\mathcal{L})} & H^2_{\text{sing}}(X,\mathbb{Z}) &
\end{array}$$

where $b_1(X)$ is the first Betti number of X. The boundary map from H^1 to H^2 provides an equivalent definition for the Chern class associated to a line bundle.

6. Line bundles and divisors

There is a close link between holomorphic line bundles and divisors on a complex manifold X, which can be defined equivalently as formal $\mathbb{Z}$-linear combinations $D = \sum_i a_i[Y_i]$ of irreducible analytic hypersurfaces $Y_i \subseteq X$ ("Weil divisors") or as global sections $\{U_i, f_i \in \mathcal{K}_X^*(U_i)\} \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ satisfying $f_i f_j^{-1} \in \mathcal{O}_X^*(U_{ij})$ for all i,j ("Cartier divisors"). The equivalence between these two viewpoints can be established by using valuation theory on analytic germs to associate to a given Cartier divisor $f = \{U_i, f_i\}$ a Weil divisor (f) which on a given $[Y]$ takes the coefficient $\text{ord}_Y(f_i)$ for some $U_i \cap Y \neq \emptyset$.

Much can be learned by studying the cohomology LES associated to the SES $0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$, which begins as follows:

$$\begin{array}{cccccccccc}
0 \longrightarrow & H^0(X,\mathcal{O}_X^*) & \longrightarrow & H^0(X,\mathcal{K}_X^*) & \longrightarrow & H^0(X,\mathcal{K}_X^*/\mathcal{O}_X^*) & \xrightarrow{\partial} & H^1(X,\mathcal{O}_X^*) & \longrightarrow ... \\
& & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & \\
& & & \mathcal{K}(X)^* & \xrightarrow{\text{principal divs}} & \text{Div}(X) & \xrightarrow{D\mapsto \mathcal{L}_D} & \text{Pic}(X) &
\end{array}$$

In particular, the boundary map from H^0 to H^1 gives us a natural group homomorphism from divisors on X to line bundles on X, equal to the trivial bundle on the principal divisors (those defined by a global meromorphic function) by exactness. This boundary map sends a divisor $D = \{U_i, f_i\}$ to the line bundle $\mathcal{L}_D = \{U_i, f_i f_j^{-1}\}$. We write $\mathcal{O}(D)$ for the associated sheaf of sections, with dual $\mathcal{O}(-D)$ by the group homomorphism property. For instance, the hyperplane $H = \{x_0 = 0\} \subseteq \mathbb{P}^n$ defines the sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$ (equivalent to the sheaf of sections of the tautological line bundle), a generator for $\mathrm{Pic}(\mathbb{P}^n)$ under tensor products.

Finally, the notion of linear system gives us a way to construct projective embeddings. A linear system is a subspace $|V| = \mathbb{C}s_0 + ... + \mathbb{C}s_N$ of $H^0(X, \mathcal{L})$ for some line bundle $\mathcal{L}$ on X. Such a $|V|$ induces a map:

$$\varphi_{(\mathcal{L},|V|)} : X - \mathrm{Bs}(\mathcal{L}, |V|) \to \mathbb{P}^N, x \mapsto [s_0(x) : ... : s_N(x)],$$

where $\mathrm{Bs}(\mathcal{L}, |V|) := \{x \in X | s_i(x) = 0 \ i = 0, ..., N\}$ is the base locus of $|V|$, and $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}|_{X - \mathrm{Bs}(\mathcal{L},|V|)}$. We call a line bundle $\mathcal{L}$ ample if we can find some linear system $|V|$ on some $\mathcal{L}^{\otimes k}$ under which $\mathrm{Bs}(\mathcal{L}^{\otimes k}, |V|) = \emptyset$ and $\varphi_{(\mathcal{L}^{\otimes k},|V|)}$ is an embedding, thus presenting X as a projective manifold.

