A PRELUDE TO THE GEOMETRIC SATAKE EQUIVALENCE

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Contents

Start with a connected reductive algebraic group G defined over $\mathbb{C}$, and let k denote an algebraically closed field of characteristic zero. Write $\mathcal{O} = \mathbb{C}[[t]]$ for the ring of formal power series with coefficients in k, with fraction field $K = \mathbb{C}((t))$. Then the geometric Satake theorem states that there is an equivalence of tensor categories:

$$(P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k), *) \simeq (\mathrm{Rep}_k(G^\vee), \otimes_k),$$

where $\mathcal{G}\mathrm{r}_G = G(K)/G(\mathcal{O})$ denotes the affine Grassmanian of G, $P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k)$ is the category of $G(\mathcal{O})$-equivariant perverse sheaves over $\mathcal{G}\mathrm{r}_G$ valued in k-vector spaces (sometimes called the Satake category and denoted by Sat_G), with monoidal structure given by the convolution product, and $\mathrm{Rep}(G^\vee)$ denotes the category of finite-dimensional algebraic representations of the Langlands dual group $G^\vee$.

The goal of these notes is to render some (and hopefully most) of the above terms less intimidating, assuming relatively modest prerequisites in the representation theory of semisimple Lie algebras and sheaf theory, so as to prepare the reader to tackle one of the available references for the proof of the geometric Satake equivalence [5], [2].

1. Tannakian reconstruction and outline

The proof of the geometric Satake theorem is structurally quite simple, once the categories in question have been conjecturally identified (something which we will later see is not that far fetched after one studies the simple objects of the categories on either side of the correspondence). Indeed, the essence of the argument relies on the abstract framework known as Tannakian formalism developed by Deligne-Milne and Rivano, which consists in part in deducing from a sufficient set of properties that a given category $\mathcal{C}$ (in our case, a category of geometric nature) may be identified with the category of representations associated to some group H. Once the equivalence has been established, one may use further known information about $\mathcal{C}$ to learn more about H. In the geometric Satake settings, the category C in question is $P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k)$, and much of the work goes into verifying that this category satisfies the hypotheses of the Tannakian reconstruction theorem. The second phase of the proof consists in transferring information across the Tannakian equivalence to eventually identify H with the Langlands dual group $G^\vee$.

We begin by enumerating some of the visible properties of the category $\mathrm{Rep}_k(G)$ for a reductive algebraic group G, as a means of motivating the statement of the Tannakian reconstruction theorem. This is an abelian, k-linear category (i.e. hom sets have a k-vector space structure), which tensor product of representations upgrades into a symmetric monoidal category.

The category $\mathrm{Rep}_k(G)$ also naturally comes equipped with a forgetful functor $\mathrm{obliv} : \mathrm{Rep}_k(G) \to \mathrm{Vect}_k^{fd}$ down to the category of finite-dimensional k-vector spaces, which can be shown to be exact, faithful and symmetric monoidal - in general, functors of this form with domain a given symmetric monoidal abelian category $\mathcal{C}$ are called **fiber functors**. Observe in particular that a representation V of G is invertible under the tensor product structure precisely when its underlying vector space is one-dimensional.

Finally, one of the key appeals of the reductive property (in the case where k is of characteristic zero) is that G being reductive is equivalent to the category $\mathrm{Rep}_k(G)$ being semisimple, meaning that every representation of G decomposes into a direct sum of irreducible representations (as the latter constitute the collection of simple objects inside $\mathrm{Rep}_k(G)$). The fact that this no longer holds in positive characteristic is one of the reasons why the geometric Satake equivalence is harder to establish in those settings.

We now state the Tannakian reconstruction theorem:

Theorem 1.1. *(Tannakian reconstruction) Let $(\mathcal{C}, \otimes)$ be an abelian, k-linear, symmetric monoidal category equipped with a fiber functor $F : \mathcal{C} \to Vect_k^{fd}$, and assume that $X \in \mathcal{C}$ is $\otimes$-invertible whenever $dim_k F(X) = 1$. Then there exists an equivalence of tensor categories compatible with fiber functors:*

$$\begin{array}{ccc} (\mathcal{C}, \otimes) & \xrightarrow{\quad\simeq\quad} & (Rep_k(H), \otimes_k) \ , \\ & {\scriptstyle F}\searrow \qquad \swarrow{\scriptstyle obliv} & \\ & (Vect_k^{fd}, \otimes_k) & \end{array}$$

where H is an affine group scheme over k.

The reader may have noticed that semisimplicity is not part of the set of properties required by Tannakian reconstruction. Indeed, this is one of the instances in which one first proves semisimplicity of $P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k)$, and then uses the Tannakian equivalence to deduce that $\mathrm{Rep}_k(H)$ is semisimple, and therefore that H is reductive. Along similar lines, the compatibility of the Tannakian equivalence with the respective fiber functors allows one to identify the simple objects of both categories, in a way which eventually allows us to identify $H = G^\vee$ via the classification of reductive algebraic groups in terms of their root datum.

The latter classification result will be the starting point of our prelude to the geometric Satake equivalence, starting with the (possibly more familiar) case of semisimple Lie algebras and then moving onto the settings of reductive algebraic group. This discussion will allow us to define the Langlands dual group, and will naturally lead to the introduction of the Weyl group associated to an algebraic group G and the associated Bruhat decomposition of the flag variety of G. From there, we move to an introduction to the category of perverse sheaves. We choose to focus on the simpler case of the category $P_{\mathscr{S}}(\mathcal{B})$ of B-equivariant perverse sheaves on the flag variety of G (equivalently, the category of perverse sheaves on $\mathcal{B}$ with respect to the Bruhat stratification), where we develop intuition for the analogous but more complex category $P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k)$ involved in the geometric Satake correspondence.

2. Recollections on semisimple Lie algebras

We begin by briefly recalling the correspondence between semisimple Lie algebras and root systems, which one usually encounters as a way to classify semisimple Lie algebras up to isomorphism by means of combinatorial methods. This theory will often inform our work in the algebraic group settings, as there is a way to functorially move between an algebraic group and its associated Lie algebra (with underlying vector space isomorphic to the Zariski tangent space at the identity).

