TOPOLOGICAL K-THEORY AND SOME OF ITS APPLICATIONS

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Abstract. This thesis is concerned with the development of topological K-theory, with a focus on the complex case. Following a treatment of the classical theory of principal bundles and classifying spaces and its connection to the study of vector bundles, we introduce the complex K-group and its extension to a generalized cohomology theory. We then proceed to construct the Chern character and the Adams operations, two important computational tools for this cohomology theory. We conclude by proving two classical theorems in topology using the language of topological K-theory: the non-existence of complex structures on S^{2n} for $n > 3$, and the resolution of the Hopf invariant one problem.

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Introduction

Inspired by the establishment of the Bott periodicity theorem in 1956, Michael Atiyah and Friedrich Hierzebruch undertook the investigation of topological K-theory as one of the first instances of a generalized cohomology theory, mirroring the properties of singular cohomology in the exotic settings of vector bundles. This development served as a catalyst in the exploration of other kinds of generalized cohomology theories, a project which was later shown to have close ties to the world of stable homotopy theory via the Brown representability theorem published by Edgar Brown in 1962 [7]. Moreover, topological K-theory quickly proved to be of great importance in tackling difficult problems in classical topology, offering new insights or drastic simplifications of existing proofs. We shall conclude the thesis with two of these success stories, after a careful development of topological K-theory as a generalized cohomology theory and of some of its important tools. Throughout this paper, we choose to focus on complex topological K-theory rather than its real counterpart, chiefly due to the nature of the applications we have in mind in Chapter 4. However, much of the elementary theory goes through nearly identically in the real case, and we make an effort to mention the analogous statements where relevant.

Chapter 1 begins with a general treatment of the elementary theory of principal G-bundles for any topological group G, leading up to the homotopy classification theorem. We then specialize to complex vector bundles via the equivalence of categories given by the Borel construction, and recover the traditional homotopy classification of rank n complex vector bundles. As the latter objects form the backbone of topological complex K-theory, we take some time to elaborate on the role of the tautological n-plane bundle over the infinite complex Grassmanian $\mathrm{Gr}_n(\mathbb{C}^\infty)$ as universal rank n complex vector bundles in Section 1.3, and devote Section 1.4 to describing a receipe for patching up standard operations at the level of vector spaces into operations on vector bundles.

Chapter 2 takes the above preliminaries as a starting point to introduce complex topological K-theory. We start with the definition of the complex K-group functor $K^0(-)$ and its reduced analog for $\tilde{K}^0(-)$ pointed spaces, and exploit the homotopy classification theorem for complex vector bundles to obtain a representability result for these functors. Next, we introduce the Bott periodicity theorem in Section 2.2 and use it to extend $\tilde{K}^0(-)$ to a generalized reduced cohomology theory in Section 2.4, after exploring the correspondence between generalized reduced cohomology theory and Ω-spectra afforded by the Brown representability theorem in Section 2.3.

Chapter 3 introduces some important tools in topological K-theory, namely the Chern character and the Adams operations. Both of these operations will be seen to extend uniquely from the prescription of a desired behavior on line bundles, together with requirements of naturality and additivity. The Adams operations on topological K-theory may be thought of as the analogs of the Steenrod square operations in singular cohomology with $\mathbb{Z}/2$ coefficients, allowing us to interpret complex K-groups as modules over an appropriate ring. The Chern character is built from Chern classes and provides a bridge between topological K-theory and singular cohomology in the form of a natural transformation $Ch : K^0(-) \to H^{ev}(-;\mathbb{Q})$ compatible with the ring structure of both sides.

Finally, Chapter 4 illustrates some of the uses that complex K-theory has found by drawing from the language and machinery introduced in the previous chapters to resolve two problems in classical topology: the non-existence of complex structures on S^{2n} for $n > 3$, and the non-existence of maps $f : S^{2n-1} \to S^n$ of Hopf invariant one for $n \neq 1, 2, 4, 8$. The latter problem has the interesting consequence that $\mathbb{R}^n$ does not admit the structure of a division algebra for n falling outside of that range. The treatment we shall give, inspired by Dugger [9], also has the advantage of indicating the relevance of the Hopf invariant one problem to understanding the stable homotopy groups of spheres - a direction first pursued by Adams [2] in 1963.

Topological K-theory was initially investigated in the late 1950s and early 1960s by Michael Atiyah and Friedrich Hierzebruch, alongside the development of cobordism theories as generalized cohomology theories. The complex versions of these two disciplines, complex topological K-theory and complex cobordism, have the additional feature of being *complex-oriented cohomology theories*, in the sense that they are multiplicative cohomology theories admitting a notion of Chern classes mirorring the usual notion in singular cohomology.

In the words of Jack Morava, “ordinary cohomology lies on a stratum of infinite codimension in the moduli space of complex-oriented theories” [19]. The subject of chromatic homotopy theory organizes this stratum into a so-called ”chromatic filtration” of complex-oriented theories, indexed by the non-negative integers. Singular cohomology occupies level zero of that filtration, followed by complex topological K-theory at level $n = 1$. Complex cobordism theory is obtained in the limit as n tends to infinity. The investigation of the intermediary landscape remains an area of active research, involving a fascinating interplay between ideas and methods from algebraic topology and algebraic geometry. It is the hope of the author that this paper, possibly complemented by the introduction to cobordism theory given in [21], will provide an accessible account of the classical framework and thereby assist the interested reader in learning about contemporary areas of research in algebraic topology.

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1. Principal G-Bundles and Classifying Spaces

1.1. **First Principles.**

In this section, we give a streamlined treatment of the theory of principal G-bundles and classifying spaces. A more comprehensive treatment of the material of this section may be found in Mitchell [18]. Vector bundles are a special class of such objects which play a central role in topological K-theory, and so later in this section we specialize our attention to these structures and provide a brief overview of standard constructions. For a classical treatment of the theory of vector bundles, see Atiyah [5] or the more recent text by Hatcher [11].

First, we lay down some preliminary notions. Let G be a topological group. Recall that a map $f : X \to Y$ between two right G-spaces is said to be *G-equivariant*, or a *G-map*, if it is compatible with the respective group actions, in the sense that $f(xg) = f(x)g$ for all $x \in X, g \in G$. Note that any space X can be viewed as a *trivial G-space* under the action $xg = x$ for all $x \in X$, $g \in G$; then, any G-map $f : Z \to X$ into a trivial G-space X has the property that the action of G on the fibers of f is well-defined.

Definition 1.1. A *principal G-bundle* $\xi : P \xrightarrow{\pi} X$ consists of a G-map $\pi : P \to X$ between a right G-space P and a trivial G-space X, together with with an open cover $X = \bigcup_\alpha U_\alpha$ such that for each α, U_α fits into a commutative triangle:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times G \\ & \searrow^{\pi} \quad \swarrow_{\pi_1} & \\ & U_\alpha & \end{array}$$

where $U_\alpha \times G$ is viewed as a right G-space under the action $(x, g)h := (x, gh)$, $\pi_1 : U_\alpha \to G$ is the projection map onto the first factor, and $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a G-equivariant homeomorphism. We call P the *total space* and X the *base space* of the bundle ξ. The open subsets U_α are called *local trivializations* of ξ.

Remark 1.2. (i) Given a principal G-bundle $\xi : P \xrightarrow{\pi} X$, for each $x \in X$, we observe the induced action of G on the fiber $\pi^{-1}(x)$ is free and transitive. This follows by direct computation from the fact that, letting U_α be a trivializing open containing x, the associated map $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ restricts to a homeomorphism $\varphi_\alpha|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times G \cong G$.
(ii) As noted above, $\pi : P \to X$ is constant on G-orbits, hence by the universal property of quotients π factors uniquely through a map $\tilde{\pi} : P/G \to X$. The fact that the action of G on P is free implies that $\tilde{\pi}$ is in fact a homeomorphism, hence X can be interpreted as the orbit space of P.

Examples 1.3. Principal G-bundles for some topological group G are a common phenomenon in topology. We give a few examples:

- The group of deck transformations of the universal cover $\tilde{X} \xrightarrow{\pi} X$ of a connected space is naturally isomorphic to the fundamental group $\pi_1(X)$, so that $\tilde{X} \xrightarrow{\pi} X$ has the structure of a principal $\pi_1(X)$-bundle.

- The Hopf fibration $\eta : S^3 \xrightarrow{\pi} S^2$ obtained via restriction of the natural projection map $\mathbb{C}^2 \to \mathbb{C}P^1 \cong S^2$ to the unit 3-sphere $S^3 \subset \mathbb{C}^2$ is a principal S^1-bundle, with action on the fibers given by phase multiplication, and local trivializations given by the opens $S^2 - \{N\}, S^2 - \{S\}$ obtained from S^2 by removing the North, resp. South pole.
- Given any Lie group G and a closed subgroup $H \leq G$, the projection map $G \xrightarrow{\pi} G/H$ has the structure of a principal H-bundle, with action on G given by right multiplication.

Principal G-bundles naturally fit into a category where a morphism between two principal G-bundles $\xi : P \xrightarrow{p} X$ and $\eta : Q \xrightarrow{p'} Y'$ consists of a G-map $g : P \to Q$ and a continuous map $f : X \to Y'$ such that the associated square commutes:

$$\begin{array}{ccc} P & \xrightarrow{g} & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

When considering a morphism of bundles over the same base space X, we shall assume unless otherwise specified that the component map $X \to X$ is the identity map. Next, we introduce a key construction on principal G-bundles. We assume the reader is familiar with the categorical notion of pullbacks.

Definition 1.4. Let $\xi : P \xrightarrow{\pi} X$ be a principal G-bundle. Given a continuous map $f : Y \to X$, we define the *pullback of* ξ *under* f to be the map $f^*\xi : f^*P \xrightarrow{\pi'} Y$ obtained from the pullback diagram in the category **Top** of topological spaces induced by the maps π and f.

Hence the pullback bundle $f^*\xi : f^*P \xrightarrow{\pi'} Y$ fits into a commutative square:

$$\begin{array}{ccc} f^*P & \longrightarrow & P \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X. \end{array}$$

Concretely, the total space f^*P is given by the set:

$$f^*P = \{(y, a) \in Y \times P \mid f(y) = p(a)\},$$

and $f^*P \xrightarrow{\pi'} Y$ is the projection map onto the first factor. It can be verified that $f^*\xi : f^*P \xrightarrow{\pi'} Y$ has the structure of a principal G-bundle, with local trivializations provided by open sets of the form $f^{-1}(U)$ where U is a local trivialization for the bundle $\xi : P \xrightarrow{\pi} X$. In particular, if $f : A \to X$ is an inclusion and $\xi : P \xrightarrow{\pi} X$ is a principal G-bundle over X, then $f^*\xi : f^*P \xrightarrow{\pi'} A$ is simply the restriction of ξ to the subspace A, which we denote by $\xi|_A$.

We thus have a way of associating to a space X the set $\mathcal{P}_G(X)$ of isomorphism classes of principal G-bundles over X, and to a map $f : X \to Y$ the induced pullback assignment $f^* : \mathcal{P}_G(Y) \to \mathcal{P}_G(X)$ at the level of bundles. Playing around with universal properties readily yields that this assignment gives a contravariant functor $\mathcal{P}_G(-) : \mathbf{Top}^{op} \to \mathbf{Set}$, i.e. we have that $(id_X)^* = id_{\mathcal{P}_G(X)}$ for any space

X, and $(fg)^* = g^*f^* : \mathcal{P}_G(Z) \to \mathcal{P}_G(X)$ for any composite of continuous maps $X \xrightarrow{g} Y \xrightarrow{f} Z$.

1.2. **A Homotopy Classification Theorem.** Next, we prove that the functor $\mathcal{P}_G(-)$ that we have just defined is homotopy invariant. The following argument follows closely the one given in [section 1.2] of Hatcher [11].

Proposition 1.5. *Given a principal G-bundle $\xi : P \xrightarrow{p} Y$ and homotopic maps $f \simeq g : X \to Y$ from a paracompact space X, the associated pullback bundles f^*P, g^*P over X are isomorphic.*

Proof. Let $h : X \times I \to Y$ be the provided homotopy between f and g, so that $f = hi_0$ and $g = hi_1$, where $i_k : X \times \{k\} \hookrightarrow X \times I$, $k = 0, 1$ denote the canonical inclusions. Consider the pullback of ξ under h:

$$\begin{array}{ccc} h^*E & \longrightarrow & E \\ \downarrow & & \downarrow \xi \\ X \times I & \xrightarrow{h} & Y. \end{array}$$

By functoriality, we have that

$$f^*E = (hi_0)^*E \cong i_0^*(h^*E),$$

hence f^*E is isomorphic to the restriction of h^*E over $X \times \{0\}$. Similarly, we see that g^*E is isomorphic to $(h^*E)|_{X\times\{1\}}$. Hence to obtain the desired result it suffices to show that given a principal G-bundle $\xi : E \to Y$ and a continuous map $h : X \times I \to Y$, the bundles over X obtained by restricting h^*E to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.

Start with an arbitrary $x \in X$, and find a family of local trivializations $\{U_t \times V_t\}_{t \in I}$, where each $U_t \times V_t$ is taken to be a basic open set of $X \times I$ containing the point (x, t), and where we may further assume that each $V_t \subset I$ is an open interval. By compactness of I, the open cover $\{V_t\}_{t\in I}$ of I admits a finite subcover $\{V_{t_1}, ..., V_{t_n}\}$. We may assume that this cover is "linearly ordered", meaning that if we write $V_{t_i} = (a_i, b_i)$ and $V_{t_{i+1}} = (a_{i+1}, b_{i+1})$, then $a_i < a_{i+1}$ and $b_i < b_{i+1}$. Now, set $U_x := \bigcap_{i=1}^n U_{t_i}$, and observe that we may inductively exploit the local trivializations above $U_x \times V_{t_i}$, $U_x \times V_{t_{i+1}}$ to obtain a local trivialization above $U_x \times (V_{t_i} \cup V_{t_{i+1}})$. In this way, we obtain a local trivialization above $U_x \times I$.

Hence in particular we obtain an open cover $\{U_x\}_{x\in X}$ of X. We use the fact that if X is a paracompact space, then every open cover $\{U_x\}$ of X admits an associated countable open cover $\{V_i\}_{i\geq 1}$ equipped with a partition of unity, with the property that each V_i is a disjoint union of opens contained in U_α's. So let $\{V_i\}_{i\geq 1}$ be as above, with associated partition of unity $\{\varphi_i\}$. The local trivializations over each $U_x \times I$ clearly induce local trivializations over each $V_i \times I$.

Now, define functions $\psi_n : X \to I$ as follows: set $\psi_0 := 1$, and inductively set $\psi_n := \psi_{n-1} - \varphi_n$ for $n \geq 1$. By construction, we have a well-defined function $1 - \Sigma_{i\geq 1}\varphi_i = 0$. For each $n \geq 0$, we may consider the graph of ψ_n, given by:

$$\Gamma_{\psi_n} = \{(x, \psi_n(x)) \mid x \in X\} \subset X \times I.$$

For each n, taking pullbacks of the bundle $h^*E \to X \times I$ along the inclusion $l_n : \Gamma_{\psi_n} \hookrightarrow X \times I$ yields corresponding restricted bundles $E_n := l_n^*(h^*E)$ over Γ_{ψ_n}. Our plan is to iteratively exhibit bundle isomorphisms $f_n : E_n \to E_{n+1}$, whose composite will provide us with a bundle isomorphism from $E_0 = (h^*E)|_{X\times\{1\}}$ to the limiting term (as n tends to infinity) $(h^*E)|_{X\times\{0\}}$.

By construction, for a given n, we have that $\text{supp}\varphi_{n+1} \subset V_{n+1}$, so that the graphs of ψ_n and ψ_{n+1} differ at most along the open subset $V_n \times I$ of $X \times I$. Hence we may define a map $f : E_n \to E_{n+1}$ to act as the identity above $(B - V_n) \times I$, and using the existing local trivialization above $V_n \times I$, we may define f_n above $V_n \times I$ to be given by:

$$f_n : (x, \psi_n(x), g) \mapsto (x, \psi_n(x) - \varphi_{n+1}(x), g) = (x, \psi_{n+1}(x), g).$$

Each $f_n : E_n \to E_{n+1}$ is then a G-equivariant, fiber-preserving homeomorphism, hence a bundle isomorphism, and local finiteness of the family $\{\varphi_i\}$ ensures that the infinite composite $f := \prod_{n\geq 0} f_n$ is a well-defined bundle isomorphism from $h^*E|_{X\times\{1\}}$ and $h^*E|_{X\times\{0\}}$, completing the proof. ☐

Thus we obtain a well-defined contravariant functor:

$$\mathcal{P}_G : \text{Ho}(\mathbf{Top})^{op} \to \mathbf{Set}$$

where Ho($\mathbf{Top}$) denotes the homotopy category of spaces and homotopy classes of continuous maps. It is defined as above by sending a space X to the set $\mathcal{P}_G(X)$ of isomorphism classes of principal G-bundles over X.

Let us say a word about functors of this kind in the realm of based spaces. Say that a functor $F : \text{Ho}(\mathbf{Top}_*)^{op} \to \mathbf{Set}_*$ (where $\mathbf{Set}_*$ denotes the category of pointed sets and based set maps) satisfies the *wedge axiom* if it sends coproducts to products, in the sense that given a wedge of spaces $\bigvee_i X_i$, the map $F(\bigvee_i X_i) \to \prod_i F(X_i)$, $x \mapsto (F(l_i)(x))_i$ induced by the natural inclusion maps $l_i : X_i \to \bigvee_i X_i$ is an isomorphism. Next, say that F satisfies the *Mayer Vietoris axiom* if whenever a space X can be expressed as a union of opens $X = U \cup V$, the square induced by the natural inclusion maps

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

has the property that whenever two elements $x \in F(U), y \in F(V)$ restrict to the same element in $F(U \cap V)$, there exists a $z \in F(X)$ such that $z|_U = x$ and $z|_V = y$. The following remarkable result was established by Brown [7] in his study of cohomology theories. A modern treatment of the proof may be found in Chapter 3 of Kochman [12].

Theorem 1.6. ***(Brown Representability Theorem)*** *Let F : Ho($\mathbf{Top}_*)^{op} \to$ $\mathbf{Set}_*$ be a contravariant functor on connected based spaces which satisfies the wedge axiom and the Mayer-Vietoris axiom. Then F is representable. That is, there exists a connected based space L and an element $u \in F(L)$ such that the natural transformation $T_u : [-, L] \to F(-)$ given by $T_u(g) := F(g)(u) \in F(X)$ for any $[g] \in [X, L]$ is a natural isomorphism.*

Remark 1.7. By the Yoneda lemma, any two connected based spaces L, L' together with classes $u \in F(L), u' \in F(L')$ with the property that $T_u : [-, L] \to F(-), T_{u'} : [-, L'] \to F(-)$ are natural isomorphisms must be homotopy equivalent.

Now, the functor $\mathcal{P}_G$ satisfies the Mayer-Vietoris axiom and the unpointed version of the wedge axiom (coproducts are disjoint unions), and a sleight of hand makes it possible to infer representability of $\mathcal{P}_G$ from the above statement. To say that $\mathcal{P}_G$ is representable amounts to saying that there exists a connected space BG and an associated principal G-bundle $EG \to BG$ such that for every space X, the set $[X, BG]$ of homotopy classes of maps from X into BG is in a natural bijection with the elements of $\mathcal{P}_G(X)$, with correspondence given by sending a map $f : X \to BG$ to the principal G-bundle $f^*EG \to X$ over X obtained by pullback of the universal principal G-bundle $EG \to BG$. In that case, we call BG a *classifying space* for principal G-bundles and refer to $EG \to BG$ as the associated *universal principal G-bundle*.