7. Kähler manifolds

Kähler manifolds are a rich class of geometric objects that are at once Hermitian, symplectic, and Riemannian manifolds. A Hermitian manifold (X, h) is called Kähler if the induced $(1, 1)$ real form $\omega = \frac{i}{2}(h - \bar{h})$ is closed, i.e. (X, ω) is symplectic as a smooth manifold. We note that if h is locally given by $h = \sum_{i,j} h_{i,\bar{j}} dz_i \otimes d\bar{z}_j$, then $w = \frac{i}{2} \sum_{i,j} h_{i,\bar{j}} dz_i \wedge d\bar{z}_j$. For the rest of this section and the next, X is assumed to be compact Kähler of complex dimension n. A volume form is then given by $\frac{1}{n!}\omega^n$ (and by Stokes this leads to an obstruction to being Kähler, namely the vanishing of any $H^{2i}(X, \mathbb{R})$ for $0 \leq i \leq n$). We also note that the symplectic form induces an operator $L = \omega \wedge - : \mathcal{A}^k \to \mathcal{A}^{k+2}$, called the Lefschetz operator.

The L^2 inner product associated to the Hermitian metric allows us to define a Hodge star operator $* : \mathcal{A}^k \to \mathcal{A}^{n-k}$, $\beta \mapsto *\beta$ via the relation $(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \overline{*\beta}$, or equivalently via $\alpha \wedge \overline{*\beta} = (\alpha, \beta)_h \frac{1}{n!}\omega^n$. This is an involution up to sign, i.e. $*^2 = (-1)^k$ on $\mathcal{A}^k$. The Hodge star operator allows us to define formal L^2-adjoints to the operators $d, \partial, \bar{\partial}, L$ under conjugation: we set

$$d^* := - * d *, \ \partial^* := - * \bar{\partial} *, \ \bar{\partial}^* := - * \partial *, \ L^* \equiv \Lambda := *^{-1} L *,$$

so that e.g. $d^* : \mathcal{A}^{k+1} \to \mathcal{A}^k$ satisfies $(d\alpha, \beta)_{L^2} = (\alpha, d^*\beta)_{L^2}$. We note that $\bar{\partial}$ was involved in defining ∂^* and vice-versa for type reasons. For instance, on $\mathcal{A}^{p,q+1}$, ∂^* is the composite $\mathcal{A}^{p+1,q} \xrightarrow{*} \mathcal{A}^{2n-q,2n-p-1} \xrightarrow{\bar{\partial}} \mathcal{A}^{2n-q,2n-p} \xrightarrow{-*} \mathcal{A}^{p,q}$. We can then define Laplacian operators via $\Delta := dd^* + d^*d$, and similarly for $\Delta_\partial, \Delta_{\bar{\partial}}$.

Crucial to this subject are the Kähler identities, which can all be derived by conjugation and $*$-adjunction from the two equations $[\Lambda, \bar{\partial}] = -i\partial^*$ and $[L, \bar{\partial}] = 0$. The latter immediately follows by the Leibniz rule and the fact that ω is closed, but the former requires work. The remaining relations are

$$[\Lambda, \partial] = i\bar{\partial}^*,\ [\bar{\partial}^*, L] = i\partial,\ [\partial^*, L] = -i\bar{\partial},\ [L, \partial] = 0,\ [\bar{\partial}^*,\ \Lambda] = 0,\ [\partial^*, \Lambda] = 0.$$

From there, one can deduce via appropriate substitutions that $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$. Finally, Δ commutes with everything, i.e. with $*, L, \Lambda, \partial, \partial^*, \bar{\partial}, \bar{\partial}^*$.

Riemann surfaces are Kähler (since $\Omega^3_X = 0$), and so are ball quotients. Projective manifolds are also Kähler, providing ties to complex algebraic geometry. This follows by restriction of the Fubini-Study form ω_{FS} on $\mathbb{P}^n$, characterized as the unique $U(n+1)$-invariant $(1,1)$-form on $\mathbb{P}^n$ such that $\int_{\mathbb{P}^1} \omega_{FS} = 1$. One can either locally define ω_{FS} explicitly (as $\omega_{FS}|_{U_i} := \partial\bar{\partial} \log\big(\sum_j |\frac{z_j}{z_i}|^2\big)$ on $U_i = \{z_i \neq 0\}$), or obtain it from the standard metric on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ under projection.