Start by fixing a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ (i.e. a maximal solvable subalgebra) containing a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ (i.e. a self-normalizing nilpotent subalgebra, which we will later always view as the Lie subalgebra corresponding to a choice of maximal torus $T \subseteq G$). Then the adjoint action ad on $\mathfrak{g}$ restricted to $\mathfrak{h}$ gives rise via simultaneous diagonalization to a splitting:

$$\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where the non-zero $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : t \cdot x = \alpha(t)x \text{ for all } t \in \mathfrak{h}\}$ are called the root spaces of $\mathfrak{g}$, and the corresponding non-zero $\alpha \in \mathfrak{h}^*$ are called the **roots** of $\mathfrak{g}$, forming a finite subset which we denote by $\Delta \subseteq \mathfrak{h}^*$ and generating a lattice $\Lambda_\Delta = \mathbb{Z} \cdot \Delta$ inside $\mathfrak{h}^*$ called the **root lattice** of $\mathfrak{g}$. The pair $(\mathfrak{h}^*, \Delta)$, where we view $\mathfrak{h}^*$ as a Euclidian real vector space with respect to the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ - which is non-degenerate for semisimple $\mathfrak{g}$ - has the combinatorial structure of a **root system**. More generally, any representation V of $\mathfrak{g}$ is subject by restriction to $\mathfrak{h}$ to a decomposition of the above form, and we call the non-trivial generalized eigenspaces V_α the **weight spaes** of V, and the associated elements $\alpha \in \mathfrak{h}^*$ the **weights** of V. Accordingly, $\mathfrak{h}^*$ is called the **space of weights** of $\mathfrak{g}$.

By a theorem of Chevalley, the root system associated to $\mathfrak{g}$ does not depend on the choice of $\mathfrak{h}$ up to isomorphism. The above combinatorial data then determines a mapping:

$$\Phi : \big\{\text{semisimple Lie algebras over } \mathbb{C}\big\} \longrightarrow (\text{Root systems}) \tag{2.1}$$

$$\mathfrak{g} \longmapsto (\mathfrak{h}^*, \Delta),$$

It turns out that if two Lie algebra $\mathfrak{g}$ and $\mathfrak{g}'$ have isomorphic root systems, then $\mathfrak{g} \simeq \mathfrak{g}'$, and that any (irreducible) root systems can be obtained from some simple

Lie algebra. From this and the uniqueness of the decomposition of a semisimple Lie algebra us unique, we have reduced the classification problem of semisimple Lie algebras to that of classifying irreducible root systems. The latter admit a simple axiomatic description and can be shown to be classified by Dynkin diagrams. (For more on this theory, the reader may consult [4] or [6].)

Using the Killing form, one may canonically associate to each root $\alpha \in \Delta$ a **coroot** $\alpha^\vee$ defined to be the element of $\mathfrak{h}$ corresponding under the isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ to the weight $\frac{2\alpha}{(\alpha,\alpha)}$, which viewed in $\mathfrak{h}^*$ satisfies $(\alpha, \alpha^\vee) = 2$. by definition, and for two roots α, β, we have that

$$(\alpha, \beta) = \alpha(\beta^\vee).$$

Together, these coroots form a finite set $\Delta^\vee \subset \mathfrak{h}$, and one may check that the pair $(\mathfrak{h}, \Delta^\vee)$ also has the structure of a root system (sometimes called the dual root system of $\mathfrak{h}$.)

It turns out that our knowledge of the root data of $\mathfrak{g}$ also greatly informs our understanding of the representation theory of $\mathfrak{g}$. We start by making some definitions. Having chosen a triple $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$, we get a resulting partition of the roots of $\mathfrak{g}$, $\Delta = \Delta^+ \cup \Delta^-$, into **positive roots** and **negative roots**. The positive roots may be defined as those roots appearing in the decomposition $\mathfrak{b} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ of the Borel subalgebra into root spaces. This decomposition behaves nicely - in particular, positive roots are closed under addition, and for each $\alpha \in \Delta$, precisely one of $\alpha, -\alpha$ is a positive root. It is often useful to choose an additive basis of the positive roots (i.e. a minimal generating set for Δ^+ under addition), which we call the **simple roots** of $\mathfrak{g}$ and denote by Δ_s (we will see later that this choice, as well as many others, is accounted for by the actions of the so-called Weyl group on the weight lattice and on the Cartan subalgebra).

We single out the two classes of weights that appear in the classification of finite-dimensional irreducible representations of $\mathfrak{g}$, both defined by looking at the pairing

$$\mathfrak{h}^* \times \Lambda_\Delta \to \mathbb{C}$$

given by $(\mu, \alpha) := \mu(\alpha^\vee)$, where $\Lambda_\Delta = \mathbb{Z} \cdot \Delta$ denotes the **root lattice** generated by Δ inside $\mathfrak{h}^*$. First, we define the **integral weight lattice** to consist of those weights whose pairing against every simple root produces an integer, i.e.

$$\Lambda_\mathfrak{g} := \{\mu \in \mathfrak{h}^* : (\mu, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta_s\}.$$

Next, given a weight $\mu \in \mathfrak{h}^*$, say that μ is a **dominant weight** if $(\mu, \alpha) \geq 0$ for every simple root $\alpha \in \Delta_s$. We may now state the following classification theorem:

Theorem 2.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then we have a 1-1 correspondence between finite-dimensional, irreducible representations of $\mathfrak{g}$ and integral, dominant weights of $\mathfrak{g}$.*

3. Root datum of an algebraic group and Langlands dual group

We now move to the algebraic group settings, and start with a connected reductive algebraic group G, with a choice of Borel subalgebra and maximal torus $T \subseteq B \subseteq G$ - which induces a triple $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ as above at the level of Lie algebras.

This time around, we may restrict the adjoint action Ad of G on $\mathfrak{g}$ to T and obtain a decomposition:

$$\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : t{\cdot}x = \alpha(t)x \text{ for all } t \in T\}$, and each $\alpha \in X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ is an element of the **character lattice** or **weight lattice** of T, i.e. an algebraic map $\alpha : T \to \mathbb{G}_m$. In the same way as above, the non-zero **root spaces** correspond to a set of **roots** $\Delta \subseteq X^*(T)$. Differentiating each root gives us a bijection between Δ and the set of roots of $\mathfrak{g}$, allowing us to define positive, negative and simple roots of G in the same way as we did in the previous section.

Furthermore, note that by algebraicity, once we choose a splitting $T \simeq (\mathbb{G}_m)^n$ for some $n \in \mathbb{N}$, we have an isomorphism $X^*(T) \simeq \mathbb{Z}^n$ (under $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$ via $(\varphi : z \mapsto z^n) \mapsto n$). Hence upon differentiating, we only access the integral weights of $\mathfrak{g}$.