We state the classification theorem in its traditional form:

Theorem 1.8. ***(Classification Theorem for Principal G-Bundles)*** *Let G be a topological group. Then there exists a classifying space BG for principal G-bundles, unique up to homotopy equivalence, such that for every space X, there is a natural bijection:*

$$[X, BG] \cong \mathcal{P}_G(X),$$

obtained by sending a map $f : X \to BG$ to the pullback under f of a fixed universal bundle $EG \to BG$.

We relegate to Appendix A the general construction of a universal principal G-bundle $EG \to BG$ for any topological group, and spend the next section discussing the cases $G = U(n)$ and $G = O(n)$, which are of central importance in complex, resp. real topological K-theory. To close our discussion of the general case, we note that the verification that a given principal G-bundle is universal is greatly simplified by the following theorem, whose proof may be found in Mitchell [18]:

Theorem 1.9. *Let $\xi : E \xrightarrow{p} B$ be a principal G-bundle whose total space E is contractible. Then ξ is a universal principal G-bundle and B is a model for the classifying space BG.*

This theorem also informs us about the homotopy type of BG, which is well-defined by uniqueness as mentioned in Remark 1.7. Since exhibiting a universal principal G-bundle as in Theorem 1.9 is always possible (as we will see in Appendix A), we can consider the fibration $G \to EG \to BG$, where EG is weakly contractible, so that the associated homotopy LES yields an isomorphism for all n:

$$\pi_{n+1}(BG) \cong \pi_n(G). \tag{1.10}$$

1.3. **Specialization to Vector Bundles.** Of particular interest to us are *vector bundles* and their corresponding classifying spaces, both in the real and complex case. We start by specializing to principal $U(n)$-bundles and describe their close ties with complex vector bundles of rank n. A completely analogous argument can be made to establish a correspondence between principal $O(n)$-bundles and real vector bundles of rank n.

Let $V_n^o(\mathbb{C}^\infty)$ be the *Stiefel manifold* of orthonormal n-frames in $\mathbb{C}^\infty$, and let $\mathrm{Gr}_n(\mathbb{C}^\infty)$ be the *Grassmanian manifold* of complex n-planes in $\mathbb{C}^\infty$. These spaces may be topologized as direct limits $\varinjlim_k V_n^o(\mathbb{C}^{n+k})$, resp. $\varinjlim_k \mathrm{Gr}_n(\mathbb{C}^{n+k})$ taken under inclusions, each of whose constituent spaces is readily seen to be compact Hausdorff, the latter as the image of the former under the canonical projection map sending an orthonormal n-frame to the n-plane it spans. The same projection map on the direct limits:

$$\pi : V_n^o(\mathbb{C}^\infty) \to \mathrm{Gr}_n(\mathbb{C}^\infty)$$

may be verified to have the structure of a principal $U(n)$-bundle ξ, with the $U(n)$ action on the fibers given by matrix multiplication. We claim that ξ is a universal principal $U(n)$-bundle.

We may then prove the following:

Proposition 1.11. *The Stiefel manifold $V_n^o(\mathbb{C}^\infty)$ is contractible, hence $\xi : V_n^o(\mathbb{C}^\infty) \xrightarrow{\pi} Gr_n(\mathbb{C}^\infty)$ is a universal principal $U(n)$-bundle and $Gr_n(\mathbb{C}^\infty)$ is a model for $BU(n)$.*

Proof. Denote by $\{e_i\}_{i\geq 1}$ the standard basis for $\mathbb{C}^\infty$. We construct a map $f : V_n^o(\mathbb{C}^\infty) \to V_n^o(\mathbb{C}^\infty)$ such that $id_{V_n^o(\mathbb{C}^\infty)} \simeq f \simeq c_x$, where $x := (\frac{1}{\sqrt{n}}e_1, ..., \frac{1}{\sqrt{n}}e_n)$ and c_x is the constant map at $x \in V_n^o(\mathbb{C}^\infty)$. Given an arbitrary element $v \in \mathbb{C}^\infty - \{0\}$, we can always find a finite number corresponding to the last non-zero entry in the expression of v as a linear combination of the e_i's. Denote this assignment by $\sigma : \mathbb{C}^\infty - \{0\} \to \mathbb{N}$. Next, let $T : V_n^o(\mathbb{C}^\infty) \to V_n^o(\mathbb{C}^\infty)$ be the linear operator sending $(v_1, ..., v_n)$ to $(v_1', ..., v_n')$, where each v_i' is obtained from v_i by shifting all entries in the basis expression of v_i by one unit to the right. Finally, define the map $f : V_n(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)$ via

$$f(v_1, ..., v_n) := T^{\max\{\sigma(v_1),...,\sigma(v_n)\}}(v_1, ..., v_n),$$

where T^k denotes the k-fold iteration of T. By construction, we may then use straight line homotopies normalized to live in $S^\infty(\sqrt{n})$ to get a homotopy $id_{V_n^o(\mathbb{C}^\infty)} \simeq f \simeq c_x$, as needed. Thus $V_n^o(\mathbb{C}^\infty)$ is contractible, and the second part of the proposition follows from Theorem 1.9. $\square$

Now, recall that a *complex vector bundle* $\xi : E \xrightarrow{\pi} X$ of rank n over a space X consists of a continuous map $\pi : E \to X$ such that the fiber over each point has the structure of an n-dimensional complex vector space, together with an open cover $\{U_\alpha\}$ of X admitting fiber-preserving homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ which restrict to linear isomorphisms on the fibers of π. There exists a general process known as the *Borel construction* which specializes to an equivalence of categories between principal $U(n)$-bundles and rank n complex vector bundles over a given base space. The Borel construction takes a principal $U(n)$-bundle $E \to X$ to an associated rank n vector bundle $E \times_{U(n)} \mathbb{C}^n \to X$, with total space given by the quotient:

$$E \times_{U(n)} \mathbb{C}^n := E \times \mathbb{C}^n / (xA, y) \sim (x, Ay),$$

ranging over all $x \in E, y \in \mathbb{C}^n, A \in U(n)$.

Since classifying spaces and universal bundles were defined categorically, they are preserved under this equivalence of categories. By investigating the structure of the balanced product $\gamma_n^\infty := V_n^o(\mathbb{C}^\infty) \times_{U(n)} \mathbb{C}^n$, we thus find that a model for the universal rank n-vector bundle is given by the *tautological vector bundle* $\gamma_n^\infty : EU(n) \to BU(n)$, where

$$EU(n) = \{(p, v) \in \mathrm{Gr}_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty \mid v \in p\}.$$

Remark 1.12. A completely analogous treatment can be given for real vector bundles, where the spaces $V_n^o(\mathbb{R}^\infty)$, resp. $\mathrm{Gr}_n(\mathbb{R}^\infty)$ are obtained via direct limit constructions as above. In these settings, the canonical projection map $\xi : V_n^o(\mathbb{R}^\infty) \xrightarrow{\pi} \mathrm{Gr}_n(\mathbb{R}^\infty)$ has the structure of a principal $O(n)$-bundle with respect to matrix multiplication on the fibers. Furthermore, the Stiefel manifold $V_n^o(\mathbb{R}^\infty)$ is contractible, hence $\xi : V_n^o(\mathbb{R}^\infty) \xrightarrow{p} \mathrm{Gr}_n(\mathbb{R}^\infty)$ is a universal principal $O(n)$-bundle and $\mathrm{Gr}_n(\mathbb{R}^\infty)$ is a model for $BO(n)$. Under the Borel construction, ξ corresponds to the universal real vector bundle $\gamma_n^\infty : EO(n) \to BO(n)$ of rank n, where

$$EO(n) := \{(p, v) \in \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty \mid v \in p\}.$$

1.4. **Standard Constructions on Vector Bundles.** We next present a general framework for exploiting the structure of known vector bundles over a fixed base space in order to create new ones. The following construction boils down to one key idea, which is to give new objects the *final topology* with respect to the appropriate collection of maps so as to obtain a vector bundle structure. The following discussion is inspired from Section (3.f) of Milnor [16]. We present the constructions for complex vector bundles, but the same arguments apply in the case of real vector bundles.

Denote by Vect_f^{iso} the category of finite dimensional complex vector spaces and linear isomorphisms. Observe that each morphism set in this category can be given the structure of a topological space as a subspace of the corresponding space of linear operators. Composition of linear isomorphisms is continuous with respect to this topology, hence we may view Vect_f^{iso} as a category enriched over **Top**. The same observation applies to the k-fold product category $(\mathrm{Vect}_f^{iso})^k$, whose morphism sets can be given the product topology. Say that a functor $T : (\mathrm{Vect}_f^{iso})^k \to \mathrm{Vect}_f^{iso}$ is *continuous* if the induced maps on morphism sets:

$$(Vect_f^{iso})(V_1, V_1') \times ... \times (Vect_f^{iso})(V_k, V_k') \xrightarrow{T} (Vect_f^{iso})(T(V_1, ..., V_k), T(V_1', ..., V_k'))$$

are all continuous.

Construction 1.13. Fix a continuous functor $T : (\mathrm{Vect}_f^{iso})^k \to \mathrm{Vect}_f^{iso}$ on k variables, and let $\xi_1, ..., \xi_k$ be given vector bundles on a connected base space X, so that each ξ_i has a well defined rank n_i. We construct a new vector bundle $T(\xi_1, ..., \xi_k)$ of rank $\dim_{\mathbb{C}} T(\mathbb{C}^{n_1}, ..., \mathbb{C}^{n_k})$ over X as follows: for each $x \in X$, define

$$F_x(\xi_1, ..., \xi_k) := T(F_x(\xi_1), ..., F_x(\xi_k)),$$

where $F_x(\xi_i)$ denotes the fiber of ξ_i over x. Then, let

$$E(\xi_1, ..., \xi_k) := \coprod_{x \in X} F_x(\xi_1, ..., \xi_k),$$

and define $\pi : E(\xi_1, ..., \xi_k) \to X$ to be the canonical projection map sending $F_x(\xi_1, ..., \xi_k)$ to x.

For a given $x \in X$, we can find local trivializations $U_1, ..., U_k$ about x corresponding to $\xi_1, ..., \xi_k$, respectively. Set $U := \cap_{i=1}^k U_i$. Now, the assignment $v \mapsto (x, v)$ provides a linear isomorphism $h_{i,x} : \mathbb{C}^{n_i} \to F_x(\xi_i)$ for each $i = 1, ..., k$. Taken as a k-tuple, the $h_{i,x}$'s are taken by T to a linear isomorphism

$$T(h_{1,x}, ..., h_{k,x}) : T(\mathbb{C}^{n_1}, ..., \mathbb{C}^{n_k}) \xrightarrow{\sim} F_x(\xi_1, ..., \xi_k).$$

Letting x range over U, the resulting maps can then be used to obtain a bijection:

$$\tilde{h} : U \times T(\mathbb{C}^{n_1}, ..., \mathbb{C}^{n_k}) \to \pi^{-1}(U)$$

via the assignment $(x, v) \mapsto T(h_{1,x}, ..., h_{k,x})(v)$. Composing each such $\tilde{h}$ with the inclusion $\pi^{-1}(U) \hookrightarrow E(\xi_1, ..., \xi_k)$ yields a family of maps:

$$h_U : U \times T(\mathbb{C}^{n_1}, ..., \mathbb{C}^{n_k}) \to E(\xi_1, ..., \xi_k),$$

where the left hand side can be given the product topology.

Now, give the set $E(\xi_1, ..., \xi_k)$ the final topology induced by the maps h_U as above. This is defined by the universal property that a map $f : E(\xi_1, ..., \xi_k) \to Z$ is said to be continuous if and only if each composite fh_U is continuous, and it is the finest topology on $E(\xi_1, ..., \xi_k)$ making all the maps h_U continuous. In particular, under this topology, each of the maps $\tilde{h}$ described above become fiber-preserving homeomorphisms restricting to linear isomorphisms on the fibers by construction. Thus $E(\xi_1, ..., \xi_k)$ is a valid total space for the vector bundle

$$T(\xi_1, ..., \xi_k) : E(\xi_1, ..., \xi_k) \xrightarrow{\pi} X$$

over X with fibers $F_b(\xi_1, ..., \xi_k)$ and local trivializations provided by the opens U and the maps $\tilde{h}$ as defined above.

Example 1.14. To give a sense of the usefulness of this construction, we list a few important examples of new vector bundles that can be obtained from existing ones by following the above recipe, all of which will be used at some later point. In the following, let ξ, resp. η denote fixed complex vector bundles over a base space X with rank n, resp. m.

- The *direct sum*, or *Whitney sum* of bundles $\xi \oplus \eta$, induced by the assignment $(V, W) \mapsto V \oplus W$, with rk $\xi \oplus \eta = n + m$;
- The *tensor product* of bundles $\xi \otimes \eta$, induced by the assignment $(V, W) \mapsto V \otimes W$, with rk $\xi \otimes \eta = n \cdot m$;
- The *hom bundle* $\mathrm{Hom}(\xi, \eta)$, induced by the assignment $(V, W) \mapsto \mathrm{Hom}_{\mathbb{C}}(V, W)$, with rk $\mathrm{Hom}(\xi, \eta) = n \cdot m$. In particular, letting ϵ^1 denote the trivial line bundle over X, we may construct the dual bundle $\xi^* := \mathrm{Hom}(\xi, \epsilon^1)$ from the assignment $V \mapsto \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$;
- The k^{th} exterior power bundle $\bigwedge^k \xi$, induced by the assignment $V \mapsto \bigwedge^k V$, with rk $\bigwedge^k \xi = \binom{n}{k}$.

2. Defining Topological K-Theory

From now on, unless otherwise specified, we work with compact Hausdorff spaces and complex vector bundles over them. This chapter is devoted to the development of topological K-theory from the building blocks laid down in the previous chapter. Namely, we will associate to the functor $\mathrm{Vect}_{\mathbb{C}}(-) : \mathrm{Ho}(\mathbf{Top})^{op} \to \mathbf{Set}$ closely associated functors $K^0(-) : \mathrm{Ho}(\mathbf{Top})^{op} \to \mathbf{Ab}$ (resp. $\tilde{K}^0(-) : \mathrm{Ho}(\mathbf{Top}_*)^{op} \to \mathbf{Ab}$)

taking values in the category of abelian groups, and assigning to a space X its (reduced) complex K-group. Then, from the homotopy classification theorem applied to $\mathrm{Vect}_{\mathbb{C}}(-) \cong \mathcal{P}_{U(n)}$ together with an additional structural property enjoyed by the spaces $\{BU(n)\}$ as a result of the Bott periodicity theorem, we will be able to extend $\tilde{K}^0(-)$ to a collection of functors $\tilde{K}^n(-) : \mathrm{Ho}(\mathbf{Top}_*)^{op} \to \mathbf{Ab}$ fitting into a reduced generalized cohomology theory known as complex topological K-theory.

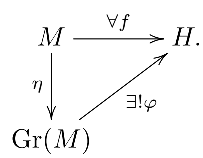
2.1. **Construction of $K^0(-), \tilde{K}^0(-)$.** We start our discussion by introducing a general procedure for associating abelian groups to abelian monoids:

Construction 2.1. (Grothendieck group) Let M be an abelian monoid. Define the *Grothendieck group* associated to M, denoted by $\mathrm{Gr}(M)$, to be the abelian group obtained as the quotient of the free abelian group on elements of M by the subgroup generated by relations of the form $[m] + [n] - [m+n]$. That is, we define:

$$\mathrm{Gr}(M) := \mathbb{Z} < [m] \mid m \in M > /([m] + [n] - [m+n], m, n \in M).$$

Observe that the quotient relation enables us to express any element of $\mathrm{Gr}(M)$ in the form $[m] - [n]$ for some $m, n \in M$, by gathering together elements of the same sign. This expression is not unique, but has the property that if $[m]-[n] = [r]-[s]$ then $[m+s] = [r+n]$.

The group $\mathrm{Gr}(M)$ comes equipped with a semigroup homomorphism $\eta : M \to \mathrm{Gr}(M)$ given by the natural inclusion $m \mapsto [m]$. In a precise sense, $\mathrm{Gr}(M)$ is the "most general" abelian group that admits such a map. Namely, it can be immediately verified that the pair $(\mathrm{Gr}(M), \eta : M \to \mathrm{Gr}(M))$ satisfies the following universal property: given any pair $(H, f : M \to H)$ where H is an abelian group and f a semigroup homomorphism, there exists a unique map $\varphi : \mathrm{Gr}(M) \to H$ making the following triangle commute:



This assignment can in fact be extended to a covariant functor

$$\mathrm{Gr}(-) : \mathbf{AbMon} \to \mathbf{Ab}$$

from the category of abelian monoids to the category of abelian groups, by sending a monoid homomorphism $\varphi\colon M \to N$ to the obvious group homomorphism $\tilde{\varphi}\colon \mathrm{Gr}(M) \to \mathrm{Gr}(N)$ given by $\tilde{\varphi}([m] - [n]) := [\varphi(m)] - [\varphi(n)]$, well defined by the comment made above. From there, we see that the data of the above diagram encodes the existence of an adjunction between the Grothendieck group functor and the forgetful functor $U : \mathbf{Ab} \to \mathbf{AbMon}$, in that we have a natural bijection, for any abelian monoid M and any abelian group H:

$$\mathbf{Ab}(\mathrm{Gr}(M), H) \cong \mathbf{AbMon}(M, UH).$$

Now, observe that for any space X, the set $\mathrm{Vect}_{\mathbb{C}}(X)$ equipped with the Whitney sum operation has the structure of an abelian monoid.

Definition 2.2. Given a space X, the *complex K-group of X*, denoted $K^0(X)$, is defined to be the Grothendieck completion of the abelian monoid $\mathrm{Vect}_{\mathbb{C}}(X)$:

$$K^0(X) := \mathrm{Gr}(\mathrm{Vect}_{\mathbb{C}}(X)).$$

Following the discussion in Construction 2.1, this assignment yields a contravariant functor:

$$K^0(-) : \mathrm{Ho}(\mathbf{Top})^{op} \to \mathbf{Ab}.$$

Remark 2.3. We call an element of $K^0(X)$ a *virtual bundle* over X. By the discussion in Construction 2.1, any virtual bundle admits an expression in the form $x - y$ for some $x, y \in \mathrm{Vect}_{\mathbb{C}}(X)$. We shall see in Corollary 2.7 that a little more is true. By convention, we sometimes denote the class of the trivial rank q bundle by $\epsilon^q \equiv q$, thereby obtaining a copy of $\mathbb{Z}$ inside each $K^0(X)$.

As a trivial first computation, since finite-dimensional vector spaces are uniquely characterized up to isomorphism by their dimension, we have over the one-point space that $\mathrm{Vect}_{\mathbb{C}}(\mathrm{pt}) \cong \mathbb{N}$, and hence that $K^0(\mathrm{pt}) = \mathrm{Gr}(\mathbb{N}) \cong \mathbb{Z}$. This leads us to the following:

Definition 2.4. Given a space X, the *reduced complex K-theory group of X*, denoted by $\tilde{K}^0(X)$, is defined to be the kernel of the map $l : \mathrm{pt} \hookrightarrow X$ induced by the inclusion of the basepoint:

$$\tilde{K}^0(X) := \ker(l^* : K^0(X) \to K^0(\mathrm{pt}) = \mathbb{Z}).$$

Remark 2.5. Note that by definition of pullbacks, the map $l^* : K^0(X) \to K^0(\mathrm{pt})$ sends a bundle ξ to the dimension of its fiber over the basepoint, i.e. the rank of the restriction of ξ to the connected component of the basepoint of X. In particular, if X is connected, $l^*(\xi)$ equals the rank of ξ.