8. Hodge theory

Hodge theory describes the interplay between the complex geometry (Dolbeault cohomology), topology (de Rham/singular cohomology), and harmonic analysis (harmonic forms) of a compact Kähler manifold. A key result from functional analysis is that the Laplacian operator Δ is an elliptic self-adjoint operator, and as such for compact X it is Fredholm and gives rise to a decomposition for each k:

$$\mathcal{A}^k = \mathcal{H}^k \oplus \Delta(\mathcal{A}^k),$$

where $\mathcal{H}^k := \ker(\Delta|_{\mathcal{A}^k})$ is finite dimensional. Using the facts that $\Delta\alpha = 0$ iff α is both closed and co-closed and that $\ker(d) \cap \operatorname{im}(d^*) = \ker(d^*) \cap \operatorname{im}(d^*) = 0$, one can quickly deduce that the natural map $\alpha \mapsto [\alpha]$ induces an isomorphism for all k:

$$\mathcal{H}^k(X) \xrightarrow{\simeq} H^k_{\text{sing}}(X; \mathbb{C}).$$

Since $\Delta = 2\Delta_\partial$ respects types, we obtain the Hodge decomposition:

$$H^k_{\text{sing}}(X; \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q},$$

with the further property that $H^{p,q} = \overline{H^{q,p}}$. Similar decompositions hold for compact Hermitian manifolds if we work with $\mathcal{H}^{p,q}_\partial$ or $\mathcal{H}^{p,q}_{\bar{\partial}}$. An immediate consequence of this result is that $H^*(X, \mathbb{C})$ is finite-dimensional for compact Kähler manifolds. Serre duality on forms, which gives an isomorphism $\mathcal{H}^{p,q}_{\bar{\partial}} \simeq (\mathcal{H}^{n-p,n-q}_{\bar{\partial}})^\vee$ via the pairing $\alpha, \beta \mapsto \int_X \alpha \wedge \beta$ (non-degenerate because $\int_X \alpha \wedge \overline{*\alpha} = ||\alpha||^2_{L^2} > 0$ for $\alpha \neq 0$), induces via the Hodge decomposition an isomorphism $H^{p,q} \simeq (H^{n-p,n-q})^\vee$. Similarly, the Hodge star operation induces an isomorphism $* : H^{p,q} \xrightarrow{\simeq} H^{n-q,n-p}$. These symmetries lead to the Hodge diamond on Hodge numbers $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$.

One can check that the triple L, Λ, H, where $H|_{\mathcal{A}^k} := (k-n)\mathrm{id}_{\mathcal{A}^k}$, induces a valid $\mathfrak{sl}_2(\mathbb{C})$-representation structure on $\mathcal{A}^* = \bigoplus_{k\geq 0} \mathcal{A}^k$ (via $e \mapsto L, f \mapsto \Lambda, h \mapsto H$). The relation $[L, \Lambda] = H$ can be established using the fact that differential identities on a Hermitian manifold can be checked to first order on the standard Hermitian metric $h_{i,j} = \delta_{i,j}$. It follows from the structure theory of $\mathfrak{sl}_2(\mathbb{C})$-modules that $\mathcal{A}^*$ decomposes as a direct sum of irreducible representations, each of which is a direct sum of lines generated by some lowest weight vector $\alpha \in \mathcal{A}^k$ such that $\Lambda\alpha = 0$, also called a primitive vector. By symmetry around 0, since $H\alpha = (k-n)\alpha = -(n-k)\alpha$, we have that $L^i\alpha \in \mathcal{A}^{k+2i}$ is non-zero for $i = 0, ..., n-k$. Thus each form decomposes uniquely as a sum of L^i's of primitive forms, so that we may write:

$$\mathcal{A}^k = \bigoplus_{i\geq 0} L^i P^{k-2i},$$

where $P^l := \{\alpha \in \mathcal{A}^l \mid \Lambda\alpha = 0\}$ denotes the space of primitive l-forms. Passing to cohomology via harmonic forms using the fact that Δ and L commute, we obtain the hard Lefschetz theorem, stating that we have an isomorphism for every $k \leq n$:

$$L^{n-k} : H^k(X, \mathbb{R}) \xrightarrow{\simeq} H^{2n-k}(X, \mathbb{R}).$$

We mention two additional results and one conjecture from Hodge theory. The Hodge-Riemann bilinear relations on a compact Kähler manifold (X^n, w) state that the assignment