In the algebraic group context, we may no longer use the Killing form to canonically define coroots. As such, we need to enlarge the combinatorics side of 2.1 above if we wish to obtain an analogous statement for reductive groups. We start by introducing the **coweight lattice** $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, where we expect the coroots of G to live, and which fits into a pairing with the weight lattice by composition:

$$X^*(T) \times X_*(T) \to \mathbb{Z} \tag{3.1}$$

$$\mathrm{Hom}(T, \mathbb{G}_m) \times \mathrm{Hom}(\mathbb{G}_m, T) \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}, \quad (\alpha, \lambda) \mapsto \lambda \circ \alpha.$$

The definition of the coroot $\alpha^\vee$ associated to a root $\alpha \in \Delta$ in the reductive group settings is based on the following result, which is inspired by the existence of $\mathfrak{sl}_2$-embeddings $\mathfrak{sl}_{2,\alpha} \to^{\simeq} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}$ associated to each root $\alpha \in \Delta$ (using the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \Delta$):

Proposition 3.2. *Let $\alpha : T \to \mathbb{G}_m$ be a root of G, and let $x \in \mathfrak{g}_\alpha - 0$. Then there exists a unique algebraic map*

$$\varphi_\alpha : SL_2 \to G$$

which maps the maximal torus of SL_2 (taken to be $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^-1 \end{pmatrix} : t \in \mathbb{G}_m \right\}$) into T, such that its differential $\mathfrak{sl}_2 \to \mathfrak{g}$ maps $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ into $\mathbb{C} \cdot x$, and such that the restriction

$$\alpha^\vee := (\varphi_\alpha)\big|_{\mathbb{G}_m} : \mathbb{G}_m \to T$$

satisfies $(\alpha, \alpha^\vee) = 2$ under the weight-coweight pairing from 3.1.

Using this proposition for each root $\alpha \in \Delta$, we obtain a set of coroots $\Delta^\vee \subseteq X_*(T)$, and we may now assemble both pieces of combinatorial data into the following mapping:

$$\Psi : \big\{\text{connected reductive groups over } \mathbb{C}\big\} \longrightarrow (\text{Root datums}) \tag{3.3}$$

$$(G, T) \longmapsto (X^*(T), \Delta, X_*(T), \Delta^\vee).$$

Once again, combinatorial structures of the form $(X^*(T), \Delta, X_*(T), \Delta^\vee)$, called root datums, can be axiomatically described, and it can be shown that a connected reductive group is determined up to isomorphism by its root datum, and that any root datum comes from some connected reductive group. A key fact is that exchanging roots with coroots and weights with coweights produces a valid root datum, which allows us to make the following:

Definition 3.4. Given a connected reductive group G with root datum as above, we define the **Langlands dual group** of G to be the connected reductive group associated to the root datum $(X_*(T), \Delta^\vee, X^*(T), \Delta)$.

Remark 3.5. This duality construction can be shown to factor through types of the Dynkin diagram classification, in such a way that the types A, D, E, F, G are self dual and the types B and C are exchanged.

Example 3.6. Let $G = SL_3$, with choice of maximal torus

$$T := \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_1 t_2 t_3 = 1 \right\}.$$

Then the elements of the weight lattice $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ are all given by maps of the form

$$\alpha : \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto t_1^a t_2^b t_3^c, \text{ for some } a, b, c \in \mathbb{Z},$$

resulting in the surjective exponent mapping $\mathbb{Z}^3 \to X^*(T)$, which one can check using the relation $t_1 t_2 t_3 = 1$ has kernel generated by $(1, 1, 1)$, so that we obtain the identification:

$$X^*(T) \simeq \mathbb{Z} < a, b, c > /(a + b + c = 0).$$

On the other hand, elements of the coweight lattice $X_*(T)$ $\mathrm{Hom}(\mathbb{G}_m, T)$ correspond to maps of the form

$$\lambda : t \mapsto \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}, \text{ for some } a, b, c \in \mathbb{Z} \text{ such that } a + b + c = 0,$$

so that we can identify $X_*(T) \simeq H \subseteq \mathbb{Z}^3$ with the hyperplane $H \simeq \mathbb{Z}^2$ inside $\mathbb{Z} < a, b, c >$ given by the condition $a + b + c = 0$. Finally, the weight-coweight pairing on SL_3 is just given by the standard inner product, sending

$$\left(\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto t_1^a t_2^b t_3^c, \quad t \mapsto \begin{pmatrix} t^{a'} & 0 & 0 \\ 0 & t^{b'} & 0 \\ 0 & 0 & t^{c'} \end{pmatrix} \right) \mapsto aa' + bb' + cc' \in \mathbb{Z}.$$

As in the Lie algebra settings, knowing the root datum of G informs us about the representation theory of G. In analogous fashion using the pairing 3.1, given $\lambda \in X^*(T)$, declare λ to be a **dominant weight** if $(\lambda, \alpha^\vee) \geq 0$ for every simple root $\alpha \in \Delta_s$. We then have the following classification theorem:

Theorem 3.7. *Let G be a connected reductive group defined over k, with choice of maximal torus T. Then irreducible representations of G up to isomorphism are in 1-1 correspondence with dominant weights $\lambda \in X^*(T)$.*

Putting this theorem side-by-side with the corresponding result for semisimple Lie algebras, and using that any reductive Lie algebra splits into a semisimple component and an abelian component $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_a$, we get the following situation by differentiating representations of G into representations of $\mathfrak{g}$:

$$\begin{array}{ccc} (\text{irrepns of } G) & \xrightarrow{\simeq} & (\text{dominant weights in } X^*(T)) \\ \downarrow & & \downarrow \\ (\text{irrepns of } \mathfrak{g}) & \xrightarrow{\simeq} & (\text{dominant integral weights of } \mathfrak{g}_s) \oplus \mathbb{Z}_{\geq 0}^{\dim \mathfrak{g}_a}. \end{array}$$

Furthermore, since the category $\mathrm{Rep}_k(G)$ is semisimple for reductive G, this theorem gives us a complete description of this category in terms of its simple objects:

$$\begin{array}{ccc} \mathrm{Rep}_k(G) & & \\ \uparrow & & \\ \{\text{irrepns of } G\} & \xrightarrow{\simeq} & \{\text{dominant weights of } G\}. \end{array}$$

We end this section by mentioning a few more notions and definitions that will be useful going forward. First, we may define a partial order on the weight lattice by declaring that $\alpha \leq \beta$ for $\alpha, \beta \in X^*(T)$ if β can be reached from α by adding a linear combination of positive roots with non-negative integer coefficients, i.e.

$$\alpha \leq \beta \text{ iff } \beta \in \alpha + \mathbb{N} \cdot \Delta^+. \tag{3.8}$$

One may similarly define a partial order on the coweight lattice using linear combinations of positive coroots with non-negative integer coefficients (where a positive coroot is one of the form $\alpha^\vee$ for $\alpha \in \Delta^+$).