Since $K^0(\mathrm{pt}) \cong \mathbb{Z}$, we always have a short exact sequence of abelian groups:

$$0 \longrightarrow \tilde{K}^0(X) \longrightarrow K^0(X) \xrightarrow{l^*} \mathbb{Z} \longrightarrow 0\ ,$$

and hence, since $\mathbb{Z}$ is projective, an isomorphism:

$$K^0(X) \cong \tilde{K}^0(X) \oplus \mathbb{Z}.$$

We next record a few useful facts about vector bundles, which are treated in detail in Section 9 of Dugger [9]:

Proposition 2.6. *Any short exact sequence of vector bundles splits, and any vector bundle $x \in \mathrm{Vect}_{\mathbb{C}}(X)$ may be expressed as a direct summand of a trivial bundle, i.e. $x \oplus y = \epsilon^q$ for some $y \in \mathrm{Vect}_{\mathbb{C}}(X), q \in \mathbb{N}$.*

In particular, we get the following convenient fact about virtual bundles (in the notation of Remark (2.3)):

Corollary 2.7. *Any virtual bundle $x - y \in K^0(X)$ may be expressed as $x - y = z - q$ for some $z \in \mathrm{Vect}_{\mathbb{C}}(X), q \in \mathbb{N}$.*

Proof. By Proposition 2.6 above, we can write $y \oplus w = \epsilon^q$ for some $w \in \mathrm{Vect}_{\mathbb{C}}(X), q \in \mathbb{N}$. Then, letting $z = x \oplus w \in \mathrm{Vect}_{\mathbb{C}}(X)$, we find that $x - y = x + w - \epsilon^q = z - \epsilon^q$ in $K^0(X)$, as needed. □

We now outline an argument showing that the functors $K^0(-)$ and $\tilde{K}^0(-)$ are representable, following the argument given in Chapter 24 of May [13]. Intuitively, from the natural isomorphisms $\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X, BU(n)]$ discussed in Chapter 1, we expect the representing space of $K^0(-)$ to be closely related to the colimit $BU := \varinjlim_n BU(n)$. Here, the direct system structure is induced from the system of inclusions $U(n) \hookrightarrow U(n+1)$ given by matrix inclusion $A \mapsto \mathrm{diag}(A,1)$, on which we apply the classifying space functor. Explicitly, each map $BU(n) \to BU(n+1)$ can be described as the classifying map associated to the rank $(n+1)$ bundle $\gamma_n^\infty \oplus \epsilon^1$ over $BU(n)$, where γ_n^∞ denotes the universal rank n bundle. Accounting for the Grothendieck group construction involved in defining $K^0(-)$, it turns out that the desired representing space is $BU \times \mathbb{Z}$.

Construction 2.8. Given a space X, let X_+ denote the based space obtained from X by adding a disjoint basepoint. We construct a map

$$g : K^0(X) \to [X_+, BU \times \mathbb{Z}],$$

as follows: given an element $x - y \in K^0(X)$, we can write $x - y = z - q$ for some $z \in \mathrm{Vect}_{\mathbb{C}}(X), q \in \mathbb{N}$ by Corollary 2.7 above. Let $f : X \to BU(n)$ be a classifying map for z. Then, composing f with the canonical inclusion $BU(n) \to BU$ to view it as a map $X \to BU$, we define $g(z - q) := (f, \mathrm{rk}z - q) : X_+ \to BU \times \mathbb{Z}$.

We readily check that g is well-defined: suppose $x - y$ admits different decompositions $x - y = z - q = w - r$ for some $z, w \in \mathrm{Vect}_{\mathbb{C}}(X), q, r \in \mathbb{N}$. Without loss of generality, assume $q \geq r$, so that we may write $z = w \oplus \epsilon^{q-r}$. Then it follows from the above description of the direct system structure on $\varinjlim_n BU(n)$ that the classifying maps of z and w viewed as maps $X \to BU$ are homotopic. Next, we may also write $z \oplus \epsilon^r = w \oplus \epsilon^q$, so that

$$\mathrm{rk}z + r = \mathrm{rk}(z \oplus \epsilon^r) = \mathrm{rk}(w \oplus \epsilon^q) = \mathrm{rk}w + q,$$

hence $g(z - q) = g(w - r)$, as needed.

Now, we claim that g gives a natural isomorphism $K^0(-) \cong [-, BU \times \mathbb{Z}]$, so that the complex K-group functor is representable by the space $BU \times \mathbb{Z}$.

Theorem 2.9. *The map $g : K^0(X) \to [X_+, BU \times \mathbb{Z}]$ defines a natural isomorphism*

$$K^0(X) \cong [X_+, BU \times \mathbb{Z}]$$

between the functors $K^0(-), [-_+, BU \times \mathbb{Z}] : Ho(\boldsymbol{Top})^{op} \to \boldsymbol{Ab}$.

Proof. We have already verified that the map g is well-defined. We check naturality: starting with a map of spaces $h : X \to Y$, we wish to show that the associated square

$$\begin{array}{ccc} K^0(Y) & \xrightarrow{h^*} & K^0(X) \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g} \\ [Y_+, BU \times \mathbb{Z}] & \xrightarrow{h^*} & [X_+, BU \times \mathbb{Z}] \end{array}$$

commutes. So let $\xi - q$ be an arbitrary element in $K^0(Y)$. Let $f : Y \to BU(n)$ be a classifying map for the bundle ξ over Y, so that $\xi = f^*\gamma_n^\infty$. Then by functoriality, we see that $(fh)^*(\gamma_n^\infty) = h^* f^* \gamma_n^\infty = h^*\xi$, hence fh is a valid classifying map for $h^*\xi$. We also have that the rank of a bundle is invariant under pullback, so that

$$g(h^*(\xi - q)) = (fh, \mathrm{rk}(h^*\xi) - q) = (fh, \mathrm{rk}\xi - q) = h^*(f, \mathrm{rk}\xi - q),$$

as needed.

Next, let $f : X_+ \to BU \times \mathbb{Z}$ be an arbitrary map given by the pair $(\tilde{f}, q)$. By compactness of X, the map $\tilde{f} : X \to BU$ factors through a map $X \to BU(n)$ for some $n \geq 0$, which corresponds under pullback to a rank n bundle ξ over X. Then, consider the virtual bundle $\xi - (n - q) \in K^0(X)$. Either $q \leq n$, in which case $g(\xi - (n-q)) = (\tilde{f}, \mathrm{rk}\xi - n + q) = (\tilde{f}, q)$ by construction, or $q \geq n$, so that $\xi - (n-q) = \xi \oplus \epsilon^{q-n}$. In this case, by a similar argument to the one given in Construction 2.8, the classifying map for $\xi \oplus \epsilon^{q-n}$ is homotopic to $\tilde{f}$ when viewed as a map $X \to BU$, hence $g(\xi-(n-q)) = (\tilde{f}, \mathrm{rk}(\xi \oplus \epsilon^{q-n})) = (\tilde{f}, q)$, proving surjectivity. Finally, suppose $z - q$ lies in the kernel of q, so that $g(z-q)$ is homotopic to the pair $(c_*, 0)$, where $c_* : X \to BU$ is the constant map on the basepoint. Then by construction, z is obtained by the pullback of some universal bundles γ_k^∞ by c_*, hence $z = \epsilon^k$ for some k, and we also have that $\mathrm{rk}z - q = 0$, so that $z = \epsilon^q$, thus $z - q = \epsilon^q - \epsilon^q = 0$, completing the proof that $g : K^0(X) \to [X_+, BU \times \mathbb{Z}]$ defines a natural isomorphism. □

It is then possible to formally deduce from the above natural isomorphism the associated result in the reduced case:

Theorem 2.10. *There is a natural isomorphism:*

$$\tilde{K}^0(X) \cong [X, BU \times \mathbb{Z}] \tag{2.11}$$

between the functors $\tilde{K}^0(-), [-, BU \times \mathbb{Z}] : Ho(\boldsymbol{Top}_*)^{op} \to \boldsymbol{Ab}$.

Proof. Given a based space X, we may add a new, disjoint basepoint to obtain the associated based space $X_+ = X \sqcup \{*\}$. Then, the based inclusion $l : S^0 \to X_+$ sending the second point of S^0 to the original basepoint of X is a cofibration, giving rise to a cofiber sequence:

$$S^0 \xrightarrow{l} X_+ \xrightarrow{j} X_+/S^0 \cong X.$$

Applying the functor $[-, BU \times \mathbb{Z}]$ to the associated Puppe sequence yields a LES:

$$\ldots \longrightarrow [\Sigma S^0, BU \times \mathbb{Z}] \longrightarrow [X, BU \times \mathbb{Z}] \xrightarrow{j^*} [X_+, BU \times \mathbb{Z}] \xrightarrow{l^*} [S^0, BU \times \mathbb{Z}],$$

where the leftmost term is $\pi_1(BU \times \mathbb{Z}) = \pi_1(BU) = \pi_0(U) = 0$ since U is connected, so that by exactness the map j^* is injective and $\ker l^* \cong [X, BU \times \mathbb{Z}]$. Now, under the identification $S^0 = \{\mathrm{pt}\}_+$ and by naturality of the isomorphism $K^0(X) \cong [X_+, BU \times \mathbb{Z}]$, the rightmost map of the above LES fits into a commutative square:

$$\begin{array}{ccc} [X_+, BU \times \mathbb{Z}] & \xrightarrow{l^*} & [S^0, BU \times \mathbb{Z}] \\ \simeq \downarrow & & \downarrow \simeq \\ K^0(X) & \xrightarrow{l^*} & K^0(pt), \end{array}$$

and hence Equation 2.11 follows by definition of $\tilde{K}^0(X) := \ker l^*$. □

[Interlude] (Connection to algebraic K-theory) The content of Proposition 2.6 suggests that the category of vector bundles over a compact Hausdorff space has properties analogous to the category of finitely generated projective modules over a commutative ring. Under consideration of the right objects, this impression can in fact be turned into a rigorous equivalence of categories. Indeed, given a vector bundle $E \xrightarrow{\pi} X$ we may look at the family of global sections, given by:

$$\Gamma(X, E) := \{s : X \to E \mid \pi s = \mathrm{id}_X\}.$$

This is a $\mathbb{C}$-vector space under pointwise addition and scalar multiplication, and it may furthermore be given the structure of a module over the ring $C(X)$ of continuous complex-valued functions on X via the assignment $(f \cdot s)(x) := f(x)s(x)$ for $f \in C(X), s \in \Gamma(X,E)$. It can be shown that $\Gamma(X,E)$ is always a finitely generated projective $C(X)$-module, and we have the following result:

Theorem 2.12. ***(Swan's theorem)*** *For any compact Hausdorff space X, the functor*

$$\Gamma : \mathit{Vect}_{\mathbb{C}}(X) \to \boldsymbol{P}_{fg}(C(X))$$

sending a vector bundle $E \xrightarrow{\pi} X$ to the $C(X)$-module of global sections $\Gamma(X,E)$ gives an equivalence of categories between $\mathit{Vect}_{\mathbb{C}}(X)$ and the category $\boldsymbol{P}_{fg}(C(X))$ of finitely-generated projective $C(X)$-modules.

The proof of this result consists of a series of straightforward checks, for which we refer the interested reader to Section 2.10 of Dugger [9]. For now, we simply observe that this equivalence of categories reinterprets $K^0(X)$ as the Grothendieck completion of the commutative monoid $\mathbf{P}_{fg}(C(X))$ under direct sums of modules, and thereby motivates the study of $\mathrm{Gr}(\mathbf{P}_{fg}(R))$ for arbitrary commutative rings. This question leads to the vast subject of algebraic K-theory, a treatment of which lies outside the scope of this paper. For an integrated treatment of these two viewpoints rich in examples, we refer the interested reader to Ravenel [20].

2.2. **Bott Periodicity.** We next introduce the Bott periodicity theorem, which will allow us to extend the K-group functor to a generalized cohomology theory. Before stating the theorem, we review some preliminaries on clutching functions associated to vector bundles over spheres.

Let $E \xrightarrow{\pi} S^n$ be a given vector bundle of rank k. Let D^n_+, resp. D^n_- denote small open neighborhoods of the northern, resp southern hemisphere of S^n, so that $D^n_+ \cap D^n_- \cong S^{n-1} \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Both D^n_+ and D^n_- are contractible, hence we can find trivializations of the restrictions $E|_{D^n_+}$, resp. $E|_{D^n_-}$, given by fiber-preserving homeomorphisms $\tilde{\varphi_+} \colon E|_{D^n_+} \to D^n_+ \times \mathbb{C}^k$, resp. $\tilde{\varphi_-} \colon E|_{D^n_-} \to D^n_- \times \mathbb{C}^k$ which restrict to linear isomorphisms on the fibers. These maps induce continuous maps $\varphi_+ \colon D^n_+ \to GL_k(\mathbb{C})$, resp. $\varphi_- \colon D^n_- \to GL_k(\mathbb{C})$ sending a given point x to the corresponding linear isomorphism $F_x E \xrightarrow{\simeq} \mathbb{C}^k$. In particular, letting $S^{n-1} \subset D^n_+ \cap D^n_-$ denote the equator of S^n, we may construct a map:

$$f : S^{n-1} \to GL_k(\mathbb{C}),$$
$$x \mapsto \varphi_+^{-1}(x)\varphi_-(x)$$

We call f as above the *clucthing function* associated to the bundle E. It can be checked that isomorphic bundles give rise to homotopic maps, and in fact that this assignment gives rise to a bijection (c.f. Section 1.3 of Cohen [8] for a proof):

Theorem 2.13. *The assignment $\mathit{Vect}^k_{\mathbb{C}}(S^n) \to [S^{n-1}, GL_k(\mathbb{C})]$ sending a rank k vector bundle over S^n to its associated clutching function is a bijection.*

As an exercise in manipulating clutching functions which will prove useful later in this section, we establish the following result:

Lemma 2.14. *Given clutching functions $f, g \colon S^1 \to GL_n(\mathbb{C})$, the associated rank n vector bundles over S^2 satisfy the following relation:*

$$E_f \oplus E_g \cong E_{fg} \oplus \epsilon^n. \tag{2.15}$$

Proof. It follows directly from the definition that the clutching function associated to the left hand side, resp. right hand side of Lemma 2.14 are given by the maps $z \mapsto \mathrm{diag}(f(z), g(z))$, resp. $z \mapsto \mathrm{diag}(f(z)g(z), I_n)$, where I_n denotes the n by n identity matrix. Hence by Theorem 2.13, it suffices to show that these two assignments are homotopic as maps $S^1 \to GL_{2n}(\mathbb{C})$. Now, $GL_{2n}(\mathbb{C})$ is path-connected (which can be see for instance via the polar decomposition), hence in particular we can find a path $\gamma : I \to GL_{2n}(\mathbb{C})$ from the identity matrix to the matrix $P := \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$. Then, the map $H\colon S^1 \times I \to GL_{2n}(\mathbb{C})$ given by:

$$H(z,t) := \mathrm{diag}(f(z), I_n) \cdot \gamma(t) \cdot \mathrm{diag}(I_n, g(z)) \cdot \gamma(t)$$

gives a homotopy $H\colon \mathrm{diag}(f(z), g(z)) \simeq \mathrm{diag}(f(z)g(z), I_n)$, as needed. $\square$

Of particular interest to us is the so-called *Bott element* $\beta \in \tilde{K}^0(S^2)$, defined to be $\beta := L - 1$, where $L \xrightarrow{\pi} S^2 \cong \mathbb{C}P^1$ denotes the tautological line bundle. By Lemma 2.14, since product of clutching functions corresponds to tensor product of bundles for line bundles, we have the relation $L \oplus L \cong L^2 \oplus \epsilon^1$. In $K^0(S^2)$, this implies that

$$\beta^2 = (L-1)^2 = L^2 + 1 - 2L = 0. \tag{2.16}$$

The statement of the Bott periodicity theorem which we choose to give relies on the existence of a product structure on K-groups associated to different spaces, analogous to the cross product in singular cohomology:

Definition 2.17. Given spaces X and Y, define the *external product* on $K^0(X)$ and $K^0(Y)$ to be the bilinear map:

$$K^0(X) \otimes K^0(Y) \xrightarrow{-\times-} K^0(X \times Y)$$

given by $x \times y := \pi_1^*(x)\pi_2^*(y)$, where $X \times Y \xrightarrow{\pi_1} X, X \times Y \xrightarrow{\pi_2} Y$ are the standard projections.

Remark 2.18. The external product is clearly natural in both factors. In particular, fixing an element $z \in K^0(Y)$ allows us to define a natural transformation $K^0(-) \xrightarrow{-\times z} K^0(- \times Y)$ given by the composite:

$$K^0(X) \xrightarrow{\mathrm{id}\otimes z} K^0(X) \otimes K^0(Y) \xrightarrow{-\times-} K^0(X \times Y).$$

Lemma 2.19. *Given spaces X and Y, the external product restricts to a reduced external product:*

$$\tilde{K}^0(X) \otimes \tilde{K}^0(Y) \xrightarrow{-\times-} \tilde{K}^0(X \wedge Y)$$

In the same way as in Remark 2.18, any element $z \in \tilde{K}^0(Y)$ induces a natural transformation $\tilde{K}^0(-) \xrightarrow{-\times z} \tilde{K}^0(- \wedge Y)$. We are now in a position to state the main theorem, a proof of which is given in Chapter 2 of Hatcher [11] via a detailed analysis of clutching functions:

Theorem 2.20. ***(Bott Periodicity)*** *The map*

$$\tilde{K}^0(X) \xrightarrow{-\times\beta} \tilde{K}^0(X \wedge S^2) = \tilde{K}^0(\Sigma^2 X)$$

induced by taking external products with the Bott element is a natural isomorphism.

As an immediate application, since $\tilde{K}^0(\mathrm{pt}) \cong \mathbb{Z}$, we find that $\tilde{K}^0(S^2)$, and more generally $\tilde{K}^0(S^{2n})$ for any even dimensional sphere is isomorphic to a copy of $\mathbb{Z}$ generated by $\beta^{\times n}$, with ring structure given by $(\beta^{\times n})^2 = (\beta^2)^{\times n} = 0$ by Equation 2.16. Next, as observed in the proof of Theorem 2.10, using Equation 1.10, we find that:

$$\tilde{K}^0(S^1) \cong [S^1, BU \times \mathbb{Z}] \cong \pi_1(BU \times \mathbb{Z}) = \pi_1(BU) = \pi_0(U) = 0,$$

since U is path-connected. It then immediately follows from Theorem 2.20 that $\tilde{K}^0(S^{2n+1}) = 0$ for all odd-dimensional spheres.