$$\alpha, \beta \mapsto i^{p-1}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \bar{\beta} \wedge^{n-(p+q)}$$

provides a non-degenerate bilinear form on primitive cohomology classes $H^{p,q}_{\text{prim}}(X)$ for each type (p, q), with application to the moduli theory of Kähler manifolds. These relations together with the Lefschetz decomposition can be used to prove the Hodge index theorem, which gives an explicit formula for the index of the intersection pairing $\alpha, \beta \mapsto \int_X \alpha \wedge \beta$ on a compact Kähler surface (X, ω) (which is in particular a 4 dimensional smooth manifold), as the pair $(2h^{2,0}(X)+1, h^{1,1}(X)-1)$.

Finally, the Hodge conjecture is a deep problem in complex (algebraic) geometry regarding complex projective manifolds. Let X be complex projective, and otice that any codimension k complex submanifold $Y \subset X$ induces via Poincarè duality an integral form of type (k, k), called an algebraic class. The Hodge conjecture asks whether every cohomology class in $H^{k,k}(X) \cap H^{2k}(X; \mathbb{Q})$ can be obtained as a $\mathbb{Q}$-linear combination of algebraic classes. The only known case of this conjecture in the general case is the Lefschetz theorem on $(1, 1)$ classes, which states that taking the Chern class of a line bundle gives a surjection $c_1 : \text{Pic}(X) \twoheadrightarrow H^{1,1}(X) \cap H^2(X; \mathbb{C})_{\mathbb{Z}}$, where $H^2(X; \mathbb{C})_{\mathbb{Z}} := \text{im}(H^2(X; \mathbb{Z}) \to H^2(X, \mathbb{C}))$.

9. Formality obstruction to being Kähler

We begin by observing that the chain complex of complex-valued differential forms $(\mathcal{A}^*(M), d)$ on a smooth manifold M, together with the wedge product operation, has the structure of a commutative differential graded algebra (cdga) over $\mathbb{C}$.[2] This means in particular that $\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha$ (where we write $|\alpha| = k$ for $\alpha \in \mathcal{A}^k$), and that d satisfies a graded Leibniz rule with respect to $\wedge$:

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d(\beta).$$

We call a cdga $\mathcal{A}^*$ is connected if the inclusion $\mathbb{C} \hookrightarrow \mathcal{A}^0$ is an isomorphism (equivalent to M being connected in the case $\mathcal{A}^* = \mathcal{A}^*(M)$), resp. simply connected if $H^1(\mathcal{A}^*) = 0$ (equivalent to $b_1(M) = 0$ for $\mathcal{A}^* = \mathcal{A}^*(M)$ by de Rham's theorem).

In the same way that one can study the homotopy type of a space by passing to a CW-replacement, the framework of abstract homotopy theory suggests that we study simply connected cdga's by passing to so-called minimal models. With respect to the model category structure on the category of non-negatively graded cdga's

[2]The theory may be developed for a general field of characteristic zero, but for our purposes it suffices to work over $\mathbb{C}$.

over $\mathbb{C}$ where weak equivalences are quasi-isomorphisms and fibrations are degree-wise surjective chain maps, a minimal model for $\mathcal{A}^*$ is a cofibrant replacement of $\mathcal{A}^*$, i.e. a cofibrant cdga $\mathcal{M}^*$ together with an acyclic fibration $\mathcal{M}^* \twoheadrightarrow \mathcal{A}^*$ factorizing the map $0 \to \mathcal{A}^*$.

Concretely, one calls a cdga $\mathcal{M}^*$ minimal if there exist elements $(x_i \in \mathcal{M}^i)_{i\in I}$, $1 \leq d_1 \leq d_2 \leq ...$, generating $\mathcal{M}^*$ as a cdga, i.e. $\mathcal{M}^* = \bigwedge^* \left(\bigoplus_{i\in I} \mathbb{C}x_i\right)$, and such that $dx_i \in < x_1, ..., x_{i-1} >_{>0}$ for each i (where the subscript "> 0" indicates that we are excluding constants). A minimal model for $\mathcal{A}^*$ is then a minimal cdga $\mathcal{M}^*$ together with a quasi-isomorphism $\mathcal{M}^* \xrightarrow{\simeq} \mathcal{A}^*$. Work of Sullivan shows that minimal models are valid cofibrant replacements with respect to the model category structure described above, and that any simply connected cdga over $\mathbb{C}$ admits a minimal model (which is then necessarily unique up to quasi-isomorphism).