Next, we introduce the notion of regular weights, which will be given a more geometric description in the following section. Given $\mu \in X^*(T)$, declare μ to be **regular** if $\mu(\alpha^\vee) \neq 0$ for every simple root $\alpha \in \Delta_s$ (where we write $\mu(\alpha^\vee)$ as shorthand for the pairing $(\mu, \alpha^\vee)$). There is one particularly important regular weight, defined by the following formula:

$$2\rho := \sum_{\alpha \in \Delta^+} \alpha.$$

This definition makes sense once we check that $2\rho(\beta^\vee) = 2$ for each simple root $\beta \in \Delta_s$. To check this, we use the fact that the simple reflection s_β associated to a given simple root $\beta \in \Delta_s$ (as defined in 4.1 below) sends the set $\Delta^+ - \{\beta\}$ to itself. From there, we can compute that:

$$2\rho(\beta^\vee) = \beta(\beta^\vee) + \sum_{\alpha \in \Delta^+ - \{\beta\}} \alpha(\beta^\vee) = 2 + \sum_{\alpha \in \Delta^+ - \{\beta\}} \alpha(\beta^\vee),$$

where the second term vanishes as a result of the following computation:

$$\begin{aligned} \sum_{\alpha \in \Delta^+ - \{\beta\}} \alpha &= \frac{1}{2}\Big(\sum_{\alpha \in \Delta^+ - \{\beta\}} (\alpha + s_\beta(\alpha)) \Big) \\ &= \frac{1}{2}\Big(\sum_{\alpha \in \Delta^+ - \{\beta\}} (2\alpha - \alpha(\beta^\vee)\beta) \Big) \\ &= \Big(\sum_{\alpha \in \Delta^+ - \{\beta\}} \alpha \Big) - \frac{1}{2}\Big(\sum_{\alpha \in \Delta^+ - \{\beta\}} \alpha(\beta^\vee) \Big)\beta. \end{aligned}$$

In the context of the geometric Satake correspondence, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ viewed as a map

$$\rho : X_*(T) \to \mathbb{Z}[1/2]$$

restricting to a map from the coroot lattice to $\mathbb{Z}$ will be useful in studying the geometry of the affine Grassmanian $\mathcal{G}\mathrm{r}_G$ - specifically, it will be involved in describing properties of the Bruhat-type decomposition of $\mathcal{G}\mathrm{r}_G$.

4. The Weyl group and the Bruhat decomposition

As we discussed above, a choice of Borel subgroup $B \subseteq G$ (or equivalently a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$) induces a notion of positivity on the set of roots of G, which in particular locates dominant weights (defined by looking at pairings against simple roots). In the Lie algebra picture, the dominant weights all live in a connected open subset of $\mathfrak{h}^*$ called the **positive/fundamental Weyl chamber** associated to Δ^+; other choices of Borel may in turn lead to different fundamental Weyl chambers. Geometrically, the set of possible Weyl chambers corresponds to the set of open connected components of the complement of $\bigcup_{\alpha\in\Delta} H_\alpha$, where each $H_\alpha \subseteq \mathfrak{h}^*$ is the hyperplane orthogonal to the root α, defined by the vanishing condition $(\mu, \alpha^\vee) = 0$. We will define the Weyl group $W \subseteq O(\mathfrak{h}^*)$, a finite group which in a sense counterbalances the choice one makes when choosing a Borel. For instance, in can be shown that W acts simply and transitively on the set of possible Weyl chambers, and that it permutes the choices of simple roots corresponding to different Borels.

We introduce two equivalent definitions of the Weyl group, one, more combinatorial, at the level of the Lie algebra $\mathfrak{g}$, and one, more conceptual, at the level of the algebraic group G.

Viewing the root system of G as living in $\mathfrak{h}^*$ by differentiating, we may associate to each simple root $\alpha \in \Delta_s$ the following endomorphism, called the **simple reflection associated to** α:

$$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$$

$$s_\alpha(x) := x - 2\frac{\kappa(\alpha, x)}{\kappa(\alpha, \alpha)}\alpha. \tag{4.1}$$

Geometrically, s_α is the reflection of $\mathfrak{h}^*$ across the hyperplane H_α orthogonal to α, exchanging α with $-\alpha$. The **Weyl group** of G is then defined to be the group generated by these simple reflections (with relations as determined inside $O(\mathfrak{h}^*)$):

$$W := < s_\alpha : \alpha \in \Delta_s > .$$

The generating set $\{s_\alpha\}_{\alpha\in\Delta_s} \subseteq W$ consisting of all simple reflections is often denoted by S.

Remark 4.2.

- The Weyl group is always finite, and it restricts to a group of permutations on the roots $\Delta \subset \mathfrak{h}^*$.
- We are now in a position to make the observation from the beginning of this section more formal: given any weight $\lambda \in X^*(T)$ viewed as an element of $\mathfrak{h}^*$, we may consider the Weyl orbit $W \cdot \lambda \subset \mathfrak{h}^*$. It can be shown that this Weyl orbit contains a unique dominant weight, given by the maximal element in $W \cdot \lambda$ with respect to the weight ordering as defined in 3.8.

- We may also provide an alternative, more geometric notion of a regular weight (viewed inside $\mathfrak{h}^*$) as being a weight living in the interior of a Weyl chamber, equivalently a weight living in the Weyl orbit of the fundamental Weyl chamber.

Example 4.3. For $\mathfrak{g} = \mathfrak{sl}_n$, the Weyl group is the symmetric group on n letters $W = S_n$, acting on the standard basis of $\mathfrak{h}^*$ by permutation $\sigma \cdot L_i = L_{\sigma(i)}$.

We now introduce a second definition of the Weyl group, this time taking place at the level of the algebraic group G and acting on weights viewed as algebraic maps $T \to \mathbb{G}_m$. Given a choice of maximal torus $T \subseteq G$, define the Weyl group of G to be the subquotient

$$W := N_G(T)/T,$$

where $N_G(T)$ denotes the normalizer subgroup of T inside G. Given an element $w \in W$, we may choose a representative $\dot{w} \in N_G(T)$ and obtain an action on T by conjugation, via $\dot{w} \cdot t := w^{-1}tw$. This lets us define associated actions

- on weights of G by precomposition:

$$\alpha \in \mathrm{Hom}(T, \mathbb{G}_m) \mapsto \dot{w} \cdot \alpha(t) := \alpha(w^{-1}tw), \tag{4.4}$$

- and on coweights of G by postcomposition:

$$\eta \in \mathrm{Hom}(\mathbb{G}_m, T) \mapsto \dot{w} \cdot \eta(t) := w^{-1}\eta(t)w. \tag{4.5}$$

Once again, one can check that this action preserves roots, and that the resulting group is a finite group isomorphic to the Weyl group as defined at the level of $\mathfrak{g}$.