We now take a moment to deduce the homotopy-theoretic version of the Bott periodicity theorem as expressed in Bott's original paper [6], as this is the version which we will mostly be needing in the next few sections. Combining Theorem 2.10, the Bott periodicity theorem and the loop-suspension adjunction leads us to a chain of natural isomorphisms:

$$[X, \Omega^2(BU \times \mathbb{Z})] \cong [\Sigma^2 X, BU \times \mathbb{Z}] \cong \tilde{K}^0(\Sigma^2 X) \cong \tilde{K}^0(X) \cong [X, BU \times \mathbb{Z}].$$

Thus, by the Yoneda lemma, it follows that we have a homotopy equivalence:

$$BU \times \mathbb{Z} \simeq \Omega^2(BU \times \mathbb{Z}).$$

Also, by compatibility of the based hom functor with products, we readily see that:

$$\Omega(BU \times \mathbb{Z}) \cong \Omega BU \times \Omega\mathbb{Z} \cong \Omega BU,$$

where the last identification follows from the fact that $\Omega\mathbb{Z} = \{\mathrm{pt}\}$ since $\mathbb{Z}$ is a discrete space. Hence the Bott periodicity theorem as stated in Theorem 2.20 also gives a homotopy equivalence:

$$BU \times \mathbb{Z} \simeq \Omega^2 BU.$$

2.3. **Spectra and Generalized Cohomology Theories.** Equipped with Bott periodicity, we are in a position to formally extend the reduced complex K-group functor defined in Section 2.1 to a reduced generalized cohomology theory. We start by recalling some facts about generalized cohomology theories, an instance of which is reduced singular cohomology with coefficients in an abelian group G.

Recall the Eilenberg-Steenrod axioms for a reduced generalized cohomology theory:

Definition 2.21. A *reduced generalized cohomology theory* consists of a sequence of functors $\tilde{h}^n : \mathrm{Ho}(\mathbf{Top}_*)^{op} \to \mathbf{Ab}$ satisfying the following axioms for all n:

- (i) **(exactness)** Given any cofibration $A \xrightarrow{i} X$, the associated cofiber sequence $A \xrightarrow{i} X \to C_i$ gives rise to an exact sequence:
$$\tilde{h}^n(C_i) \to \tilde{h}^n(X) \to \tilde{h}^n(A);$$
- (ii) **(suspension)** There exists a natural isomorphism of functors:
$$\Sigma : \tilde{h}^n(X) \xrightarrow{\simeq} \tilde{h}^{n+1}(\Sigma X);$$
- (iii) **(additivity)** given a wedge sum $X = \vee_i X_i$, the inclusion maps $X_i \xrightarrow{l_i} X$ give rise to an isomorphism:
$$\tilde{h}^n(X) \xrightarrow{\simeq} \prod_i \tilde{h}^n(X_i);$$

- (iv) **(weak equivalence)** given a weak equivalence $X \xrightarrow{f} Y$, the induced map $\tilde{h}^n(Y) \xrightarrow{f^*} \tilde{h}^n(X)$ is an isomorphism.

Remark 2.22. Any reduced generalized cohomology theory determines and is determined by an unreduced generalized cohomology theory, as defined in Chapter 18 of May [13]. In particular, the construction we present in Section 2.4 for reduced topological K-theory as a reduced generalized cohomology theory extending $\tilde{K}^0(-)$ in degree 0 corresponds to an unreduced theory extending $K^0(-)$.

Now, we may view each $\tilde{h}^n$ as taking values in the category of sets. By definition, $\tilde{h}^n$ is then a homotopy invariant, contravariant functor which commutes with coproducts. It can further be checked that the Mayer-Vietoris axiom is satisfied as a result of the above axioms. Hence, by the Brown representability theorem stated in Section 1.2, each functor $\tilde{h}^n$ is representable, i.e. admits a natural isomorphism:

$$[X, T_n] \xrightarrow{\sim} \tilde{h}^n(X),$$

for some based space T_n which is unique up to homotopy equivalence.

Now using the loop-suspension adjunction together with the suspension axiom, we get a sequence of natural isomorphisms:

$$[X, \Omega T_{n+1}] \cong [\Sigma X, T_{n+1}] \cong \tilde{h}^{n+1}(\Sigma X) \cong \tilde{h}^n(X),$$

implying that the functor $\tilde{h}^n$ is also represented by the space ΩT_{n+1}. It follows by uniqueness that we must have a homotopy equivalence:

$$T_n \xrightarrow{\sim} \Omega T_{n+1}, \tag{2.23}$$

corresponding under the loop-suspension adjunction to a unique map (up to homotopy) $\Sigma T_n \to T_{n+1}$.

Such a sequence of based spaces $\{T_n\}_{n\geq 0}$ equipped with *structure maps* $\sigma_n : \Sigma E_n \to E_{n+1}$ for each n is known as a *spectrum*. Spectra constitute a central object of study in stable homotopy theory. In what follows, we are mainly concerned with the notion of Ω-*spectra*: we say that a spectrum $\{T_n\}_{n\geq 0}$ is an Ω-spectrum if the adjoint map $\tilde{\sigma}_n : T_n \to \Omega T_{n+1}$ of each structure map is a weak homotopy equivalence.

Thus, the Brown representability theorem implies that any reduced generalized cohomology theory is represented by an Ω-spectrum. A converse to the above statement can readily be established, once one knows what kind of structure to look for. Namely, the sequence of functors $\tilde{h}^n(-) := [-, E_n] : \mathbf{Top}_*^{op} \to \mathbf{Set}$ associated with any given Ω-spectrum gives rise to a valid generalized cohomology theory. First observe that these functors are clearly homotopy invariant, and that the isomorphism

$$\tilde{h}^n(X) = [X, E_n] \cong [X, \Omega^2 E_{n+2}]$$

for all $n \geq 0$ guarantees that each $\tilde{h}^n(X)$ has an abelian group structure, hence that we in fact get a sequence of functors $\tilde{h}^n : \mathrm{Ho}(\mathbf{Top}_*)^{op} \to \mathbf{Ab}$.

To check that these functors do indeed determine a reduced cohomology theory, we make use of the fact, essentially due to the CW approximation theorems, that a reduced generalized cohomology theory as in Definition 2.21 determines and is determined by its restriction to based CW-complexes, where the weak equivalence axiom is trivially satisfied by the Whitehead theorem and for which the exactness axiom is replaced by the requirement that for any pair of CW-complexes (X, A) and for all $n \geq 0$, the resulting sequence

$$\tilde{h}^n(X/A) \to \tilde{h}^n(X) \to \tilde{h}^n(A)$$

is exact (see Chapter 19 of May [13] for more details). All other verifications are more or less immediate: the exactness axiom holds by general exactness of the functor $[-, Z]$ applied to a cofiber sequence for any based space Z, the suspension axiom follows from naturality of the loop-suspension adjunction, and the additivity axiom follows from compatibility of the hom functor with coproducts.

Therefore, there is a bijective correspondence between Ω-spectra and reduced generalized cohomology theories given by the assignment:

$$\{E_n\}_{n\geq 0} \mapsto \left(\tilde{h}^n : \mathrm{Ho}(\mathbf{Top}_*)^{op} \to \mathbf{Ab}, X \mapsto [X, E_n]\right)_{n\geq 0}.$$

Remark 2.24. (i) It should be noted that, under the proper notion of morphisms of Ω-spectra and cohomology theories, this bijection on objects does not produce an equivalence of categories. This is due to the existence of *hyperphantom maps* between spectra producing the zero map on the associated cohomology theories.
(ii) Some of the existing literature requires that a cohomology theory be defined for negative n as well, in which case we may extend the above correspondence via the assignment $X \mapsto [X, \Omega^{-n}E_0]$ for $n < 0$.

2.4. **Extending $\tilde{K}^0(-)$ to a Reduced Generalized Cohomology Theory.** We are now ready to introduce the Ω-spectrum corresponding to (complex) topological K-theory. Recall that Bott periodicity gives us a homotopy equivalence $\Omega^2 BU = \Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}$. We may then formulate the following definition.

Definition 2.25. The *complex K-theory spectrum KU* is defined to be the Ω-spectrum whose components are given by:

$$KU_n := \begin{cases} BU \times \mathbb{Z} & \text{if } n \text{ is even,} \\ \Omega BU & \text{if } n \text{ is odd.} \end{cases}$$

For even n, the adjoint of the structure map $\tilde{\sigma}_n : KU_n \to \Omega KU_{n+1}$ is defined to be the Bott homotopy equivalence $BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2 BU$, and for odd n it is defined to be the homeomorphism $\Omega BU \xrightarrow{\cong} \Omega(BU \times \mathbb{Z})$.

We then define *complex topological K-theory* to be the reduced generalized cohomology theory associated to this Ω-spectrum, by defining for all $n \geq 0$ and based space X:

$$\tilde{K}^n(X) := [X, KU_n].$$

The 0^{th} component of this cohomology theory coincides with our earlier definition of the reduced complex K-theory group, under the natural isomorphism $\tilde{K}^0(X) \cong [X, BU \times \mathbb{Z}]$ discussed in Section 2.1. Next, for $n = 1$, we find by the loop-suspension adjunction that:

$$\tilde{K}^1(X) = [X, \Omega BU] \cong [X, \Omega(BU \times \mathbb{Z})] \cong [\Sigma X, BU \times \mathbb{Z}] = \tilde{K}^0(\Sigma X).$$

Now, complex K-theory is a 2-periodic cohomology theory, in the sense that, for all n:

$$\tilde{K}^{n+2}(X) = [X, KU_{n+2}] = [X, KU_n] = \tilde{K}^n(X). \tag{2.26}$$

Hence, the above two cases suffice to determine the behavior of this cohomology theory for all n.

3. Adams Operations and the Chern Character

In this chapter, we spend some time developing tools that will allow us to get a better grasp on topological K-theory as a generalized cohomology theory, eventually leading to the resolution of the Hopf invariant one problem in Chapter 4. In Section 3.1, we construct the *Adams operations*, which are cohomology operations on topological K-theory which can be viewed as the analogues of the Steenrod operations in singular cohomology. Next, in Section 3.2, we use Chern classes to define the *Chern character*, which will give us a way to translate data from topological K-theory into the world of singular cohomology via a natural transformation compatible with the Adams operations.

3.1. Construction of the Adams operations.

By a cohomology operation on K-theory, we mean a natural transformation $F : K^0(-) \to K^0(-)$ from the K-group functor to itself, viewed as a functor $K^0(-) : \text{Ho}(\mathbf{Top}_*)^{op} \to \mathbf{Set}$ (that is, we do not necessarily require the component maps $F_X : K^0(X) \to K^0(X)$ to be group homomorphisms). In order to carry out the construction of the Adams operations, we will be needing a key lemma whose proof may be found in Chapter 2 of Hatcher [11]:

Lemma 3.1. ***(Splitting principle)*** *Let $\xi : E \to X$ be a vector bundle of rank n over a compact Hausdorff base space. Then there exists a map $p : F(E) \to X$ from a compact Hausdorff space $F(E)$ such that the induced map on K-groups $p^* : K^0(X) \to K^0(F(E))$ is injective, and such that the image of ξ under p splits as a direct sum of lines bundles, i.e. $p^*E = \bigoplus_{i=1}^n L_i$ for some line bundles $L_1, ..., L_n \in K^0(F(E))$.*

Remark 3.2. Notice that the splitting principle may be slightly extended so as to account for a pair of vector bundles $\xi : E \xrightarrow{\pi} X, \xi' : E' \xrightarrow{\pi} X$. Namely, starting with the map $p : F(E) \to X$ provided by applying the splitting principle to E, with p^* injective and $p^*\xi$ splitting as a sum of line bundles, we may apply the splitting principle again with respect to the vector bundle $p^*\xi' \in K^0(F(E))$ to obtain a map $q : F(F(E)) \to F(E)$ such that q^* is injective and $q^*p^*\xi' = (pq)^*\xi'$ splits as a sum of line bundles. Then the composite $(pq)^*$ is injective, and by compatibility of pullbacks with taking Whitney sums, $(pq)^*\xi$ also splits as a sum of line bundles in $K^0(F(F(E)))$, as needed.

Theorem 3.3. ***(Adams Operations)*** *There exists a unique collection of cohomology operations on K-theory $(\psi^k : K^0(-) \to K^0(-))_{k \geq 0}$ satisfying the following properties:*

- *(i) Each $\psi^k : K^0(X) \to K^0(X)$ is a group homomorphism;*
- *(ii) For any line bundle L, $\psi^k(L) = L^k$;*
- *(iii) Each $\psi^k : K^0(X) \to K^0(X)$ is a ring homomorphism;*
- *(iv) $\psi^k\psi^l = \psi^l\psi^k = \psi^{kl}$, for all $k, l \geq 0$;*

- *(v) For all primes p, $x \in K^0(X)$, we have $\psi^p(x) \equiv x^p (mod\, p)$, where by $x \equiv y (mod\, p)$ we mean that $x = y + pz$ for some $z \in K(X)$;*
- *(vi) For all $m, k \geq 0$, $x \in \tilde{K}^0(S^{2m}) \cong \mathbb{Z}$, we have $\psi^k(x) = k^m x$.*

We call $\psi^k : K^0(-) \to K^0(-)$ the k^{th} Adams operation.

Remark 3.4. Part (iv) of the above theorem relies on the fact that naturality of the Adams operations with respect to the inclusion map $l : \{\text{pt}\} \hookrightarrow X$ guarantees that we obtain induced natural transformations:

$$\psi^k : \tilde{K}^0(-) \to \tilde{K}^0(-).$$

We readily observe that properties (i) and (ii) already suffice to characterize the Adams operations uniquely. For suppose that $\psi^k, \varphi^k : K^0(-) \to K^0(-)$ are two natural transformations satisfying these two properties. Then agreement of ψ^k, φ^k is immediate on line bundles, and by additivity we also get that

$$\psi^k(\oplus_{i=1}^k L_i) = \oplus_{i=1}^k \psi^k(L_i) = \oplus_{i=1}^k L_i^k = \oplus_{i=1}^k \varphi^k(L_i) = \varphi^k(\oplus_{i=1}^k L_i),$$

giving agreement of ψ^k and φ^k for any finite sum of line bundles. Finally, for an arbitrary vector bundle $\xi : E \xrightarrow{\pi} X$, we may use the splitting principle to find a space $F(E)$ and a map $p : F(E) \to X$ as in Lemma 3.1, such that $p^*\xi = \oplus_{i=1}^n L_i$ for some line bundles $L_1, ..., L_n \in K^0(F(E))$, so that by naturality of ψ^k and φ^k we see that

$$p^*\psi^k(\xi) = \psi^k(p^*\xi) = \psi^k(\oplus_{i=1}^n L_i) = \varphi^k(\oplus_{i=1}^n L_i) = \varphi^k(p^*\xi) = p^*\varphi^k(\xi);$$

from there, injectivity of p^* allows us to conclude that $\psi^k(\xi) = \varphi^k(\xi)$, hence that $\psi^k = \varphi^k$, as claimed. Also note that once existence is established, property (iv) follows immediately from the fact that, without loss of generality, the composite $\psi^k\psi^l$ is a natural transformation, a group homomorphism on each component, and on line bundles satisfies $\psi^k\psi^l(L) = \psi^k(L^l) = (L^l)^k = L^{lk}$, hence by uniqueness we must have that $\psi^k\psi^l = \psi^{kl}$, resp. $\psi^l\psi^k = \psi^{kl}$.

For the remainder of this section, we set out to prove the existence of Adams operations satisfying all of the above properties. Our approach is inspired by the elegant treatment in Chapter 11 of Wirthmüller [23]. The essential ingredient in the following construction is the fact that each K-theory ring $K^0(X)$ has the structure of a λ-*ring* induced from the exterior power operations $E \mapsto \Lambda^k(E)$ induced for each $k \geq 0$ from the k^{th} exterior power operation at the level of vector spaces following Construction 1.13. These operations satisfy certain properties following from the properties of the standard exterior product on vector spaces: $\Lambda^0(E) = 1$, $\Lambda^k(E) = 0$ whenever $k > \text{rk } E$, and on direct sums

$$\Lambda^k(E \oplus F) = \bigoplus_{i+j=k} \Lambda^i(E) \otimes \Lambda^j(F). \tag{3.5}$$

Construction 3.6. Observe that the collection of exterior power operations may be used to associate to any vector bundle on a space X a formal power series with values in $K(X)$. Namely, we define

$$\lambda_t : \text{Vect}(X) \to K^0(X)[[t]],$$

$$\lambda_t(E) := \sum_{k \geq 0} \Lambda^k(E) t^k.$$

Since the constant coefficient of $\lambda_t(E)$ always equals 1, this assignment in fact maps into the multiplicative group of formal power series with coefficients in $K^0(X)$:

$$\lambda_t : \mathrm{Vect}_{\mathbb{C}}(X) \to K^0(X)[[t]]^{\times},$$

where the product on $K^0(X)[[t]]^{\times}$ is given by formal product of power series. Then, we see from Equation 3.5 above that λ_t is in fact a semigroup homomorphism, hence factors through a unique group homomorphism

$$\lambda_t : K^0(X) \to K^0(X)[[t]]^{\times}$$

by the defining universal property of the Grothendieck group. We also denote this homomorphism by λ_t, so that on virtual bundles we have $\lambda_t(E - F) = \lambda_t(E)\lambda_t(F)^{-1}$.

Equipped with this homomorphism, consider the following power series:

$$\psi_t : K^0(X) \to K^0(X)[[t]]$$

$$\psi_t(E) := \psi^0(E) - t\frac{d}{dt}\log\lambda_{-t}(E), \tag{3.7}$$

where $\psi^0(E)$ denotes the trivial bundle over X whose rank coincides with the rank of E over each connected component of X. We then define the Adams operations $(\psi^k \colon K(-) \to K(-))_{k\geq 0}$ to be the coefficients of the resulting power series expansion:

$$\psi_t(E) := \sum_{k\geq 0} \psi^k(E)t^k.$$

The properties of the Adams operations listed in Theorem 3.3 may then be verified through direct manipulations of formal power series, together with the use of the splitting principle stated in Lemma 3.1. We first observe that naturality of λ_t with respect to pullbacks implies naturality of the Adams operations by construction.

Starting with a line bundle $L \in K^0(X)$, we directly compute that:

$$\begin{aligned}\psi_t(L) &= 1 - t\frac{d}{dt}\log(1 - Lt) = 1 - t(1 - Lt)^{-1}\frac{d}{dt}(1 - Lt)\\ &= 1 + Lt(\sum_{k\geq 0} L^k t^k) = \sum_{k\geq 0} L^k t^k = \sum_{k\geq 0} \psi^k(L)t^k,\end{aligned}$$

verifying property (i).

Next, given two line bundles $L, P \in K^0(X)$, using the properties of the logarithm, we find that:

$$\begin{aligned}\psi_t(L \oplus P) &= 2 - t\frac{d}{dt}\log\lambda_{-t}(L \oplus P) = 2 - t\frac{d}{dt}\log(\lambda_{-t}(L)\lambda_{-t}(P))\\ &= (1 - t\frac{d}{dt}\log\lambda_{-t}(L)) + (1 - t\frac{d}{dt}\log\lambda_{-t}(P))\\ &= \sum_{k\geq 0}(\psi^k(L) + \psi^k(P))t^k = \sum_{k\geq 0}\psi^k(L \oplus P)t^k,\end{aligned}$$

and likewise for finite sums of line bundles. Then, if $E, E' \in K^0(X)$ are arbitrary vector bundles, we may use the extended splitting principle discussed in Remark 3.2

to find a space Y and a map $p : Y \to X$ such that $p^* : K^0(X) \to K^0(Y)$ is injective and such that $p^*E = \oplus_{i=1}^n L_i$, $p^*E' = \oplus_{j=1}^m P_j$ both split as direct sums of line bundles. We then have by naturality:

$$p^*(\psi^k(E \oplus E')) = \psi^k(p^*E \oplus p^*E') = \psi^k((\oplus_{i=1}^n L_i) \oplus (\oplus_{j=1}^m P_j)) \\ = \psi^k(\oplus_{i=1}^n L_i) \oplus \psi^k(\oplus_{j=1}^m P_j) = \psi^k(p^*E) \oplus \psi^k(p^*E') \\ = p^*(\psi^k(E) \oplus \psi^k(E')), \tag{3.8}$$

whence it follows from injectivity of p^* that $\psi^k(E \oplus E') = \psi^k(E) \oplus \psi^k(E')$, verifying property (ii). As observed earlier, property (iv) now also follows by uniqueness.