An interesting obstruction to a compact manifold M admitting a Kähler metric arises in this framework. Call a cdga $\mathcal{A}^*$ formal if it is equivalent to a cdga $(\mathcal{B}^*, 0, \wedge)$ under a zigzag of quasi-isomorphisms (i.e. if $\mathcal{A}^* \simeq \mathcal{B}^*$ in the derived category), and call M formal if $\mathcal{A}^*(M)$ is formal. Notice that $\mathcal{A}^*$ is formal iff a minimal model of $\mathcal{A}^*$ is formal. This allows us to quickly verify that the sphere S^n is formal for odd n, since it admits a minimal model $\mathcal{M}^*$ consisting of a single generator x in degree n, with $x \wedge x = (-1)^n x \wedge x = -x \wedge x$ since n is odd, hence $x \wedge x = 0$ and $\mathcal{M}^*$ is clearly formal. The key result of this section is that every compact Kähler manifold X is formal. This can be shown using the operator $d^c := -i(\partial - \bar{\partial}) : \mathcal{A}^*(X) \to \mathcal{A}^{*+1}(X)$ and an analogue of the $\partial\bar{\partial}$ lemma for d and d^c, which allows us to obtain the following zigzag of quasi-isomorphisms:

$$(\mathcal{A}^*(X), d) \xrightarrow{\simeq} (\ker d^c, d) \xleftarrow{\simeq} (H^*_{d^c}(X), 0).$$

10. Harmonic theory on Hermitian vector bundles

In order to best capture the geometry of a manifold via vector bundle theory, it is useful to equip the bundles $E \to M$ with a Hermitian metric h, given by a C^∞-map of vector bundles $h : E \otimes_{\mathbb{C}} \bar{E} \to \underline{\mathbb{C}}_M$ (or equivalently $h : E \to \bar{E}^*$ by adjunction) such that for each $x \in X$, h_x defines a Hermitian inner product on the fiber E_x. With respect to a local trivialization $U \subset M$ of E with frame $e_1, ..., e_r$ dual to $e^1, ..., e^r$ (where $r = \mathrm{rk}(E)$), we can express

$$h|_U = \sum_{i,j} h_{i,\bar{j}} e^i \otimes \bar{e^j}$$

via the data of smooth functions $h_{i,\bar{j}} \in C^\infty(U)$.

Existence of a Hermitian metric on an arbitrary complex vector bundle follows from a partition of unity argument starting with the standard Hermitian metric on $\mathbb{C}^r$. In the case of a complex line bundle $\mathcal{L} \to M$ with local trivializations $\psi_i : \mathcal{L}|_{U_i} \xrightarrow{\simeq} U_i \times \mathbb{C}$ and given global sections $s_1, ..., s_k \in \mathcal{A}^0(\mathcal{L}) = \Gamma(M, \mathcal{L})$ with no common zeroes, we can construct a Hermitian metric on $\mathcal{L}$ explicitly by patching together the local formulas $h|_{U_i}(u, v) := \frac{\psi_i(u)\overline{\psi_i(v)}}{\sum_{j=1}^k |\psi_i(s_j)|^2}$.

Given a Hermitian vector bundle over a Riemannian n-manifold $(E, h) \to (M, g)$, we write $\mathcal{A}^{p,q}(E) := \Gamma(X, \Lambda^{p,q}TX \otimes E)$. We can define a Hodge-* operator:

$$\bar{*}_E : \Lambda^{p,q}TX \otimes E \to \Lambda^{n-p,n-q}TX \otimes \bar{E}^*$$

via the formula $\bar{*}_E(\alpha \otimes s) := *\bar{\alpha} \otimes h(s)$, giving a map on global sections $\mathcal{A}^{p,q}(E) \xrightarrow{\bar{*}_E} \mathcal{A}^{n-p,n-q}(E)$. If E is holomorphic, we can also extend the notion of a $\bar{\partial}$-operator:

$$\bar{\partial}_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E).$$

These operators allow us to do harmonic theory "relative to E": we can define a formal adjoint $\bar{\partial}_E^* := -\bar{*}_E \bar{\partial}_E \bar{*}_E$, a Laplacian operator $\Delta_E := \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E$, and corresponding harmonic forms $\mathcal{H}^{p,q}(X, E) \subseteq \mathcal{A}^{p,q}(E)$.