Example 4.6. Consider SL_3, with maximal torus

$$T = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_1t_2t_3 = 1 \right\}.$$

Then $W = S_3$ acts by permuting the coordinates of T via permutation matrices (adjusted if necessary to have determinant 1). We may consider for instance the map $s_{1,2} : T \to T$ given by exchanging t_1 and t_2 in the above notation. The resulting action on the character lattice $X^*(T)$ identified with a quotient of $\mathbb{Z}^3$ is given by $(a, b, c) \mapsto (b, a, c)$. This particular element of the Weyl group is a simple reflection corresponding to the simple root $\alpha = t_1t_2^{-1}$ of SL_3.

Having introduced the Weyl group in two equivalent ways, we now proceed to state some of the useful additional structures that it carries. First, interpreting W as generated by S, we may define a **length function** on W,

$$l : W \to \mathbb{N},$$

given by setting $l(w) = r$, where r is the minimal length of an expression $w = s_{\alpha_1}...s_{\alpha_r}$ in **reduced form**, i.e. as a product of simple generators $s_{\alpha_i} \in S$. For instance, $l(1) = 0$ and for each simple root $\alpha \in \Delta_s$, $l(s_\alpha) = 1$.

We may also equip W with a partial order, sometimes called the **Bruhat order** (for reasons that will shortly become clear), by declaring that $w \leq \eta$ for $w, \eta \in W$ whenever η can be expressed as an expansion of w, taken to mean that one can reach η from w in some reduced forms by inserting (not necessarily consecutive) simple reflections to w. This is best conveyed by means of an example: $w \leq \eta$ for $w = s_1s_5s_3$ and $\eta = s_2s_1s_5s_4s_3$.

With these notions in hand, we are in a position to state one of the main results through which W informs us about the geometry of G. As mentioned above, the perspective $W = N_G(T)/T$ induces a map via representatives $W \to G, w \to \dot{w}$. One may then act on the right and left by B on the elements $\dot{w} \in G$. The **Bruhat decomposition theorem** states that the resulting double cosets induce a decomposition of G into disjoint locally closed subsets:

$$G = \bigsqcup_{w \in W} B\dot{w}B.$$

Of particular interest to us is the induced decomposition on the **flag variety** of G, which we denote by $\mathcal{B} = G/B$ (see interlude below for some recollections on the flag variety associated to an algebraic group).

[**Interlude**] (Brief reminder about the flag variety) The flag variety $\mathcal{B} = G/B$ can be interpreted as a parameter space for the set of all Borel subgroups of G. Indeed, we can consider the action of G on the space of Borels in G by conjugation (by a basic property of Borel subgroups). Fixing a Borel subgroup $B \subseteq G$, we have that $\text{Stab}_G(B) = B$, hence the action induces a map

$$G/B \to \{\text{Borel subgroups } B' \subseteq G\}$$

$$gB \mapsto gBg^{-1}$$

which turns out to be a bijection.

Fun 4.7. The terminology "flag variety" comes from the fact that when one takes $G = GL_n$, there is a correspondence between Borel subgroups of GL_n and complete flags in $\mathbb{C}^n$ (i.e. sequences of nested subspaces $0 = F_0 \subseteq F_1 \subseteq ... \subseteq F_n\ \mathbb{C}^n$ with $\dim_{\mathbb{C}} F_i = i$). Under this perspective, the usual Borel in GL_n consisting of upper triangular matrices corresponds to the standard complete flag $0 \subseteq \mathbb{C}e_1 \subseteq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subseteq ... \subseteq \mathbb{C}^n$ with respect to the standard basis for $\mathbb{C}^n$.

Remarks 4.8.
- $\mathcal{B}$ can always be given the structure of a projective variety. For instance, for $G = SL_2$, $\mathcal{B} \simeq \mathbb{P}^1$
- The Borel-Weil theorem constructs the irreducible representations of an algebraic group as global sections of line bundles over the corresponding flag variety. This is an early instance of a result in geometric representation theory, established in 1953.

With respect to $\mathcal{B}$, the Bruhat decomposition becomes:

$$\mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}_w,$$

where $\mathcal{B}_w = B\dot{w}B/B$. This decomposition is well-behaved, notably in the following ways:

- Each orbit can be identified with an affine space of dimension equal to the length of the indexing Weyl group element: $\mathcal{B}_w \simeq \mathbb{A}^{l(w)}$ for each $w \in W$. In particular, each $\mathcal{B}_w$ is simply connected and locally closed (open in its closure);
- One can show that we have a containment relation of the form $\mathcal{B}_w \subseteq \bar{\mathcal{B}}_\eta$ for $w, \eta \in W$ if and only if $w \subseteq \eta$ with respect to the Bruhat order.

In what follows, we choose to discuss the category of perverse sheaves over the flag variety with respect to the Bruhat decomposition - a setting in which the ‘base space” is simpler than the affine Grassmanian involved in the geometric Satake correspondence. We do this so that the reader's attention can be devoted to looking at the categorical structures involved - which turn out to be structurally similar to the Satake category.

We close this section by mentioning that the affine Grassmanian $\mathcal{G}r_G$ also admits a Bruhat-type decomposition into simply connected orbits, indexed by the *dominant coweights* $X_*(T)$ of G, and that the category of perverse sheaves with respect to the resulting stratification is equivalent to the Satake category. This fact, together with the analogue of the classification of simple objects in $P_{\mathscr{S}}(\mathcal{B})$, tells us that the categories Sat_G and $\mathrm{Rep}_k(G^\vee)$ both have simple objects indexed by $X_*(T)$, a first indication that the geometric Satake correspondence "stands a chance" in the first place. Namely, we obtain a chain of correspondences:

$$\big\{\text{Simple objects of } \mathrm{Rep}(G^\vee)\big\} \leftrightarrow X^*(T^\vee)^+ \leftrightarrow X_*(T)^+ \leftrightarrow \big\{\text{Simple objects of } \mathrm{Sat}_G\big\},$$

where e.g. $X_*(T)^+$ denotes the space of dominant coweights on G.

5. The derived category of constructible sheaves $\mathcal{D}^b_{\mathscr{S}}(\mathcal{B})$

The collection $\{\mathcal{B}_w\}_{w\in W}$ forms the **strata** of a **stratification** on $\mathcal{B}$, by which we mean a disjoint cover by smooth, connected, locally closed subvarieties satisfying the property that for any two $w \neq \eta \in W$, either $\mathcal{B}_w \cap \bar{\mathcal{B}_\eta} = \emptyset$ or $\mathcal{B}_w \subseteq \bar{\mathcal{B}_\eta}$ (in fact, this much is true of the orbits of any action of an algebraic group G on a smooth variety X with finitely many orbits).