Similarly, starting with two line bundles $L, P \in K^0(X)$, since $L \otimes P$ is also a line bundle, we compute:

$$\psi_t(L \otimes P) = \sum_{k \geq 0} (LP)^k t^k = \sum_{k \geq 0} L^k P^k t^k = \sum_{k \geq 0} \psi^k(L)\psi^k(P) t^k = \sum_{k \geq 0} \psi^k(L \otimes P) t^k,$$

giving multiplicativity of the Adams operations on line bundles. From there, we find that on finite sums of line bundles:

$$\psi_t((\oplus_{i=1}^n L_i) \otimes (\oplus_{j=1}^m P_j)) = \psi_t(\oplus_{i,j=1}^{n,m}(L_i \otimes P_j)) = \sum_{i,j=1}^{n,m} \psi_t(L_i \otimes P_j) \\ = \sum_{i,j=1}^{n,m} \sum_{k \geq 0} \psi^k(L_i)\psi^k(P_j) t^k = \sum_{k \geq 0} (\sum_{i,j=1}^{n,m} \psi^k(L_i)\psi^k(P_j)) t^k \\ = \sum_{k \geq 0} \psi^k(\oplus_{i=1}^n L_i)\psi^k(\oplus_{j=1}^m P_j) t^k = \sum_{k \geq 0} \psi^k((\oplus_{i=1}^n L_i) \otimes (\oplus_{j=1}^m P_j)) t^k,$$

as needed. The verification of property (iii) for the general case then follows from the splitting principle in the same way as in Equation 3.8 above.

To establish property (v), we first observe that for p: prime and $\oplus_{i=1}^n L_i$ a sum of line bundles, we immediately have:

$$\psi^p(\oplus_{i=1}^n L_i) = \oplus_{i=1}^n L_i^p \equiv (\oplus_{i=1}^n L_i)^p \pmod p,$$

and the general case follows from the splitting principle.

Finally, we verify property (vi). Recall that by the Bott periodicity theorem, $\tilde{K}^0(S^{2r}) \cong \mathbb{Z}$ is generated by the r^{th} external product $\beta^{\times r}$ of the Bott element $\beta = (1 - L) \in \tilde{K}^0(S^2)$. Now, since L is a line bundle, we have that

$$\psi^k(L - 1) = \psi^k(L) - 1 = L^k - 1 = (L - 1)(\sum_{j=0}^{k-1} L^j) = k(L - 1),$$

since each $L^j = ((L - 1) + 1)^j \equiv 1 (\text{mod } L - 1)$, and hence it follows from the fact that ψ^k is a ring homomorphism and a natural transformation that

$$\psi^k((L - 1)^{\times r}) = (\psi^k(L - 1))^{\times r} = k^r (L - 1)^{\times r},$$

completing the proof.

3.2. **Chern Classes and the Chern Character.** We now set out to introduce the Chern character, which provides us with a useful connection between complex K-groups and singular cohomology. The Chern character will be defined as a "polynomial over the Chern classes", and so we take a moment to recall the main properties of the latter:

Theorem 3.9. ***(Chern classes)*** *Given a rank n complex vector bundle $\xi\colon E \xrightarrow{\pi} X$, there is a unique family of cohomology classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$, satisfying the following properties:*

- *$c_0(E) = 1$, the unit in $H^*(B;\mathbb{Z})$, and $c_i(E) = 0$ for $i > n$;*
- *(Naturality) Chern classes commute with pullbacks: given a map $f : Y \to X$ and a bundle E over X, we have that $c_i(f^*E) = f^*(c_i(E))$;*
- *(Whitney product formula) Given vector bundles E and E' over the same base space, we have:*
$$c_i(E \oplus E') = \sum_{p+q=i} c_p(E) \cup c_q(E');$$
- *Letting γ_1^∞ denote the tautological line bundle over $\mathbb{C}P^\infty$, we have that $c_1(\gamma_1^\infty)$ generates $H^2(\mathbb{C}P^\infty)$.*

We refer to $c_i(E) \in H^{2i}(X;\mathbb{Z})$ as the i^{th} Chern class of the bundle ξ.

Remarks 3.1. (i) The naturality axiom ensures that each $c_i(-)$ specifies the data of a natural transformation $c_i(-) : \mathrm{Vect}_{\mathbb{C}}(-) \to H^*(-;\mathbb{Z})$, taking values in even degree cohomology groups, i.e. factoring through $H^{ev}(-;\mathbb{Z}) = \oplus_{i\in\mathbb{Z}} H^{2i}(-;\mathbb{Z})$.
(ii) Given a rank n bundle $\xi : E \xrightarrow{\pi} X$, we may view the formal sum $c(E) = 1 + c_1(E) + ... + c_n(E)$ as an element in the multiplicative group of units $H^*(X;\mathbb{Z})^\times$, which we refer to as the *total Chern class* of the bundle ξ. Then, the Whitney product formula for vector bundles E, E' over X simplifies to:
$$c(E \oplus E') = c(E)c(E').$$
(iii) Given two line bundles L, L', we also have the useful relation:
$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

In constructing the Chern character, it will be useful to keep in mind the explicit construction of the Chern classes of a vector bundle. Let $\gamma_k^\infty : EU(k) \xrightarrow{\pi} BU(k)$ denote the universal rank k vector bundle, where $BU(k) = Gr_k(\mathbb{C}^\infty)$. Under the canonical isomorphism $(\mathbb{C}^\infty)^k \cong \mathbb{C}^\infty$, we have an inclusion map $S : (\mathbb{C}P^\infty)^k \to Gr_k((\mathbb{C}^\infty)^k) \cong BU(k)$ sending k-lines to the k-plane they span in $(\mathbb{C}^\infty)^k$. Then, it can be checked that this map appears in the pullback square:

(3.10)
$$\begin{array}{ccc} \oplus_{i=1}^k \pi_i^*(L) & \longrightarrow & EU(k) \\ \downarrow & & \downarrow \\ (\mathbb{C}P^\infty)^k & \xrightarrow{S} & Gr_k((\mathbb{C}^\infty)^k) = BU(k), \end{array}$$

where $\pi_i : (\mathbb{C}P^\infty)^k \to \mathbb{C}P^\infty$ are the projection maps for $i = 1, ..., k$ and $L \to \mathbb{C}P^\infty$ is the tautological line bundle over $\mathbb{C}P^\infty$.

Next, S induces a map in cohomology:

(3.11)
$$S^* : H^*(BU(k);\mathbb{Z}) \to H^*((\mathbb{C}P^\infty)^k;\mathbb{Z}) \cong \mathbb{Z}[x_1, ..., x_k \mid |x_i| = 2],$$

where the isomorphism $H^*((\mathbb{C}P^\infty)^k;\mathbb{Z}) \cong \mathbb{Z}[x_1,...,x_k \mid |x_i| = 2]$ is obtained from the Künneth theorem together with the standard fact that $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x \mid |x_i| = 2]$, so that each generator x_i in Equation 3.11 is obtained via pullback of the generator of the cohomology ring of the i^{th} component of $(\mathbb{C}P^\infty)^k$, i.e. $x_i = \pi_i^* c_1(L)$. Next, consider the standard action of Σ_k, the symmetric group on k letters, on the product $(\mathbb{C}P^\infty)^k$, given by permutation of the variables. Passing to integral cohomology, we get an induced action of Σ_k on $H^*((\mathbb{C}P^\infty)^k;\mathbb{Z})$ permuting the generators. We then have the identification:

$$H^*((\mathbb{C}P^\infty)^k;\mathbb{Z})^{\Sigma_k} \cong \mathbb{Z}[\sigma_1,...,\sigma_k],$$

where $H^*((\mathbb{C}P^\infty)^k;\mathbb{Z})^{\Sigma_k}$ is the subgroup of Σ_k-fixed points, and where each σ_i denotes the i^{th} elementary symmetric polynomial on k variables. Now, for each $\theta \in \Sigma_k$, since the inclusion $S \circ \theta : (\mathbb{C}P^\infty)^k \to BU(k)$ produces a bundle isomorphic to $\oplus_{i=1}^k \pi_i^*(L)$ under pullback of γ_k^∞, we get by the homotopy classification theorem that $S\theta \simeq S$, and hence that $\theta^* S^* = S^*$ at the level of cohomology. Therefore, the map $S^* : H^*(BU(k);\mathbb{Z}) \to H^*((\mathbb{C}P^\infty)^k;\mathbb{Z})$ factors through a map:

$$S^* : H^*(BU(k);\mathbb{Z}) \to H^*((\mathbb{C}P^\infty)^k;\mathbb{Z})^{\Sigma_k} \cong \mathbb{Z}[\sigma_1,...,\sigma_k], \tag{3.12}$$

One may then show that the above map is an isomorphism, so that $H^*(BU(k);\mathbb{Z})$ is isomorphic to the polynomial algebra over the elementary symmetric polynomials in k variables. Under this isomorphism, we may then define the i^{th} Chern class of γ_k^∞ for $i = 1,...,k$ via the relation:

$$c_i(\gamma_k^\infty) := (S^*)^{-1}(\sigma_i).$$

Note that this is well-defined since each x_j has degree 2, so that σ_i has degree $2i$. Next, given an arbitrary rank k vector bundle $E \to X$, let $f : X \to BU(k)$ be a classifying map and define

$$c_i(E) := f^*(c_i(\gamma_k^\infty)),$$

which is well-defined by the homotopy classification theorem and homotopy invariance of cohomology. This construction may then be verified to satisfy the properties of Theorem 3.9.

Equipped with the notion of Chern classes taking values in singular cohomology, we start moving towards a natural transformation $\varphi : K^0(-) \to H^*(-)$, which we require to be a ring homomorphism in each component. Intuitively, we start with Chern classes and look for an appropriate polynomial over them. In particular, we expect φ to factor through $H^{ev}(-;\mathbb{Q}) = \oplus_{i\in\mathbb{Z}} H^{2i}(-;\mathbb{Z})$. The relation $c_1(L \otimes L') = c_1(L) + c_1(L')$ motivates the definition $\varphi(L) = \exp(c_1(L)) = \sum_{r\geq 0} \frac{1}{r!} c_1(L)^r$ on line bundles, in which case we must allow φ to take values in $H^{ev}(-;\mathbb{Q})$. We then have the following:

Theorem 3.13. ***(Chern Character)*** *There exists a unique natural transformation $Ch : K^0(-) \to H^{ev}(-;\mathbb{Q})$ satisfying the following properties:*

- *(i) Each $Ch : K^0(X) \to H^{ev}(X;\mathbb{Q})$ is a group homomorphism;*
- *(ii) For any line bundle L, $Ch(L) = exp(c_1(L))$;;*
- *(iii) Each $Ch : K^0(X) \to H^{ev}(X;\mathbb{Q})$ is a ring homomorphism;*
- *(iv) For all $m \geq 0$, $Ch : \tilde{K}^0(S^{2m}) \to \tilde{H}^{ev}(S^{2m};\mathbb{Q})$ maps $\tilde{K}^0(S^{2m})$ isomorphically onto $H^{ev}(S^{2m};\mathbb{Z}) \subset H^{ev}(S^{2m};\mathbb{Q})$.*

We refer to $Ch : K^0(-) \to H^{ev}(-;\mathbb{Q})$ as the Chern character.

Remark 3.14. (i) Part (iv) of the above theorem relies on the fact that the Chern character induces a natural transformation:

$$Ch : \tilde{K}^0(-) \to \tilde{H}^{ev}(-;\mathbb{Q})$$

in the same way as in Remark 3.4 (ii) Via the suspension isomorphism and Bott periodicity, the Chern character may also be extended to complex topological K-theory in every degree in the form of a map

$$Ch : K^*(-) \to \oplus_{p\in\mathbb{Z}} H^{*+2p}(-).$$

We shall see that Chern character is given on an arbitrary rank k vector bundle $E \to X$ by:

$$Ch(E) = \sum_{r\geq 0} \frac{1}{r!} s_r(E), \tag{3.15}$$

where $s_r(E) := S_r(c_1(E), ..., c_k(E))$ is defined to be the r^{th} Newton polynomial evaluated on the Chern classes of E (recall that the r^{th} Newton polynomial is the unique polynomial $S_r \in \mathbb{Z}[\sigma_1, ..., \sigma^k]$ satisfying $S_r(\sigma_1, ..., \sigma_k) = x_1^r + ... + x_k^r$, where $\sigma_i \in \mathbb{Z}[x_1, ..., x_k]$ is the i^{th} elementary symmetric polynomial). This assignment at the level of $\mathrm{Vect}_{\mathbb{C}}(X)$ factors by additivity through the desired map $Ch : K^0(X) \to H^{ev}(X;\mathbb{Q})$.

As with the Adams operations, specifying the behavior of such a map on line bundles and requiring it to be additive suffices to give uniqueness. Indeed, suppose that we disposed of such a natural transformation $\varphi : K^0(-) \to H^{ev}(-;\mathbb{Q})$. Then the inclusion map $S : (\mathbb{C}P^\infty)^k \hookrightarrow BU(k)$ as in Equation 3.10 fits into a commutative square:

$$\begin{array}{ccc} K^0((\mathbb{C}P^\infty)^k) & \xleftarrow{S^*} & K^0(BU(k)) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ H^{ev}((\mathbb{C}P^\infty)^k;\mathbb{Q}) & \xleftarrow{S^*} & H^{ev}(BU(k);\mathbb{Q}). \end{array}$$

As mentioned earlier, the bottom horizontal map is an isomorphism onto the subgroup $H^{ev}((\mathbb{C}P^\infty)^k;\mathbb{Q})^{S_k}$ of S_k-fixed points, hence in particular it is a monomorphism. From the pullback square in Equation 3.10, we also have that $S^*\gamma_k^\infty = \oplus_{i=1}^k \pi_i^*(L)$. Hence, by additivity and naturality of φ, we find that:

$$\begin{aligned} S^*\varphi(\gamma_k^\infty) = \varphi(S^*\gamma_k^\infty) = \varphi(\oplus_{i=1}^k \pi_i^*(L)) = \sum_{i=1}^k \pi_i^*(\varphi(L)) \\ = \sum_{i=1}^k \pi_i^*(\exp(c_1(L))) = \sum_{i=1}^k \exp(\pi_i^* c_1(L)) = \sum_{r\geq 0} (x_1^r + ... + x_k^r), \end{aligned}$$

where we set $x_i := \pi_i^* c_1(L)$ for $i = 1, ..., k$ as in the notation of Equation 3.11. Recall that the Chern classes of γ_k^∞ were defined via the relation $S^* c_i(\gamma_k^\infty) = \sigma_i$ under the isomorphism

$$S^* : H^*(BU(k);\mathbb{Z}) \xrightarrow{\simeq} H^*((\mathbb{C}P^\infty)^k;\mathbb{Z}) \cong \mathbb{Z}[\sigma_1, ..., \sigma_k].$$

Thus, we have that:

$$S^*\left(\sum_{r\geq 0}\frac{1}{r!}s_r(\gamma_k^\infty)\right) = \sum_{r\geq 0}\frac{1}{r!}S^*(S_r(c_1(\gamma_k^\infty),...,c_k(\gamma_k^\infty))$$
$$= \sum_{r\geq 0}\frac{1}{r!}S_r(\sigma_1,...,\sigma_k) = \sum_{r\geq 0}(x_1^r+...+x_k^r).$$

Hence, by injectivity of S^*, we get from the above two computations that the behavior of φ on the universal rank k-bundle is forced to be:

$$\varphi(\gamma_k^\infty) = \sum_{r\geq 0}\frac{1}{r!}s_r(\gamma_k^\infty).$$

Next, for an arbitrary vector bundle $E \to X$, let $f : X \to BU(k)$ be an associated classifying map, so that $f^*\gamma_k^\infty = E$. Also recall that the Chern classes of E were defined by the relation $c_i(E) := f^*c_i(\gamma_k^\infty)$. Then, by naturality of φ, we see that:

$$\varphi(E) = \varphi(f^*\gamma_k^\infty) = f^*(\varphi(\gamma_k^\infty)) = f^*\left(\sum_{r\geq 0}\frac{1}{r!}s_r(\gamma_k^\infty)\right)$$
$$= \sum_{r\geq 0}\frac{1}{r!}S_r(f^*c_1(\gamma_k^\infty),...,f^*c_k(\gamma_k^\infty)) = \sum_{r\geq 0}\frac{1}{r!}s_r(E),$$

completing the verification of uniqueness.

Hence it suffices to verify that the map $Ch : K^0(-) \to H^{ev}(-;\mathbb{Q})$ given by Equation 3.15 is natural, and that it is compatible with sums and products of vector bundles. Naturality follows readily from naturality of the Chern classes: given a map $Y \xrightarrow{f} X$ and a rank k bundle $E \to X$, we have that:

$$f^*Ch(E) = f^*\left(\sum_{r\geq 0}\frac{1}{r!}S_r(c_1(E),...,c_k(E))\right) = \sum_{r\geq 0}\frac{1}{r!}S_r(c_1(f^*E),...,c_k(f^*E)) = Ch(f^*E).$$

Next, for additivity, first observe that by the Whitney product formula, given line bundles $L_1,...,L_k$, we have that:

$$c(\oplus_{i=1}^k L_i) = \prod_{i=1}^k c(L_i) = \prod_{i=1}^k(1+c_1(L_i)) = \sum_{i=1}^k \sigma_i(c_1(L_1),...,c_1(L_k)),$$

i.e. $c_j(\oplus_{i=1}^k L_i) = \sigma_j(c_1(L_1),...,c_1(L_k))$. Hence, on sums of line bundles, we find that:

$$Ch(\oplus_{i=1}^k L_i) = \sum_{r\geq 0}\frac{1}{r!}s_r(\oplus_{i=1}^k L_i) = \sum_{r\geq 0}\frac{1}{r!}S_r(\sigma_1,...,\sigma_k)$$
$$= \sum_{r\geq 0}\frac{1}{r!}(c_1(L_1)^r+...+c_1(L_k)^r) = \sum_{i=1}^k \exp(c_1(L_i)) = \sum_{i=1}^k Ch(L_i),$$

as needed. Next, given arbitrary bundles $E, E' \to X$, we may use the splitting principle stated in the previous section to find a map $q : F \to X$ such that $q^* : K^0(X) \to K^0(F)$ is injective and $q^*E = \oplus_{i=1}^k L_i, q^*E' = \oplus_{j=1}^l P_j$ both split as sums of line bundles. Then, by naturality, we get that:

$$\begin{aligned} q^*(Ch(E \oplus E')) &= Ch(q^*E \oplus q^*E') = Ch(\oplus_{i=1}^k L_i \oplus \oplus_{j=1}^l P_j) \\ &= Ch(\oplus_{i=1}^k L_i) + Ch(\oplus_{j=1}^l P_j) = q^*(Ch(E) + Ch(E')), \end{aligned}$$

hence by injectivity it follows that $Ch(E \oplus E') = Ch(E) + Ch(E)$.