Many of the earlier results from Hodge theory carry over to these settings: one can for instance establish a Hodge decomposition:

$$\mathcal{A}^{p,q}(E) = \mathcal{H}^{p,q} \oplus \bar{\partial}_E \mathcal{A}^{p,q-1}(E) \oplus \bar{\partial}_E^* \mathcal{A}^{p,q+1}(E),$$

as well as a more general version of Serre duality for any complex vector bundle:

$$H^{p,q}(E) = H^{n-p,n-q}(E^*)^*,$$

where $H^{p,q}(E) := H^q(\mathcal{A}^{p,*}, \bar{\partial}_E)$. Perhaps most recognizable to the algebraic geometer is the special case of this identity for $p = n$, with $K_X = \Omega_X^n$ denoting the canonical line bundle on X:

$$H^q(X, E \otimes K_X) \simeq H^{n-q}(X, E^*)^*.$$

11. Connections and Chern-Weil theory

Intuitively, a connection on a complex vector bundle $E \to X$ gives us "a way to parallel transport paths on the base to the total space." In practice, we define a connection on E to be the data of a map:

$$\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$$

satisfying a Leibniz rule with respect to the C^∞-module structure: $\nabla(fs) = f\nabla s + df \otimes s$. Locally, we can write $\nabla = d + A$ in terms of the exterior derivative plus a matrix-valued 1-form $A \in \mathcal{A}^1(\mathrm{End}(E))$. The structure of a connection can be transported along operations on bundles such as direct sum and tensor products.

If (E, h) is equipped with a Hermitian metric, we say that a connection ∇ on E is Hermitian if it locally satisfies the relation

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2),$$

for given sections $s_1, s_2 \in \Gamma(U, E)$. Writing $\nabla = d + A$ and $H = (h_{i,\bar{j}})$, this yields the local condition $A^t H + H A^t$. Next, if E is holomorphic, decomposing $\nabla = \nabla^{0,1} + \nabla^{1,0}$ into types (w.r.t. $\bar{\partial}_E$), we say ∇ is compatible with the complex structure if $\nabla^{0,1} = \bar{\partial}_E$. There always exists a unique connection on a holomorphic Hermitian bundle (E, h) which is compatible with both h and $\bar{\partial}_E$, called the Chern connection of E. This is the analogue of the Levi-Civita connection from Riemannian geometry.

To each pair (E, ∇), we can associate a curvature 2-form $F_\nabla \in \mathcal{A}^2(\mathrm{End}(E))$ via the formula $F_\nabla = \nabla \circ \nabla$ (where we define $\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ via $\nabla(\alpha \otimes s) := d\alpha \otimes s + (-1)^k \alpha \otimes \nabla s$). Locally, we can write $F_\nabla = dA + A \wedge A$, where the wedge operation is distributed along matrix multiplication. The Bianchi identity for the curvature form is a useful vanishing result stating that $\tilde{\nabla} F_\nabla = 0$, where $\tilde{\nabla}$ is the connection on $\mathrm{End}(E)$ associated to ∇. The curvature form plays well with operations on connections $(\oplus, \otimes, (-)^*)$.