Notation 5.1. In what follows, sheaves are assumed to take values in finite-dimensional k-vector spaces, where we fixed k to denote an algebraically closed field of characteristic zero. This can be done in much more generality for k a Noetherian ring of finite global dimension and arbitrary characteristic, but we choose not to go there for the sake of simplicity.

In attempting to study $\mathcal{B}$ from a geometric perspective, one may naturally be led to studying the category of **constructible sheaves** on $\mathcal{B}$ with respect to the Bruhat stratification $\mathscr{S}$. For the purposes of the geometric Satake correspondence, the right category to look at can be reached by first enlarging our geometric category to the **derived category of constructible sheaves** $\mathcal{D}^b_{\mathscr{S}}(\mathcal{B})$ (whose objects are bounded complexes of sheaves with constructible cohomology sheaves), and then restricting down to the abelian heart $P_{\mathscr{S}}(\mathcal{B})$ of $\mathcal{D}^b_{\mathscr{S}}(\mathcal{B})$ with respect to an appropriate t-structure called the perverse t-structure. This section and the next are aimed at making this paragraph more palatable to the uninitiated reader.

Recall that a sheaf $\mathcal{F}$ on $\mathcal{B}$ is said to be **constructible** with respect to a given stratification $\mathscr{S}$ if the restriction of $\mathcal{F}$ to each stratum $\mathcal{B}_s, s \in \mathscr{S}$, is a local system of k-vector spaces. Notice that in the event that a stratum $\mathcal{B}_s$ is simply connected (which is always the case for the Bruhat stratification of $\mathcal{B}$), the equivalence of categories

$$\mathrm{LocSys}(\mathcal{B}_s) \simeq \mathrm{Rep}(\pi_1(\mathcal{B}_s)) \tag{5.2}$$

tells us that $\mathcal{F}\big|_{\mathcal{B}_s}$ must actually be a free sheaf of finite rank - a fact which will greatly simplify the study of simple objects of $P_{\mathscr{S}}(\mathcal{B})$ in Section 7.

Remark 5.3. The reader may have noticed that we are setting out to study the *stratified category* of perverse sheaves on $\mathcal{B}$ with respect to the Bruhat stratification, whereas the geometric Satake correspondence suggests one should be looking at B-*equivariant* sheaves on $\mathcal{B}$. These two categories are related by the existence of a fully-faithful forgetful functor

$$\mathrm{obliv} : P_B(\mathcal{B}) \to P_{\mathscr{S}}(\mathcal{B}),$$

which in the case of the flag variety with the Bruhat decomposition may fail to be an equivalence (whether or not this is the case needs to be verified). Thus, we can think of $P_B(\mathcal{B})$ as a full subcategory of $P_{\mathscr{S}}(\mathcal{B})$. In the case of the Satake category, the corresponding forgetful functor is actually an equivalence, and so working in the stratified category makes no difference (see corollary 4.8 in [2]). For the purposes of these introductory notes, we choose to work with the stratified category $P_{\mathscr{S}}(\mathcal{B}) \subseteq \mathcal{D}^b(\mathcal{B})$, as it is simpler to define.

6. Perverse t-structure and the category $P_{\mathscr{S}}(\mathcal{B}) = \mathcal{D}^b_{\mathscr{S}}(\mathcal{B})^{\heartsuit^{\text{perv}}}$

Let $\mathcal{C}$ be a **triangulated category** - for intuition, the reader should feel free to think of $\mathcal{C} = \mathcal{D}(\mathcal{A})$ as the derived category associated to an abelian category $\mathcal{A}$, consisting of cochain complexes in $\mathcal{A}$ with morphisms taken up to chain homotopy equivalence and quasi-isomorphisms formally inverted.

Idea 6.1. A t-structure on $\mathcal{C}$ gives us a way of "taking cohomology in $\mathcal{C}$".

Example 6.2. To illustrate the above idea, we introduce the **standard t-structure** on the derived category $\mathcal{C} = \mathcal{D}(R)$ associated to the abelian category of R-modules a given ring R. We may single out two full subcategories defined in terms of the grading of the constituent complexes, namely:

$$\mathcal{C}^{\leq 0} = \{A^\bullet \in \mathcal{C} : H^i(A^\bullet) = 0 \text{ for all } i > 0\},$$
$$\mathcal{C}^{\geq 0} = \{A^\bullet \in \mathcal{C} : H^i(A^\bullet) = 0 \text{ for all } i < 0\}.$$

The respective inclusions $\mathcal{C}^{\leq 0} \hookrightarrow \mathcal{C}$, $\mathcal{C}^{\geq 0} \hookrightarrow \mathcal{C}$ admit adjoint functors called **truncation functors**:

$$\mathcal{C}^{\geq 0} \xleftarrow{\tau^{\geq 0}} \mathcal{C} \xrightarrow{\tau^{\leq 0}} \mathcal{C}^{\leq 0}.$$

Composing these gives rise to a functor

$$H^0 = \tau^{\geq 0} \circ \tau^{\leq 0} : \mathcal{C} \to \mathcal{C}^{\heartsuit} \simeq R\text{-mod}$$

where the intersection $\mathcal{C}^{\heartsuit} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$, called the **heart** of $\mathcal{C}$ with respect to the standard t-structure, can be identified with the underlying abelian category R-mod. As the notation suggests, this functor viewed as a functor $\mathcal{C} \to R$-mod sends an object $A^\bullet \in \mathcal{C}$ to the cohomology group $H^0(A^\bullet)$, and precomposing with shifts produces the other cohomology groups $H^n(A^\bullet)$.