Finally, to check multiplicativity, recall the formula $c_1(L \otimes L') = c_1(L) + c_1(L')$ for the first Chern class of the tensor product of two line bundles, which yields:

$$\begin{aligned} Ch(L \otimes L') &= \exp(c_1(L \otimes L')) = \exp(c_1(L) + c_1(L')) \\ &= \exp(c_1(L))\exp(c_1(L')) = Ch(L)Ch(L') \end{aligned}$$

Then, on sums of line bundles $\oplus_{i=1}^k L_i, \oplus_{j=1}^l P_j$, we get that, by additivity:

$$\begin{aligned} Ch(\oplus_{i=1}^k L_i \otimes \oplus_{j=1}^l P_j) &= Ch(\oplus_{i,j}(L_i \otimes P_j)) = \sum_{i,j} Ch(L_i \otimes P_j) \\ &= \sum_{i,j} Ch(L_i)Ch(P_j) = \Big(\sum_{i=1}^k Ch(L_i)\Big)\Big(\sum_{j=1}^l Ch(P_j)\Big). \end{aligned}$$

The result for arbitrary bundles then follows from the splitting principle in a similar way as above.

Finally, we verify the claim made in Theorem 3.13 (iv): first recall that the Bott element $\beta = 1 - L$, where $L \xrightarrow{\pi} \mathbb{C}P^1 \cong S^2$ is the tautological line bundle, is a generator of $\tilde{K}^0(S^2)$, and that by Bott periodicity the m^{th} external power $\beta^{\times m}$ is a generator for $\tilde{K}^0(S^{2m})$. Now, by looking at the pullback under the inclusion map $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$, it follows from Theorem 3.9 that $c_1(L)$ is a generator of $\tilde{H}^{ev}(S^2;\mathbb{Z})$. Next, since $\tilde{H}^{ev}(S^2;\mathbb{Z}) = \mathbb{Z}_{(2)}$, higher powers of $c_1(L)$ vanish, and so:

$$Ch(\beta) = Ch(1 - L) = Ch(1) - Ch(L) = 1 - \exp(c_1(L)) = -c_1(L),$$

hence Ch maps $\tilde{K}^0(S^2)$ isomorphically onto $\tilde{H}^{ev}(S^2;\mathbb{Z}) \subset \tilde{H}^{ev}(S^2;\mathbb{Q})$. The general case follows from naturality and multiplicativity of the Chern character, from which we see that:

$$Ch(\beta^{\times m}) = (Ch(\beta))^{\times m} = (-1)^m c_1(L)^{\times m},$$

where $\pm c_1(L)^m$ is a generator of $\tilde{H}^{ev}(S^{2m};\mathbb{Z})$, as needed.

4. Applying Topological K-Theory to Problems in Classical Topology

We now begin to explore some of the bearings the theory we have developed so far has on problems in classical topology. The first section is devoted to proving that no sphere of dimension greater than 6 can be given the structure of a complex manifold, and the following two sections are concerned with the Hopf invariant one problem. Both proofs are quite simple in essence with the tools we have at hand. We should note that the applications presented here only scratch the surface of what can be done - perhaps the most famous application we are omitting is the establishment of a sharp bound on the maximal possible number of pointwise linearly independent vector fields on spheres, first proven by Adams in 1961 [4]

4.1. **Non-Existence of Complex Structures on Spheres.** Starting with the observation that the 2-sphere S^2 can be realized as a complex 1-manifold under the homeomorphism $S^2 \cong \mathbb{C}P^1$, one may wonder whether this is true for spheres of different (necessarily even) dimensions - i.e, for which n can the smooth manifold structure on S^{2n} be enriched into that of a complex manifold? In this section, we answer this question in the negative for $n > 3$. It turns out that the same conclusion holds for $n = 2$; the case $n = 3$ is still an open problem.

We start with an observation that follows relatively quickly from Theorem 3.13:

Lemma 4.1. *Given any rank $n \geq 2$ complex vector bundle $E \xrightarrow{S^2 n}$, we have that*

$$(n-1)! \mid c_n(E)$$

inside $\tilde{H}^{2n}(S^{2n};\mathbb{Z}) \cong \mathbb{Z}$.

Proof. Recall from part (iv) of Theorem 3.13 that the image of the $Ch : S^{2n} \to \tilde{H}^{ev}(S^{2n};\mathbb{Q})$ lives entirely in $\tilde{H}^{ev}(S^{2n};\mathbb{Z})$, i.e.

$$Ch(E) = \sum_{r \geq 0} \frac{1}{r!} Q_r(c_1(E), ..., c_n(E)) \in \tilde{H}^{ev}(S^{2n};\mathbb{Z}). \tag{4.2}$$

Next, by degree considerations, since $\tilde{H}^{ev}(S^{2n};\mathbb{Z})$ is concentrated in degree $2n$, all terms in the above sum vanish except for $\frac{1}{n!}Q_n(c_1(E), ..., c_n(E))$. Now, it can be computed inductively that, for $n \geq 2$:

$$Q_n(c_1(E), ..., c_n(E)) = \sum_{i=1}^{n-1} c_i(E) Q_{n-i}(c_1(E), ..., c_n(E)) + (-1)^{n+1} n c_n(E)$$

(see Lemma B.1 in Dugger [9] for a quick verification of this fact). Since $c_i(E) \in \tilde{H}^{2i}(S^{2n};\mathbb{Z})$ must vanish for $i < n$, the above equation simplifies to:

$$Q_n(c_1(E), ..., c_n(E)) = (-1)^{n+1} n c_n(E).$$

Hence, by Equation 4.2 above, we get that $\frac{(-1)^{n+1}}{(n-1)!} c_n(E) \in \tilde{H}^{2n}(S^{2n};\mathbb{Z})$, and the result follows. □

Equipped with this lemma, the proof of the main result boils down to an exercise in characteristic classes:

Theorem 4.3. *If $n > 3$, then there is no complex vector bundle $T \xrightarrow{\pi} S^{2n}$ of rank n whose underlying real vector bundle of rank $2n$ is the tangent bundle induced from a smooth manifold structure on S^{2n}.*

Proof. Suppose we had such a bundle $T \xrightarrow{\pi} S^{2n}$. Then the associated Euler class $e(T) \in \tilde{H}^{2n}(S^{2n};\mathbb{Z})$ satisfies the equation

$$< e(T), [S^{2n}] >= \chi(S^{2n}) = \sum_{i=0}^{2n} (-1)^i \text{rk } H^i(S^{2n};\mathbb{Z}) = 2,$$

where $[S^{2n}] \in H_{2n}(S^{2n};\mathbb{Z})$ denotes the fundamental class. But we also know that the Euler class equals the top Chern class of T (for a quick proof up to sign, use naturality w.r.t the classifying map of T and the fact that $c_n(\gamma_n^\infty) \in H^*(BU(n);\mathbb{Z})$ is a generator by construction to get that $e(T) = \lambda c_n(T)$ for some $\lambda \in \mathbb{Z}$, and use naturality again together with the Gysin sequence in degree 0 associated to the universal rank n bundle to see that $c_n(T) = \mu e(T)$ for some $\mu \in \mathbb{Z}$. The result follows by combining the two relations). From there, by Lemma 4.1, it follows that:

$$2 =< e(T), [S^{2n}] >=< c_n(T), [S^{2n}] >= (n-1)!k$$

for some $k \in \mathbb{Z}$. Hence $\frac{2}{(n-1)!}$ must be an integer, which requires that $n \leq 3$, completing the proof. □

4.2. **The Hopf Invariant One Problem.** In 1928, Heinz Hopf first defined the Hopf invariant of a smooth map $f : S^3 \to S^2$ in terms of the notion of linking numbers of f, and used it to demonstrate that the Hopf fibration $\eta : S^3 \to S^2$ mentioned in Examples 1.3 is not null-homotopic. It was later realized that the notion of Hopf invariant could be defined for a wider range of maps, and had a bearing on classical problems in algebra and topology, as specified by the following proposition, a proof of which can be found in Section 2.3 of Hatcher [11]:

Proposition 4.4. *The following are equivalent:*

- *(i) There exists a map $f : S^{2n-1} \to S^n$ with Hopf invariant one;*
- *(ii) The vector space $\mathbb{R}^n$ admits a division algebra structure;*
- *(iii) The sphere S^{n-1} admits an H-space structure.*

Thus, determining the non-existence of maps $f : S^{2n-1} \to S^n$ of Hopf invariant one is equivalent to proving the non-existence of division algebra structures on $\mathbb{R}^n$, as well as the non-existence of H-space structures on S^{n-1} (where the data of an H-space structure on a space X consists of a continuous map $\mu \colon X \times X \to X$ and a distinguished element $e \in x$ such that $\mu(-, e) = \mu(e, -) = \mathrm{id}_X$). We already know that maps of Hopf invariant one exist for $n = 1, 2, 4, 8$, corresponding respectively to the usual real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. Strikingly, this list of admissible dimensions turns out to be exhaustive:

Theorem 4.5. *There exists a map $f : S^{2n-1} \to S^n$ with Hopf invariant one if and only if $n \in \{1, 2, 4, 8\}$.*

This section introduces the notion of Hopf invariant and explores some of its properties, and the next section builds towards a proof of the above theorem.

We start by presenting two definitions for the Hopf invariant of a map $f \colon S^{2n-1} \to S^n$, and verify that they coincide. The first formulation is homotopy theoretic in nature, while the second is inspired from the language of differential forms. As mentioned in the opening of this section, a more geometric definition may also be given; we choose to omit it from this discussion, and refer the interested reader to a treatment by Steenrod [22].

Construction 4.6. Let $f : S^{2n-1} \to S^n$ be any continuous map. We may form the *mapping cone* associated to f, denoted by X_f and obtained via the following pushout square:

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{f} & S^n \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \alpha} \\ D^{2n} & \xrightarrow[\beta]{} & X_f, \end{array}$$

where we identify the cone on S^{2n-1} with a $2n$-disk D^{2n}. Viewing S^n as a CW-complex with one 0-cell and one n-cell, the above diagram immediately gives X_f the structure of a CW-complex with one cell each in dimensions 0, n, and $2n$. Cellular cohomology then tells us that the cohomology ring of X_f with integral coefficients consists of a copy of $\mathbb{Z}$ in dimensions 0, n, $2n$. Hence, if we let a, resp. b denote generators for $H^n(X_f;\mathbb{Z})$, resp. $H^{2n}(X_f;\mathbb{Z})$, then the product structure of $H^*(X_f;\mathbb{Z})$ as a graded ring is fully determined by an integer, which we denote by $H(f)$, such that:

$$a \cup a := H(f)b.$$

We observe a few things: $H(f)$ as above is well-defined up to sign, and the latter behavior can be controlled by making a consistent choice of orientation. Since homotopic maps $f \simeq g : S^{2n-1} \to S^n$ yield homotopy equivalent mapping cones $X_f \simeq X_g$, the integer $H(f)$ only depends on f up to homotopy, hence induces an assignment $H(-) : \pi_{2_n-1}(S^n) \to \mathbb{Z}$, which we will later verify to be a group homomorphism. Also observe that for odd n, graded commutativity of the cup product implies that

$$H(f)b = a \cup a = (-1)^{|a|^2} a \cup a = -H(f)b,$$

and hence that $H(f) = 0$. Thus the value of $H(f)$ can only be non-zero for maps f into even-dimensional spheres.

Construction 4.7. Starting again with a continuous map $f : S^{2n-1} \to S^n$, let $[x]$ denote a generator of the top integral cohomology group of S^n. Then the pullback of x under f yields an element $f^*(x) \in H^n(S^{2n-1};\mathbb{Z}) = 0$, so that we may express $f^*(x) = \delta y$ as the coboundary of some $(n-1)$-cochain $y \in C^{n-1}(S^{2n-1})$. Letting $[S^{2n-1}] \in H_{2n-1}(S^{2n-1};\mathbb{Z})$ denote the fundamental class of S^{2n-1}, we may then obtain an integer, which we denote by $\tilde{H}(f)$, via the following dual pairing:

$$\tilde{H}(f) :=< [S^{2n-1}], y \cup \delta y > .$$

Again, we observe that $\tilde{H}(f)$ is well-defined up to sign, independently of the choices of x and y made above.

Passing to singular cohomology with real coefficients via the UCT and applying de Rham's theorem, we see that this construction translates into a definition in the language of differential forms. Namely, if we let $[\omega] \in H_{dR}(S^n)$ be a top degree form such that $\int_{S^n} \omega = 1$ and write the pullback form $f^*\omega = d\eta$ for some $\eta \in \Omega^{n-1}(S^{2n-1})$ by exactness, we have that

$$\tilde{H}(f) = \int_{S^{2n-1}} \eta \wedge d\eta.$$

Proposition 4.8. *Given any continuous map $f : S^{2n-1} \to S^n$, the integers $H(f)$ and $\tilde{H}(f)$ defined above are equal. We define the Hopf invariant of the map f to be the integer $H(f)$.*

Proof. Let us recall the pushout square from Construction 4.6:

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{f} & S^n \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \alpha} \\ D^{2n} & \xrightarrow[\beta]{} & X_f. \end{array} \tag{4.9}$$

As above, denote by a, resp. b chosen generators for $H^n(X_f;\mathbb{Z})$, resp. $H^{2n}(X_f;\mathbb{Z})$. Since $[\beta^* a] \in H^n(D^{2n}) = 0$, we may write $\beta^* a = \delta z$ for some cochain $z \in C^{n-1}(D^{2n})$. We then compute:

$$\begin{aligned}
H(f) &= H(f) < [D^{2n}], \beta^* b > \\
&=< [D^{2n}], \beta^*(a \cup a) > && \text{by definition of H(f)} \\
&=< [D^{2n}], \beta^* a \cup \beta^* a) > && \text{by naturality of the cup product} \\
&=< [D^{2n}], \delta z \cup \delta z) > \\
&=< [D^{2n}], \delta(z \cup \delta z) > \\
&=< \partial[D^{2n}], z \cup \delta z > \\
&=< l_*[S^{2n-1}], z \cup \delta z > \\
&=< [S^{2n-1}], l^* z \cup \delta l^* z > && \text{by duality and naturality.}
\end{aligned}$$

Now, by commutativity of the diagram in Equation 4.9, if we set $y := l^* z \in C^{n-1}(S^{2n-1})$, then we get that

$$\delta y = \delta l^* z = l^* \delta z = l^* \beta^* a = f^*(\alpha^* a),$$

where $\alpha^* a$ is a valid generator for $H^n(S^n;\mathbb{Z})$, hence by Construction 4.6 we get that:

$$\tilde{H}(f) =< [S^{2n-1}], \delta y \cup y >=< [S^{2n-1}], l^* z \cup \delta l^* z >= H(f),$$

as needed. □

We establish a few elementary facts about the Hopf invariant:

Proposition 4.10. *The map $H(-)\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ sending a map f to its Hopf invariant is a group homomorphism.*

Proof. First recall that the group structure on $\pi_{2n-1}(S^n)$ is given by sending two maps $f, g : S^{2n-1} \to S^n$ to the composite

$$f + g : S^{2n-1} \to S^{2n-1} \vee S^{2n-1} \xrightarrow{f \vee g} S^n,$$

where the first map is the pinch map collapsing the equator of S^{2n-1} to a point. Now, we may consider the intermediate mapping cone $X_{f\vee g}$, obtained by using the data of the map $f \vee g : S^{2n-1} \vee S^{2n-1} \to S^n$ to glue two $2n$-cells to S^n. We then get a quotient map

$$q : X_{f+g} \to X_{f\vee g}$$

obtained by gluing together all of the equators of the $(2n-1)$-spheres forming the $2n$-cell of X_{f+g}, corresponding to an equatorial $(2n-1)$-disk. The map q restricts to a homeomorphism on the n-cells of X_{f+g} and $X_{f\vee g}$, so that $q^*(a_{f\vee g}) = a_{f+g}$ in the usual notation. Next, considering the pullback maps induced by the inclusions $X_f \hookrightarrow X_{f\vee g}, X_g \hookrightarrow X_{f\vee g}$ tells us that:

$$a_{f\vee g}^2 = H(f) b_f + H(g) b_g,$$

where b_f, b_g denote the generators of the $H^{2n}(X_{f\vee g};\mathbb{Z})$. It follows by definition of the Hopf invariant of $f+g$ and linearity of the pullback that:

$$\begin{aligned}
H(f+g) b_{f+g} = a_{f+g}^2 = q^*(a_{f\vee g}^2) &= H(f) q^*(b_f) + H(g) q^*(b_g) \\
&= (H(f) + H(g)) b_{f+g},
\end{aligned}$$

as needed. □

Proposition 4.11. *For any even n, there exists a map $f : S^{2n-1} \to S^n$ with $H(f) = 2$. Hence, for even n, the image of $H(-)\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ always contains a copy of $\mathbb{Z}$ as a direct summand.*

Proof. Given an even integer n, we construct a map $f : S^{2n-1} \to S^n$ as follows: identifying S^n with the quotient $I^n/\partial I^n$ of the unit cube in $\mathbb{R}^n$, we may view the product $S^n \times S^n$ as the quotient of I^{2n} where we make opposite side identifications on $\partial I^n \times I$ and on $I^n \times \partial I^n$. Under this quotient, ∂I^{2n} is homeomorphic to $S^n \vee S^n$. Hence, viewing $S^{2n-1} \subset I^{2n} \subset \mathbb{R}^{2n}$, we may compose the radial projection map $S^{2n-1} \to \partial I^{2n}$ with the restricted quotient map $\partial I^{2n} \to S^n \times S^n$ to obtain a map $S^{2n-1} \to S^n \vee S^n$. Composing further with the fold map $S^n \vee S^n \to S^n$ results in a map:

$$f : S^{2n-1} \to S^n \vee S^n \to S^n.$$

One can then verify that the mapping cone of f is given by the James reduced product $X_f = S^n \times S^n/(x, *) \sim (*, x)$ with respect to the basepoint $* \in S^n$. Now, consider the projection map $q : S^n \times S^n \to X_f$, inducing a map in cohomology $q^* : H^*(X_f; \mathbb{Z}) \to H^*(S^n \times S^n; \mathbb{Z})$. By the Künneth formula, we know that

$$H^*(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}_{(0)} \oplus \mathbb{Z}^2_{(n)} \oplus \mathbb{Z}_{(2n)}$$

has two generators a_1, a_2 in degree n and one generator b_1 in degree $2n$, with cup product structure given by $a_1^2 = a_2^2 = 0,\ a_1 a_2 = b_1$. Furthermore, q takes each cell of $S^n \times S^n$ homeomorphically onto a cell of X_f. Thus, letting $a \in H^n(X_f)$, $b \in H^{2n}(X_f)$, we have by cellular cohomology that $q^*(a) = a_1 + a_2$, $q^*(b) = b_1$, so that:

$$\begin{aligned} H(f)a_1a_2 = H(f)b_1 = H(f)q^*(b) &= q^*(a^2) \\ &= (a_1 + a_2)^2 = a_1^2 + a_1a_2 + a_2a_1 + a_2^2 = 2a_1a_2, \end{aligned}$$

since n is even, so that $H(f) = 2$, as needed. The second part of the proposition follows from the fact that $H(-) : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a group homomorphism. □

4.3. **Reduction to an Ext Computation.** We now start working towards a resolution of the Hopf invariant one problem. The reader acquainted with elements of stable homotopy theory may take the initiative of probing the situation by studying the structure of the cohomology ring $H^*(X_f; \mathbb{Z}/2)$ defined in Construction 4.6 as a module over the mod 2 Steenrod algebra $\mathcal{A}$. This approach leads to the following realization: in the notation of Construction 4.6, if the map f has Hopf invariant one, then we get that

$$Sq^n(a) = a \cup a = b.$$

It follows from the fact that there are no non-trivial cohomology classes in $H^i(X_f; \mathbb{Z}/2)$ for $n < i < 2n$ that the element $Sq^n \in \mathcal{A}$ must be *indecomposable*. But the indecomposable elements of $\mathcal{A}$ consist precisely of Steenrod squares of the form Sq^{2^k} for some k, whence we get the following:

Theorem 4.12. *If a map $f : S^{2n-1} \to S^n$ has Hopf invariant one then n must be a power of two.*

Restricting the range of possible values of n any further becomes a very difficult problem if one continues to work exclusively with singular cohomology. In

1960, Adams published a complex proof along those lines involving secondary cohomology operations and spectral sequence arguments [1]. On the other hand, it is possible to view the problem through the lens of topological K-theory by means of the Chern character, in which case the use of Adams operations makes the problem much more tractable. This approach was first presented in a much shorter paper of Adams and Atiyah in 1964 [3]. In this section, inspired by the treatment given in Chapter 5 of Dugger [9], we aim to present the K-theoretic approach in slightly different settings leading to a more general problem in homological algebra.