Chern-Weil theory uses the language of connections to define characteristic classes. Start with a complex vector bundle with connection (E, ∇) of rank r over M. Let $P \in \mathbb{C}[M_r(\mathbb{C})]$ be a homogeneous polynomial of degree k on the space of r by r

matrices, invariant under the conjugation action of $GL_r(\mathbb{C})$. We may view P as an element of the space of symmetric functions $\mathrm{Sym}^k(M_r(\mathbb{C}))^{GL_r(\mathbb{C})}$. One may further enable the P_k's to take in and return differential forms. We then define

$$\eta_P(E) := [P(\frac{i}{2\pi}F_\nabla)] \in H^{2k}_{\mathrm{dR}}(M;\mathbb{C}).$$

This assignment can be shown to be independent of the choice of connection, producing a map of graded $\mathbb{C}$-algebra, for a fixed complex vector bundle E over M:

$$\eta_-(E) : \mathrm{Sym}^*(M_r(\mathbb{C}))^{GL_r(\mathbb{C})} \to H^*_{\mathrm{dR}}(M;\mathbb{C})$$

called the Chern-Weil homomorphism.

For instance, we could start with the characteristic polynomial $\det(A-tI)$ (evaluated at $t=-1$) as a conjugation-invariant function on $M_r(\mathbb{C})$, and express it as a sum of homogeneous polynomials P_k of degree k for $0 \le k \le r$. We then define the k^{th} Chern class of E to be the cohomology class $c_k(E) := \eta_{P_k}(E) \in H^{2k}_{\mathrm{dR}}(M;\mathbb{C})$ associated to P_k under the Chern-Weil homomorphism. This definition can be shown to coincide with the standard construction of $\{c_k(E)\}_k \subseteq H^*(M;\mathbb{Z})$ via appropriate pullbacks from the cohomology ring of the classifying space for complex vector bundles (the identification can be established by checking that the collection $\{[P_k(F_\nabla)]\}_{k,E}$ satisfies the axioms characterizing the Chern classes). It is customary to define a total Chern class via the formula $c(E) = \sum_{k\ge 0} c_k(E) \in H^*(X;\mathbb{C})$, and to write $c_k(X) := c_k(TX)$ - and likewise for other kinds of characteristic classes.

Other examples of characteristic classes which can be obtained in this way include the Chern character $\mathrm{ch}(E) = \sum_{k\ge 0} \mathrm{ch}_k(E)$, associated to the homogeneous components of $\mathrm{tr}(e^A)$, and the Todd class $\mathrm{td}(E) = \sum_{k\ge 0} \mathrm{td}_k(E)$ via the various coefficients of the power series $\frac{\det(tA)}{\det(I-e^{-tA})} = \sum_{k\ge 0} P_k(A)t^k$.

This algorithm applies more generally to principal G-bundles over M for a complex Lie group G, where one may similarly define a Chern-Weil homomorphism:

$$\mathrm{Sym}^*(\mathfrak{g})^G \to H^*_{\mathrm{dR}}(M;\mathbb{C}),$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, and G acts on $\mathfrak{g}$ via the adjoint action.

The theory of characteristic classes allows for the formulation of results encoding deep connections between the topology and geometry of complex manifolds, such as the Hirzebruch-Riemann-Roch theorem, which states that every holomorphic vector bundle E over a compact complex n-manifold X is subject to the identity:

$$\chi(E) = \int_X \mathrm{ch}(E)\mathrm{td}(X),$$

where $\chi(E) := \sum_{i=0}^n h^i(X,E)$ is the holomorphic Euler characteristic of X with coefficients in E, and the RHS denotes the sum of the evaluations of the fundamental class $[X] \in H_n(X;\mathbb{Z})$ against top degree terms of the product $\mathrm{ch}(E)\mathrm{td}(X)$ inside $H^*(X;\mathbb{C})$. For instance, given a holomorphic line bundle over a Riemann surface $\mathcal{L} \to C$, we have that $\mathrm{ch}(\mathcal{L}) = 1 + c_1(\mathcal{L})$ and $\mathrm{td}(C) = 1 + \frac{1}{2}c_1(TC)$, so that

$$\int_C \mathrm{ch}(E)\mathrm{td}(C) =< [C], c_1(\mathcal{L}) > + \frac{1}{2} < [C], c_1(TC) >= \deg(\mathcal{L}) + \frac{1}{2}\chi_{\mathrm{top}}(X),$$

where $\chi_{\mathrm{top}}(C) = 1 - g(C)$ denotes the topological Euler characteristic, and we therefore recover the classical Riemann-Roch theorem for curves:

$$h^0(\mathcal{L}) - h^1(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g(C).$$