The general definition of a t-structure on a triangulated category $\mathcal{C}$ is modeled on the properties of the above example. Rather than a formal definition (which one may find for instance in [1]), we cite two of the key facts that make finding valid t-structures on triangulated categories a worthwhile task:

- The heart $\mathcal{C}^{\heartsuit} = \tau^{\geq 0} \circ \tau^{\leq 0}(\mathcal{C})$ with respect to a given t-structure is always an **abelian category**.
- The associated projection functor ${}^tH^0 : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ is always a **cohomological functor**, meaning that it sends distinguished triangles
$$X \to Y \to Z \to X[1]$$
in $\mathcal{C}$ to exact sequences in $\mathcal{C}^{\heartsuit}$, and consequently to "cohomology long exact sequences" of the form
$$... \to^t H^0(X) \to^t H^0(Y) \to^t H^0(Z) \to^t H^0(X[1]) =^t H^1(X) \to ...$$

Definition 6.3. The category of $\mathscr{S}$-perverse sheaves on the flag variety $\mathcal{B}$ is defined to be the heart of the triangulated category $\mathcal{D}^b_{\mathscr{S}}(\mathcal{B})$ with respect to the perverse t-structure:
$$P_{\mathscr{S}}(\mathcal{B}) := \big(\mathcal{D}^b_{\mathscr{S}}(\mathcal{B})\big)^{\heartsuit^{\mathrm{perv}}},$$
where the perverse t-structure on $\mathcal{D}^b_{\mathscr{S}}(\mathcal{B})$ can be defined by declaring that:
$$\mathcal{F} \in^p \mathcal{D}^b_{\mathscr{S}}(\mathcal{B})^{\leq 0} \text{ iff } i_w^*\mathcal{F} \in \mathcal{D}^b_{loc}(\mathcal{B}_w)^{\leq -\dim \mathcal{B}_w} \text{ for all } w \in \mathscr{S}$$
$$\mathcal{F} \in^p \mathcal{D}^b_{\mathscr{S}}(\mathcal{B})^{\geq 0} \text{ iff } i_w^!\mathcal{F} \in \mathcal{D}^b_{loc}(\mathcal{B}_w)^{\geq -\dim \mathcal{B}_w} \text{ for all } w \in \mathscr{S},$$
where $i_w : \mathcal{B}_w \hookrightarrow \mathcal{B}$ denote the stratum inclusion.

In words, the perversity condition for belonging to e.g. ${}^p\mathcal{D}^b_{\mathscr{S}}(\mathcal{B})^{\leq 0}$ requires the pullback of F to each stratum of the Bruhat decomposition to have local systems as cohomology sheaves, and for the cohomology sheaves to be concentrated below degree $-\dim \mathcal{B}_w$, where as we explicitly said earlier $\dim \mathcal{B}_w = l(w)$ is given by the length of the corresponding Weyl group element.

Remark 6.4. This is not the standard definition for the perverse t-structure with respect to the category $\mathcal{D}^b_{\mathscr{S}}(X)$ associated to a general stratified space X. Rather, it is a simpler variant which relies on the fact (which may require confirmation) that elements of $\mathcal{D}^b_{\mathscr{S}}(\mathcal{B}) \subseteq \mathcal{D}^b_c(\mathcal{B})$ are closed under taking Verdier duals, something which in turn is satisfied provided that the Bruhat decomposition of $\mathcal{B}$ is a Whitney stratification.

7. IC sheaves as simple objects inside $P_{\mathscr{S}}(\mathcal{B})$

Because the strata of the Bruhat decomposition on $\mathcal{B}$ are well-behaved (simply-connected orbits under an algebraic group action), one can check that $\mathcal{D}^b_B(\mathcal{B})$ is closed under the derived functors $i_*, i_!, i^*, i^!$ associated to the locally closed inclusion of a stratum $i : \mathcal{B}_w \hookrightarrow \mathcal{B}$.

Construction 7.1. (IC sheaves on $\mathcal{B}$) Because each Bruhat stratum $\mathcal{B}_w$ is simply connected, we get from 5.2 that $\underline{k}$ is the unique irreducible local system on $\mathcal{B}_w$, which we may view as a perverse sheaf on $\mathcal{B}_w$. Let $i : \mathcal{B}_w \hookrightarrow \mathcal{B}$ be the associated locally closed embedding, and define the **IC sheaf associated to** $\mathcal{B}_w$ to be the perverse sheaf on $\mathcal{B}$ given by:
$$\mathrm{IC}_w := i_{!*}\underline{k} \ = \ {}^pH^0\big(\mathrm{im}(i_!\underline{k} \to i_*\underline{k})\big)$$
Recall that for a morphism $f : X \to Y$, $f_!$ denotes the derived functor associated to the direct image with proper support functor, given by:
$$f_! : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$$
$$f_!\mathcal{F}(U) := \{s \in \mathcal{F}(f^{-1}(U)) : \mathrm{supp}(s) \subseteq X \text{ is proper}\}$$

This interpretation makes evident the fact that we have an inclusion $f_!\mathcal{F} \hookrightarrow f_*\mathcal{F}$ (which is an equality when f is a proper morphism).

Theorem 7.2. *For each $w \in W$, the IC sheaf IC_w is a simple object in $P_{\mathscr{S}}(\mathcal{B})$, and together these constitute all the simple objects in $P_{\mathscr{S}}(\mathcal{B})$ up to isomorphism.*

One can then investigate whether or not the category $P_{\mathscr{S}}(\mathcal{B})$ is semisimple using this statement. The familiar viewpoint in the derived category of R-modules $\mathcal{D}(R)$ that extensions between two R-modules correspond to chain maps of degree one between their image in $\mathcal{D}(R)$ holds with respect to any t-structure on a triangulated category $\mathcal{C}$, in the sense that we always have an isomorphism:

$$\mathrm{Ext}^1_{\mathcal{C}^\heartsuit}(X,Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y[1]).$$

Hence, to check whether the category $P_{\mathscr{S}}(\mathcal{B})$ is semisimple, it suffices to check the following vanishing condition on hom sets, for every pair $w, \eta \in W$:

$$\mathrm{Hom}_{P_{\mathscr{S}}(\mathcal{B})}(\mathrm{IC}_w, \mathrm{IC}_\eta[1]) = 0. \tag{7.3}$$

In the Satake category Sat_G, the Bruhat-type decomposition of the affine Grassmanian induces a classification of the simple objects by IC sheaves indexed by dominant elements of $X_*(T)$, and we can show that Sat_G is semisimple using the analogue of criterion 7.3 above. Thus, the IC sheaves fully determine the objects of the Satake category.

8. Convolution product on $P_B(\mathcal{B})$

We now return to the category $P_B(\mathcal{B})$ of B-equivariant sheaves on the flag variety $\mathcal{B}$ and introduce the convolution product, which will equip it with a monoidal structure. Taking to heart the analogy that sheaves are categorified analogues of functions, we start with the toy example of the convolution structure on the ring of functions on a finite group, something which will get us quite close to the sheaf theoretic definition.