Having verified in Construction 4.6 that the Hopf invariant one problem is only interesting for maps into even dimensional spheres, we assume from now on that n is even and write $n = 2r$. Let $S^{2n-1} \xrightarrow{f} S^n \to X_f$ be the cofiber sequence associated with a given map $f : S^{2n-1} \to S^n$. Then the associated Puppe sequence

$$S^{4r-1} \longrightarrow S^{2r} \longrightarrow X_f \longrightarrow S^{4r} \longrightarrow S^{2r+1} \longrightarrow \Sigma X_f \longrightarrow ...$$

induces a long exact sequence of reduced K-groups:

$$... \longrightarrow \tilde{K}^0(S^{2r+1}) \longrightarrow \tilde{K}^0(S^{4r}) \longrightarrow \tilde{K}^0(X_f) \longrightarrow \tilde{K}^0(S^{2r}) \longrightarrow \tilde{K}^0(S^{4r-1}).$$

Now, we saw in Chapter 2 that $\tilde{K}^0$ vanishes on odd-dimensional spheres, so that we obtain a short exact sequence:

$$0 \longrightarrow \tilde{K}^0(S^{4r}) \longrightarrow \tilde{K}^0(X_f) \longrightarrow \tilde{K}^0(S^{2r}) \longrightarrow 0. \tag{4.13}$$

The above SES may in fact be interpreted in the category of modules over an appropriate ring. Indeed, if we let $\mathcal{B} := \mathbb{Z}[\psi^2, \psi^3, \psi^5, ...]$ denote the monoid ring over $\mathbb{Z}$ generated by the Adams operations subject to the relations of Theorem 3.3 (iv), we see that each K-group $\tilde{K}^0(X)$ comes equipped with a $\mathcal{B}$-module structure as a result of the fact that the ψ^k's are group homomorphisms. Then, naturality of the Adams operations implies that the maps in Equation 4.13 are valid $\mathcal{B}$-module homomorphisms.

Let $\mathbb{Z}(r)$ denote the $\mathcal{B}$-module consisting of a copy of $\mathbb{Z}$ generated by an element x, with the property that $\psi^k(x) = k^r x$ for all $k \geq 0$. Then by Theorem 3.3 (vi), $\tilde{K}^0(S^{2r}) = \mathbb{Z}(r)$ as $\mathcal{B}$-modules. Hence $\tilde{K}^0(X_f)$ fits into a SES of $\mathcal{B}$-modules:

$$0 \longrightarrow \mathbb{Z}(2r) \longrightarrow \tilde{K}^0(X_f) \longrightarrow \mathbb{Z}(r) \longrightarrow 0.$$

By the usual correspondence between $\mathrm{Ext}^1_{\mathcal{B}}(A, B)$ and isomorphism classes of extensions of B by A, it follows that we may associate to $\tilde{K}^0(X_f)$ a unique element of the group $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(2r), \mathbb{Z}(r))$.

We are thus naturally led to the slightly more general problem of computing the element of $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(s), \mathbb{Z}(r))$ associated to a SES of $\mathcal{B}$-modules

$$0 \longrightarrow \mathbb{Z}(s) \xrightarrow{i} X \xrightarrow{j} \mathbb{Z}(r) \longrightarrow 0. \tag{4.14}$$

In a later discussion, we will clarify the utility of this generalization in the context of stable homotopy theory. For now, we start an investigation of this problem which will rather quickly lead us to a resolution of the Hopf invariant one problem.

Choose generators $b \in \mathbb{Z}(s)$, $\tilde{a} \in \mathbb{Z}(r)$, and let $a \in X$ be a preimage of $\tilde{a}$ under the surjective map j. For simplicity of notation, we identify b with its image in X under the injective map i. Let $k \geq 0$, and observe that by naturality of ψ^k we have that

$$j(\psi^k a) = \psi^k \tilde{a} = k^r \tilde{a} = j(k^r a),$$

hence by exactness it follows that $\psi^k a - k^r a = P_k b$, i.e. that

$$\psi^k a = k^r a + P_k b$$

for some integer $P_k \in \mathbb{Z}$. In particular, the structure of X as a $\mathcal{B}$-module is fully determined by a tuple of integers $(P_2, P_3, ...)$ defined via the above relations. By Theorem 3.3 (vi), we have that $\psi^k \psi^l(a) = \psi^l \psi^k(a)$ for all $k, l \geq 0$. Now, a quick computation yields:

$$\begin{aligned} \psi^k \psi^l(a) &= \psi^k(l^r a + P_l b) = l^r \psi^k(a) + P_l \psi_k(b) \\ &= l^r(k^r a + P_k b) + P_l k^s b, \end{aligned}$$

$$\begin{aligned} \psi^l \psi^k(a) &= \psi^l(k^r a + P_k b) = k^r \psi^l(a) + P_k \psi^l(b) \\ &= k^r(l^r a + P_l b) + P_k l^s b. \end{aligned}$$

Equating the two terms, we find that

$$(k^r - k^s)P_l = (l^r - l^s)P_k.$$

In particular, setting $k = 2$, we see that specifying the value of P_2 determines the entire tuple via the relations

$$P_l = \frac{l^r - l^s}{2^r - 2^s} P_2. \tag{4.15}$$

It follows that a valid $\mathcal{B}$-module structure on X as in Equation 4.14 is determined by any choice of integer $P_2 \in Z_{r,s}$, where we define

$$Z_{r,s} := \{P \in \mathbb{Z} \mid \frac{l^r - l^s}{2^r - 2^s} P \in \mathbb{Z} \text{ for all } l\}.$$

Note that $Z_{r,s}$ is an ideal of $\mathbb{Z}$, non-empty since $2^r - 2^s \in Z_{r,s}$. We thus get a surjective mapping

$$\varphi : Z_{r,s} \longrightarrow \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(s), \mathbb{Z}(r)) ,$$

obtained by sending an element $P \in Z_{r,s}$ to the $\mathcal{B}$-module X_P fitting in a SES as in Equation 4.14 with $\mathcal{B}$-module structure given by the tuple $(P, \frac{3^r - 3^s}{2^r - 2^s} P, ...)$.

Proposition 4.16. *The map $\varphi : Z_{r,s} \to Ext^1_{\mathcal{B}}(\mathbb{Z}(s), \mathbb{Z}(r))$ defined above is a group homomorphism.*

Proof. First recall that the abelian group structure on $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(s), \mathbb{Z}(r))$ viewed as isomorphism classes of extensions is given by the Baer sum operation: starting with two short exact sequences of $\mathcal{B}$-modules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

$$0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C \to 0,$$

we first construct $B \times_C B'$ from the pullback square

$$\begin{array}{ccc} B \times_C B' & \longrightarrow & B \\ \downarrow & & \downarrow g \\ B' & \xrightarrow[g']{} & C, \end{array}$$

then define the Baer sum to be the $\mathcal{B}$-module $Y_{B,B'}$ obtained as the coequalizer of the diagram:

$$A \underset{(0,f')}{\overset{(f,0)}{\rightrightarrows}} B \times_C B' \longrightarrow X_{B,B'}.$$

Concretely, we then have that:

$$X_{B,B'} = \{(b, b') \in B \oplus B' \mid g(b) = g'(b')\} / \big\{(f(a), -f'(a)) \mid a \in A\big\}.$$

In particular, under the maps

$$A \xrightarrow{\bar{f}} X_{B,B'}, a \mapsto (f(a), 0) = (0, f'(a')),$$

$$X_{B,B'} \xrightarrow{\bar{g}} C, (b, b') \mapsto g(b) = g'(b'),$$

$X_{B,B'}$ fits into the SES:

$$0 \longrightarrow A \xrightarrow{\bar{f}} X_{B,B'} \xrightarrow{\bar{g}} C \longrightarrow 0$$

In our settings, we are interested in the Baer sum of the short exact sequences

$$0 \longrightarrow \mathbb{Z}(s) \xrightarrow{i_P} X_P \xrightarrow{j_P} \mathbb{Z}(r) \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}(s) \xrightarrow{i_Q} X_Q \xrightarrow{j_Q} \mathbb{Z}(r) \longrightarrow 0,$$

corresponding to the $\mathcal{B}$-modules X_P, X_Q as defined above, for given elements $P, Q \in Z_{r,s}$. Denote this extension by $X_{P,Q}$, so that we have a short exact sequence:

$$0 \longrightarrow A \xrightarrow{\bar{i_P}} X_{P,Q} \xrightarrow{\bar{j_P}} C \longrightarrow 0\ .$$

Then as a set, we have that:

$$X_{P,Q} = \{(x, y) \in X_P \oplus X_Q \mid j_P(x) = j_Q(y)\} / \big\{(i_P(x), -i_Q(x)) \mid x \in \mathbb{Z}(s)\big\}.$$

In particular, we may look at the class of the element (a_P, a_Q), belonging to $X_{P,Q}$ since $j_P(a_P) = j_Q(a_Q) = \tilde{a}$ by construction. The isomorphism class of $X_{P,Q}$ in $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(s), \mathbb{Z}(r))$ is then determined by the integer N such that $\psi^2(a_P, a_Q) = 2^r(a_P, a_Q) + N(b, 0)$. We verify that:

$$\begin{aligned} \psi^2(a_P, a_Q) &= (\psi^2(a_P), \psi^2(a_Q)) = (2^r a_P + Pb, 2^r a_Q + Qb) \\ &= 2^r(a_P, 0) + 2^r(0, a_Q) + P(b, 0) + Q(0, b) \\ &= 2^r(a_P, a_Q) + (P + Q)(b, 0), \end{aligned}$$

since $(0, b) = (b, 0)$ in $X_{P,Q}$. It then readily follows that $X_{P,Q} \cong X_{P+Q}$, as needed. $\square$

The work we have done so far actually suffices to resolve the Hopf invariant one problem, and so we choose to halt this slightly more general discussion and conclude the proof of the Hopf invariant one problem. It is worth remarking that the Ext computation in the general case is not only of algebraic but also of topological interest, in that it allows us to learn something about the stable homotopy groups of spheres. Indeed, by a similar reasoning as above, for any $k \in \mathbb{N}$, the mapping cone of a map $f : S^{2r+2k-1} \to S^{2r}$ fits into a SES of $\mathcal{B}$-modules:

$$0 \longrightarrow \tilde{K}^0(S^{2r+2k}) \longrightarrow \tilde{K}^0(X_f) \longrightarrow \tilde{K}^0(S^{2r}) \longrightarrow 0,$$

and hence corresponds to an element of $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r+k), \mathbb{Z}(r))$. We thus obtain a group homomorphism:

$$A(-) : \pi_{2k-1+2r}(S^{2r}) \to \mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(r+k), \mathbb{Z}(r)).$$

Acquiring more information about the target group and the kernel of this map can thus provide insight into the structure of the stable homotopy group $\pi_{2k-1}(\mathbb{S})$. For a more in-depth discussion of the general computation of $\mathrm{Ext}^1_{\mathcal{B}}(\mathbb{Z}(s), \mathbb{Z}(r))$, we refer the interested reader to Section 31 of Dugger [9].

Let us come back to the SES of $\mathcal{B}$-modules associated with a given map $f\colon S^{2n-1} \to S^n$:

$$0 \longrightarrow \mathbb{Z}(2r) \xrightarrow{\ i\ } \tilde{K}^0(X_f) \xrightarrow{\ j\ } \mathbb{Z}(r) \longrightarrow 0. \tag{4.17}$$

As before, choose generators $b \in \mathbb{Z}(2r)$, $\tilde{a} \in \mathbb{Z}^r$, identify $ib \equiv b$ in $\tilde{K}^0(X_f)$, and let $a \in \tilde{K}^0(X_f)$ be such that $j(a) = \tilde{a}$. Then from the ring structure of $\tilde{K}^0(S^{2r})$ described in Section 2.2, we see that $j(a^2) = (j(a))^2 = 0$, so that by exactness $a^2 = hb$ for some integer $h \in \mathbb{Z}$. Let us verify that this integer is well-defined up to sign. Injectivity of i ensures that the resulting $b \in \tilde{K}^0(X_f)$ is unique up to sign. Now, suppose we had chosen another $a' \in \tilde{K}^0(X_f)$ such that $j(a') = \tilde{a}$. Then by exactness we have that $a' = a + \eta b$ for some $\eta \in \mathbb{Z}$, so that

$$a'^2 = a^2 + 2\eta ab + \eta^2 b^2 = hb + 2\eta ab,$$

by graded-commutativity and the fact that b^2 vanishes in $\tilde{K}^0(S^{4r})$. Further, we note that $j(ab) = j(a)ji(b) = 0$, hence by exactness $ab = \nu b$ for some $\nu \in \mathbb{Z}$. Applying a to both sides gives that

$$0 = hb^2 = a^2 b = \nu ab,$$

hence $\nu = 0$ and $a'^2 = hb$, as needed.

We now claim that $h = H(f)$ corresponds to the Hopf invariant of f. To see this, we have recourse to the Chern character introduced in Section 4.4. Recall that the Chern character provides a natural transformation $Ch : \tilde{K}^0(-) \to \tilde{H}^{\mathrm{ev}}(-;\mathbb{Q})$. We may apply the Chern character to the SES in Equation 4.13, so that by naturality, we obtain a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}^0(S^{4r}) & \xrightarrow{\ i\ } & \tilde{K}^0(X_f) & \xrightarrow{\ j\ } & \tilde{K}^0(S^{2r}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Ch} & & \downarrow{\scriptstyle Ch} & & \downarrow{\scriptstyle Ch} & & \\
0 & \longrightarrow & \tilde{H}^{\mathrm{ev}}(S^{4r};\mathbb{Q}) & \xrightarrow{\ i\ } & \tilde{H}^{\mathrm{ev}}(X_f;\mathbb{Q}) & \xrightarrow{\ j\ } & \tilde{H}^{\mathrm{ev}}(S^{2r};\mathbb{Q}) & \longrightarrow & 0
\end{array}$$

Now, by Theorem 3.13 (iv), the Chern character is injective on even dimensional spheres, and takes the generator of $\tilde{K}^0(S^{2r})$ to the generator of $\tilde{H}^{2r}(S^{2r};\mathbb{Z}) \subset \tilde{H}^{\mathrm{ev}}(S^{2r};\mathbb{Q})$. Hence the 5-lemma applied to the factorization of the above diagram through $\tilde{H}^{ev}(-;\mathbb{Z})$ implies that the middle vertical map induces an isomorphism $Ch : \tilde{K}^0(X_f) \xrightarrow{\simeq} \tilde{H}^{ev}(X_f;\mathbb{Z})$. It follows by commutativity that $Ch(b) \in H^{4r}(X_f;\mathbb{Z})$, $Ch(a) \in H^{2r}(X_f;\mathbb{Z})$ are valid generators, where $b, a \in \tilde{K}^0(X_f)$ are as above. Hence by definition of the Hopf invariant as in Construction 4.6, we have that $H(f)$ satisfies the relation $Ch(a)^2 = H(f)Ch(b)$, and since the Chern character is a ring homomorphism we get that

$$H(f)Ch(b) = Ch(a)^2 = Ch(a^2) = Ch(hb) = hCh(b),$$

whence it follows that $h = H(f)$, as claimed.

Therefore, if the map f has Hopf invariant one, we get that

$$\psi^2(a) \equiv a^2 = hb = H(f)b \equiv b \pmod 2.$$

Next, if we let $P \in Z_{2r,r}$ be the integer determining the $\mathcal{B}$-module structure of $\tilde{K}^0(X_f)$ via $\psi^2(a) = 2^r a + Pb$, it follows that

$$Pb \equiv \psi^2(a) \equiv b \pmod 2,$$

hence P must be odd. Then, Equation 4.15 implies that for all odd primes l, we must have:

$$\frac{l^{2r} - l^r}{2^{2r} - 2^r} P = \frac{l^r}{2^r} \frac{l^r - 1}{2^r - 1} P \in \mathbb{Z},$$

and thus necessarily

$$2^r \mid l^r - 1$$

for all odd primes l. Specializing to the case $l = 3$ suffices to complete the proof of Theorem 4.5 using elementary modular arithmetic:

Lemma 4.18. *Suppose $r \in \mathbb{Z}_{\geq 0}$ satisfies $2^r \mid 3^r - 1$. Then $r \in \{0, 1, 2, 4\}$.*

Proof. Write $r = 2^l m$ for m odd. We proceed by induction on l and show that the highest power of 2 dividing $3^r - 1$ is 2^{l+2} for $l \geq 1$, and 2 for $l = 0$. From there, we see that $2^r \mid 3^r - 1$ implies that

$$2^r \leq 2^{l+2} \leq 4 \cdot 2^l m = 4r,$$

and hence that $r \leq 4$. The result then follows by manual verification of the cases $r = 0, 1, 2, 3, 4$.

So first suppose $l = 0$, so that $r = m$ is odd. Note that $2 \mid 3 - 1$, so 2 is a valid candidate. Now, write $m = 2k + 1$ for some k. Then we find that $3^m - 1 \equiv (3^2)^k \cdot 3 - 1 \equiv 2 \pmod 4$, hence 2^2 cannot divide $3^m - 1$. Next, for $l = 1$, $r = 2m$, so that $3^r - 1 = (3^m - 1)(3^m + 1)$. The first factor is divisible by at most 2 by the above reasoning; next, we compute that, writing $m = 2k + 1$, we have $3^m + 1 = (3^2)^k \cdot 3 + 1 \equiv 4 \pmod 8$, so that the second factor is divisible by at most 2^2, and therefore the highest power of 2 that can divide $3^r - 1$ is 2^3, as needed. For the general case, we have that

$$3^{2^{l+1} m} - 1 = (3^{2^l m} - 1)(3^{2^l m} + 1).$$

The first factor is divisible by at most 2^{l+2} by induction hypothesis, and we compute that $3^{2^l m} + 1 \equiv ((3^2)^{2^{l-1}})^m + 1 \equiv 2 \pmod 4$, hence the second factor is divisible by at most 2, and therefore the highest power of 2 that can divide $3^r - 1$ is 2^{l+3}, completing the proof. □

Appendix A. Constructing Classifying Spaces at the Level of Categories

The purpose of this appendix is to describe a remarkably simple conceptual formulation of a model for the classifying space BG of principal G-bundles for any topological group G, which we introduced in Chapter 1. The approach we adopt is close in spirit to the *bar construction*, and we refer the interested reader to Milnor [16] for a treatment via the *join construction*.