Example 8.1. Let G be a finite group, and write $\mathbb{C}[G]$ for the ring of complex-valued functions on G. Given two functions $f, g \in \mathbb{C}[G]$, their convolution product is traditionally defined as the function $f * g \in \mathbb{C}[G]$ given pointwise by the formula:

$$f * g(x) := \frac{1}{|G|} \sum_{y \in G} f(xy^{-1})g(y).$$

Letting $m : G \times G \to G$ denote the multiplication map on G and writing $f \times g \in \mathbb{C}[G \times G]$ for the function given by pointwise multiplication, the above expression may be rewritten as:

$$\frac{1}{|G|} \sum_{y \in G} f(xy^{-1})g(y) = \frac{1}{|G|} \sum_{y,z \in G : m(y,z)=x} f(y)g(z) = \frac{1}{|G|} \sum_{(y,z) \in m^{-1}(x)} (f \times g)(y,z).$$

The convolution of f and g may then be repackaged into functorial language as:

$$f * g = m_*(p_1^* f \otimes p_2^* g),$$

where $m_* : \mathbb{C}[G \times G] \to \mathbb{C}[G]$ denotes the (normalized) pushforward on functions induced by m, sometimes informally referred to as "integration along the fibers of

m", $\mathbb{C}[G] \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathbb{C}[G \times G]$ denote the pullback operations associated to the natural projection maps $G \times G \underset{p_2}{\overset{p_1}{\rightrightarrows}} G$, and the tensor of two functions is given by pointwise multiplication.

Although this toy computation does not take into account the equivariance considerations needed to define convolution on $P_{\mathscr{S}}(\mathcal{B})$, one can already see in the settings $H \subseteq G$ of a subgroup contained in a finite group that the space $\mathbb{C}[G/H]$, equivalently given by the space of right H-invariant functions on G, is not closed under the convolution product, while functions on the double coset $\mathbb{C}[H\backslash G/H]$ are closed under this operation. This may give some motivation as to why we are choosing to look at B-equivariant sheaves on G/B in the first place.

Construction 8.2. We choose to carry the construction in the ambient derived category $\mathcal{D}^b_B(\mathcal{B})$. Appropriate insertions of the perverse cohomological functor ensures that this restricts to a well-defined operation on $P_{\mathscr{S}}(\mathcal{B})$. This construction section 6.2 of [2], and is very much an outline - a proper construction would need to be more careful about shifts and the way in which the equivalence of categories associated to q in 8.3 below is obtained.

Start with two elements $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b_B(\mathcal{B})$. We can form their **exterior product** $\mathcal{F} \boxtimes^{\mathbb{L}} \mathcal{G} \in \mathcal{D}^b_{B\times B}(\mathcal{B} \times \mathcal{B})$ analogously to the toy settings:

$$\mathcal{F} \boxtimes^{\mathbb{L}} \mathcal{G} := (p_1^* \mathcal{F}) \otimes^{\mathbb{L}} (p_2^* \mathcal{G}),$$

where we view $\mathcal{B} \times \mathcal{B}$ as a $B \times B$-space under component-wise left multiplication. Before taking their pushforward under m, we need to ensure equivariance. To this end, consider the following diagram:

(8.3)
$$\begin{array}{ccc} & G \times \mathcal{B} & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathcal{B} \times \mathcal{B} & & G \times^B \mathcal{B} \\ & & \downarrow m \\ & & \mathcal{B} \end{array}$$

Here p and q are projection maps, where we are taking:

- $G \times \mathcal{B}$ as a $B \times B$-space under the action $(b_1, b_2) \cdot (x, yB) = (b_1 g b_2^{-1}, b_2 h B)$;
- $G \times^B \mathcal{B}$ as the induction space obtained from $G \times \mathcal{B}$ with respect to the given action, itself viewed as a B-space under the action $b \cdot (x, yB) = (xb^{-1}, byB)$.

Now, consider the pullback $p^*(\mathcal{F} \boxtimes^{\mathbb{L}} \mathcal{G}) \in \mathcal{D}^b_{B\times B}(G \times \mathcal{B})$ under the map p, well defined because p is smooth and equivariant. Next, it can be shown that the map q induces under pullback an equivalence of categories:

$$q^* : \mathcal{D}^b_B(G \times^B \mathcal{B}) \xrightarrow{\simeq} \mathcal{D}^b_{B\times B}(G \times \mathcal{B}),$$

and so one may define the sheaf $\mathcal{F} \tilde{\boxtimes} \mathcal{G} \in$ to be the unique element satisfying the relation

$$q^* \mathcal{F} \tilde{\boxtimes} \mathcal{G} \simeq p^*(\mathcal{F} \boxtimes^{\mathbb{L}} \mathcal{G}).$$

From there, we define the **convolution product** of $\mathcal{F}$ and $\mathcal{G}$ to be the following B-equivariant sheaf on $\mathcal{B}$:

$$\mathcal{F} * \mathcal{G} := m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

One can then check that this construction restricts to a valid monoidal structure on $P_{\mathscr{S}}(\mathcal{B})$ with multiplicative unit the IC sheaf IC_e associated to the identity element $e \in W$, and an essentially identical argument gives the Satake category Sat_G a similar structure. Convolution interacts nicely with IC sheaves, and one has for instance that if $w = s_1 ... s_r$ in reduced form, then $\mathrm{IC}_{s_1} * ... * \mathrm{IC}_{s_r}$ is a semisimple complex, and IC_w appears in the direct sum decomposition of $\mathrm{IC}_{s_1} * ... * \mathrm{IC}_{s_r}$ with multiplicity 1.

Finally, we conclude these notes by mentioning remarkable connections to Hecke algebras and to the classical Satake equivalence. In the settings of these notes, upon taking the Grothendieck ring of $P_{\mathscr{S}}(\mathcal{B})^{ss}$ (only allowing for semisimple objects) with respect to its monoidal structure, one recovers the Hecke algebra $\mathcal{H}$ associated to the Coxeter group (W, S) (as defined for instance in chapter 7 of [3]):

$$K_0(P_{\mathscr{S}}(\mathcal{B})^{ss}, *) \simeq \mathcal{H}.$$

An analogous result occurs in the case of the Satake category $\mathrm{Sat}_G = P_{G(\mathcal{O})}(\mathcal{G}r_G)$, where if we consider the enlargement in which shifts are allowed, the corresponding Grothendieck ring produces the **spherical Hecke algebra** $\mathcal{H}_{sph}$. On the other hand, one can show that the Grothendieck ring of $\mathrm{Rep}(G^\vee)$ is isomorphic to the W-fixed points on the group ring $\mathbb{Z}[X_*(T)]$, where the action of W on $\mathbb{Z}[X_*(T)]$ is induced from the action on the coweight lattice of G introduced in 4.5.

Allowing for shifts on both sides, the geometric Satake equivalence therefore decagorifies under K_0 to the classical Satake equivalence:

$$\mathcal{H}_{sph} \simeq \mathbb{Z}[q^{\pm 1/2}][X_*(T)]^W.$$

For a definition of the spherical Hecke algebra and more information about this equivalence, the reader may consult chapter 9 of [1].

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