Throughout this exposition, we shall make use of simplicial sets as a bridge allowing us to transfer data from the settings of small categories to the settings of spaces. This translation process is allowed by the existence of functors fitting into the diagram:

$$\mathbf{Cat} \underset{\Pi}{\overset{\mathscr{N}}{\rightleftarrows}} \mathbf{sSet} \underset{\mathscr{S}}{\overset{|.|}{\rightleftarrows}} \mathbf{Top}, \tag{A.1}$$

subject to the adjunctions:

$$\mathbf{Cat}(\Pi K, \mathscr{C}) \cong \mathbf{sSet}(K, \mathscr{N}\mathscr{C}),$$

$$\mathbf{Top}(|K|, X) \cong \mathbf{sSet}(K, \mathscr{S}X).$$

In the above, $\mathscr{N}, \Pi, |.|, \mathscr{S}$ refer to the *nerve, fundamental category, geometric realization* and *singular* functors, respectively, A more in-depth discussion of these adjunctions may be found in [14]. We shall only consider the forward direction, i.e. going from categories to spaces under the composite $|\mathscr{N}|$. However, we will occasionally rely on the existence of these adjunction pairs to obtain desirable properties of the functors under consideration.

A.1. Refresher on Simplicial Sets, Nerves and Geometric Realizations.

For completeness, we provide a streamlined treatment of the necessary notions about simplicial sets. For building intuition about these objects, we refer the reader to the introductory treatment by Friedman [10].

One impetus for considering simplicial sets is to provide a flexible yet tractable combinatorial framework in which to do homotopy theory. A concise way to arrive to this framework is by first defining the category Δ whose objects are the totally ordered finite sets $[n] = \{0, ..., n\}$, and whose morphisms are given by monotonic increasing set maps $\mu : [n] \to [m]$, i.e. those satisfying $i < j \implies \mu(i) \leq \mu(j)$. This is a small category whose morphism set is generated by two families of monotonic maps called the *face* and *degeneracy* maps, denoted respectively by $\delta_i : [n] \to [n+1]$ and $\sigma_i : [n+1] \to [n]$, indexed over all $n \geq 0$ and $0 \leq i \leq n$. The ith face map in degree n is the inclusion of $[n]$ into $[n+1]$ which skips the element $i \in [n+1]$, and the ith degeneracy map in degree n is the projection of $[n+1]$ onto $[n]$ which sends both elements $i, (i+1) \in [n+1]$ to the element $i \in [n]$ and subsequently sends each $j > i$ in $[n+1]$ to $(j-1) \in [n]$. Further, these maps satisfy certain relations under composition which we will only need in the formulation of some definitions:

$$\delta_i\delta_j = \delta_{j-1}\delta_i \ , \ i < j;$$
$$\delta_i\sigma_j = \begin{cases} \sigma_{j-1}\delta_i \ , \ i < j \\ id \ , \ i = j \text{ or } i = (j+1) \\ \sigma_j\delta_{i-1} \ , \ (j+1) < i \end{cases} \qquad . \tag{A.2}$$

One can then show that every monotonically increasing map can be expressed as a composite of face maps and degeneracy maps.

Now, we define a *simplicial set* K to be a contravariant functor from the category Δ to the category of sets, i.e. $K : \Delta^{op} \to \mathbf{Set}$. On objects, such a functor corresponds to a choice of set $K_n = K[n]$ for each $n \geq 0$, to which we refer as the *n-simplices* of K. On morphisms, by the above discussion, it suffices to specify associated set maps $d_i : K_{n+1} \to K_n$ for each face map $\delta_i : [n] \to [n+1]$ and $s_i : K_n \to K_{n+1}$ for each degeneracy map $\sigma_i : [n+1] \to [n]$ for all $n \geq 0$, $0 \leq i \leq n$, in such a way that the relations in Equation A.2 are satisfied in the image. The action of K on the remaining morphisms in Δ is then determined by functoriality. We call an n-simplex $k \in K_n$ *degenerate* if it lies in the image of a degeneracy map; otherwise, we say k is a *non-degenerate* n-simplex.

We may then consider the category **sSet** with simplicial sets as objects and natural transformations between them as morphisms. Once again, it is enough to require an assignment $(f_n : K_n \to L_n)_{n\geq 0}$ to commute with face maps and degeneracy maps in order for it to be a valid natural transformation.

There is a systematic way to associate in a functorial way a simplicial set $\mathcal{N}\mathscr{C}$ to any small category $\mathscr{C}$. Composing this map with the geometric realization functor will allow us to associate a *topological space* to a given small category $\mathscr{C}$.

Construction A.3. Let $\mathscr{C}$ be a small category. Define the *nerve* of $\mathscr{C}$ to be the simplicial set whose 0-simplices are given by the set $ob\mathscr{C}$, and whose n-simplices are given by n-tuples of composable morphisms in $\mathscr{C}$. That is, to every chain of morphisms $a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_3 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_n$ in $\mathscr{C}$, we associate an n-simplex $(f_1|...|f_n) \in \mathcal{N}\mathscr{C}_n$. Next, define the face maps $d_i : \mathcal{N}\mathscr{C}_n \to \mathcal{N}\mathscr{C}_{n-1}$ and degeneracy maps $s_i : \mathcal{N}\mathscr{C}_n \to \mathcal{N}\mathscr{C}_{n+1}$ as follows:

$$s_i : (f_1|...|f_n) \mapsto (f_1|...|f_i|id|f_{i+1}|...|f_n),$$
$$d_i : (f_1|...|f_n) \mapsto \begin{cases} (f_2|...|f_n) \ , \ i = 0 \\ (f_1|...|f_{i+1}f_i|...|f_n) \ , \ 1 \leq i \leq (n-1) \\ (f_1|...|f_{n-1}) \ , \ i = n. \end{cases}$$

In the above definition for s_i, id refers to the identity morphism on domain of f_{i+1} (or equivalently the codomain of f_i). It can be verified that these maps satisfy the relations in Equation A.2, and hence that they generate a valid simplicial set structure on $\mathcal{N}\mathscr{C}$.

Further, given a functor $F : \mathscr{C} \to \mathscr{D}$ between categories, we may obtain an induced simplicial map $\mathcal{N}F : \mathcal{N}\mathscr{C} \to \mathcal{N}\mathscr{D}$ given in degree n by the assignment:

$$\mathcal{N}F_n : (f_1|...|f_n) \mapsto (F(f_1)|...|F(f_n)).$$

One may check that functoriality of F makes the map $\mathcal{N}F$ commute with face and degeneracy maps, so that $\mathcal{N}F$ defines a valid simplicial map. It readily appears that the resulting assignment $\mathcal{N} : \mathbf{Cat} \to \mathbf{sSet}$ at the level of categories is in fact a covariant functor, as desired.

Remark A.4. Specializing to a one object groupoid category (as we shall do in the next section), the simplicial set structure of the corresponding nerve corresponds to what is known as the *bar construction*, a closely related approach to achieving the construction of classifying spaces for principal bundles.

Next, we come to the second step indicated in the diagram in Equation A.1. Namely, we shall specify a way to functorially map simplicial sets to topological spaces. The construction is very similar to the analogous geometric realization functor for simplicial complexes, with which the reader may already be familiar.

Construction A.5. For each $n \geq 0$, denote by Δ_n^t the standard topological n-simplex, i.e. the subset of $\mathbb{R}^{n+1}$ characterized as the convex hull of the standard orthonormal basis in $\mathbb{R}^{n+1}$. That is,

$$\Delta_n^t := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \ \sum_{i=0}^{n} t_i = 1\}.$$

Together, these objects fit into a covariant functor $\Delta_*^t : \Delta \to \mathbf{Top}$ taking the ith face map $\delta_i : [n] \to [n+1]$ to the map $\delta_i : \Delta_n^t \to \Delta_{n+1}^t$ given by $\delta_i : (t_0, ..., t_n) \mapsto (t_0, ..., t_{i-1}, 0, t_i, ..., t_n)$, resp. the ith degeneracy map $\sigma_i : [n] \to [n-1]$ to the map $\sigma_i : \Delta_n^t \to \Delta_{n-1}^t$ given by $\sigma_i : (t_0, ..., t_n) \mapsto (t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+1}, ..., t_n)$. Now, let K be a simplicial set, with face and degeneracy maps denoted by d_i, resp. s_i. Viewing each K_n as a topological space endowed with the discrete topology, we define the *geometric realization* of K to be the space $|K|$ given by:

$$|K| := \coprod_{n \geq 0} K_n \times \Delta_n^t / (\sim),$$

where $(d_i k, v) \sim (k, \delta_i v)$ and $(s_i k, v) \sim (k, \sigma_i v)$ wherever it makes sense. In particular, thinking of the discrete set K_n as an indexing set for as many copies of the topological n-simplex Δ_n^t as there are elements in K_n, this equivalence relation ensures that every equivalence class in TK contains a unique point lying in the interior of a topological n-simplex indexed by a non-degenerate element of K_n. Further, it can be verified that a simplicial map $f : K \to L$ induces a continuous map $|f| : |K| \to |L|$ under the assignment $(|f|)[(k, x)] := [f(k), x]$. This assignment turns geometric realization into a covariant functor $|.| : \mathbf{sSet} \to \mathbf{Top}$.

A.2. **Construction at the Level of Categories.** We introduce the functor between categories which we claim will correspond to a universal principal G-bundle after application of the composite functor $|\mathcal{N}| : \mathbf{Cat} \to \mathbf{Top}$. The striking simplicity of the construction at this level will hopefully help motivate this rather abstract approach to obtaining classifying spaces.

Given a group G, we may consider the associated one object groupoid category. This is the category G with a unique object $\{*\}$ and unique morphism set the underlying set of G, with composition law given by the group structure of G and identity morphism the arrow $* \xrightarrow{id} *$ corresponding to the identity element of G. Next, define a small category $\mathscr{E}$ as follows: $\mathscr{E}$ has the underlying elements of G as objects, and for each pair of objects g, h in $\mathscr{E}$, the hom set $\mathscr{E}(g, h)$ is a one-point set. Observe that by uniqueness of the morphisms in $\mathscr{E}$, the data of an n-simplex $(g_1 \to g_2 \to ... \to g_n \to g_{n+1})$ in the associated nerve $\mathcal{N}\mathscr{E}$ may be captured by an $(n+1)$-tuple $(g_1|...|g_{n+1})$, where the components denote elements of the underlying category, rather than the conventional n-tuple of morphisms introduced in the previous section.

Now, define a functor $F : \mathscr{E} \to G$ acting on objects of $\mathscr{E}$ as the one point projection to the unique object $*$ in G, and sending the unique morphism in $\mathscr{E}(g, h)$ to the morphism hg^{-1} in $G(*, *)$ for each pair of objects g, h in $\mathscr{E}$. Then, the composite functor $|\mathcal{N}| : \mathbf{Cat} \to \mathbf{Top}$ takes F to a continuous map between spaces

$$|\mathcal{N}F| : |\mathcal{N}\mathscr{E}| \to |\mathcal{N}G|.$$

The remainder of these notes will be devoted to studying this map and the intermediary simplicial map $\mathcal{N}F : \mathcal{N}\mathscr{E} \to \mathcal{N}G$, with the intention to show that $|\mathcal{N}F|$ is a universal principal G-bundle and hence that $|\mathcal{N}G|$ is a valid model for BG.

A.3. **Construction at the Level of Simplicial Sets.** The nerve functor $\mathcal{N}$: $\mathbf{Cat} \to \mathbf{sSet}$ assigns to the functor $F : \mathscr{E} \to G$ a simplicial map $\mathcal{N}F : \mathcal{N}\mathscr{E} \to \mathcal{N}G$. The behavior of this map on n-simplices is given component-wise by $\mathcal{N}F$: $(g_1|...|g_{n+1}) \mapsto (g_2g_1^{-1}|...|g_{n+1}g_n^{-1})$. Now, we may ask about the fiber under $\mathcal{N}F$ of a given n-simplex $(g_1|...|g_n)$ in $\mathcal{N}G$. Since $\mathcal{N}F$ is a simplicial map, any element in the pre-image of $(g_1|...|g_n)$ must lie in $\mathcal{N}\mathscr{E}_n$. So suppose $(x_1|...|x_{n+1}) \in \mathcal{N}\mathscr{E}_n$ satisfies

$$(x_2x_1^{-1}|...|x_{n+1}x_n^{-1}) = (g_1|...|g_{n+1}).$$

Then we get equations $x_2 = g_1x_1$, $x_3 = g_2g_1x_1$, ..., $x_{n+1} = g_n...g_1x_1$ which fully determine the entries $x_2, ..., x_{n+1}$ once x_1 has been given. Now, notice that any value $h \in G$ for $x_1 \in ob\mathscr{E} = G$ determines a distinct element $(h|g_1h|...|g_n...g_1h)$ in the fiber of $(g_1|...|g_n)$ under $\mathcal{N}F$. Thus the fiber of $(g_1|...|g_n)$ looks precisely like a copy of G. Further, any such fiber can be given a free right G-action under the assignment:

$$(\mathcal{N}F)^{-1}(g_1|...|g_n) \times G \to (\mathcal{N}F)^{-1}(g_1|...|g_n)$$
$$((h|g_1h|...|g_n...g_1h), k) \mapsto (hk|g_1(hk)|...|g_n...g_1(hk)).$$

We highlight one more restriction on elements in the preimage of a non-degenerate n-simplex. This will be helpful later in studying the fibers at the level of geometric realizations:

Lemma A.6. *The simplicial map $\mathcal{N}F$ takes non-degenerate simplices to non-degenerate simplices.*

Proof. Suppose an n-simplex in the image of $\mathcal{N}F$ is degenerate. Then without loss of generality, we may assume it contains an identity morphism $* \xrightarrow{e} *$ at the ith position and is the image of the n-simplex $(g_1|...|g_{n+1})$ in $\mathcal{N}\mathscr{E}$. Hence, we get that $g_{i+1}g_i^{-1} = e$, i.e. $g_i = g_{i+1}$, so that the ith morphism in $(g_1|...|g_{n+1})$ corresponds to the unique identity morphism $g_i \xrightarrow{id} g_i$, whence $(g_1|...|g_{n+1})$ was degenerate to begin with. □

A.4. **From Adjunctions to Homotopy Equivalences.** We now come to a useful feature of the passage from small categories to spaces via the composite functor $|\mathcal{N}| : \mathbf{Cat} \to \mathbf{Top}$, namely that natural transformations between functors in **Cat** induce homotopic maps at the level of spaces, and that as a result adjoint pairs of functors between categories $\mathscr{C}$ and $\mathscr{D}$ induce homotopy equivalences between the geometric realizations of the corresponding nerves $|\mathcal{N}\mathscr{C}| \simeq |\mathcal{N}\mathscr{D}|$.

To speak of a homotopy between geometric realizations of nerves of categories, it is convenient to formulate an analog of the unit interval at the level of categories. Denote by $\mathcal{I}$ the category with two objects $[0], [1]$ and a unique non-identity morphism $\alpha : [0] \to [1]$. We claim that $|\mathcal{N}\mathcal{I}|$ is homeomorphic to the unit interval $I \subset \mathbb{R}$. Indeed, all but one 1-simplex (α) and two 0-simplices $[0], [1]$ are degenerate, hence upon taking the quotient of $\coprod_n \mathcal{N}\mathcal{I}_n \times \Delta^t_n$ by face and degeneracy maps we are left precisely with a copy of $\Delta^t_1 \cong I$. We proceed to prove:

Proposition A.7. *Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors related by a natural transformation $\eta : F \to G$. Then there exists a homotopy $h : |\mathcal{N}\mathscr{C}| \times I \to |\mathcal{N}\mathscr{D}|$ between the corresponding maps $|\mathcal{N}F|$ and $|\mathcal{N}G|$.*

Proof. Let $\mathcal{I}$ be the category with two objects and one non-identity morphism as above. Consider the product category $\mathscr{C} \times \mathcal{I}$. Then $\mathscr{C} \times \mathcal{I}$ can be thought of as two disjoint copies of $\mathscr{C}$ linked to each other by morphisms of the form (f, α), where α is the unique non-trivial morphism of $\mathcal{I}$ and f is some morphism in $\mathscr{C}$. We may then define a functor $\tilde{h} : \mathscr{C} \times \mathcal{I} \to \mathscr{D}$ by letting $\tilde{h}|_{\mathscr{C} \times \{[0]\}} := F$, $h|_{\mathscr{C} \times \{[1]\}} := G$, and setting $\tilde{h}(id_x \times \alpha) := \eta_x$ for each $x \in ob\mathscr{C}$. Finally, we extend functorially the action of $\tilde{h}$ on composite morphisms of the form $(f \times \alpha) = (f \times id_{[1]})(id_x \times \alpha) = (id_y \times \alpha)(f \times id_{[0]})$ for each $f : x \to y$ in $\mathscr{C}$.

From functoriality of F and G and commutativity of the induced diagrams:

$$\begin{array}{ccccccc}
(x,[0]) & \xrightarrow{f \times id_{[0]}} & (y,[0]) & & F(x) & \xrightarrow{\tilde{h}(f \times id_{[0]})} & F(y) \\
{\scriptstyle id_x \times \alpha}\downarrow & \searrow & \downarrow{\scriptstyle id_y \times \alpha} & \longmapsto & {\scriptstyle \eta_x}\downarrow & \searrow & \downarrow{\scriptstyle \eta_y} \\
(x,[1]) & \xrightarrow[f \times id_{[1]}]{} & (y,[1]) & & G(x) & \xrightarrow[\tilde{h}(f \times id_{[1]})]{} & G(y),
\end{array}$$

provided by naturality of η, it follows that the resulting assignment $\tilde{h} : \mathscr{C} \times \mathcal{I} \to \mathscr{D}$ is a well-defined functor.

Now, we have that $|\mathcal{N}(\mathscr{C} \times \mathcal{I})| \cong |\mathcal{N}\mathscr{C}| \times |\mathcal{N}\mathcal{I}| \cong |\mathcal{N}\mathscr{C}| \times I$, so that at the level of spaces $\tilde{h}$ induces a map $h : |\mathcal{N}\mathscr{C}| \times I \to |\mathcal{N}\mathscr{D}|$ satisfying $h|_{|\mathcal{N}\mathscr{C}| \times \{0\}} = |\mathcal{N}F|$ and $h|_{|\mathcal{N}\mathscr{C}| \times \{1\}} = |\mathcal{N}G|$, thereby providing the desired homotopy. □

Corollary A.8. *Suppose there exists an adjoint pair of functors $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$. Then the spaces $|\mathcal{N}\mathscr{C}|$ and $|\mathcal{N}\mathscr{D}|$ are homotopy equivalent.*

Proof. One way to express this adjunction is in the form of two natural transformations $\eta : GF \to Id_{\mathscr{D}}$, $\gamma : FG \to Id_{\mathscr{C}}$, where $Id_{\mathscr{C}}$, $Id_{\mathscr{D}}$ denote the identity functors on the respective categories. Hence, assuming that natural transformations between functors provide homotopies between the associated continuous maps, and using functoriality of the assignment $|.| \circ \mathcal{N} : \mathbf{Cat} \to \mathbf{Top}$, we get homotopies:

$$|\mathcal{N}G| \circ |\mathcal{N}F| \xrightarrow{\simeq} id_{|\mathcal{N}\mathscr{C}|},$$
$$|\mathcal{N}F| \circ |\mathcal{N}G| \xrightarrow{\simeq} id_{|\mathcal{N}\mathscr{D}|}.$$

This corresponds precisely to the data of a homotopy equivalence between the spaces $|\mathcal{N}\mathscr{C}|$ and $|\mathcal{N}\mathscr{D}|$. □

In particular, we get the following result "for free":

Corollary A.9. *Let $\mathscr{C}$ be a category with either an initial object or a terminal object. Then $|\mathcal{N}\mathscr{C}|$ is a contractible space.*

Proof. Assume $\mathscr{C}$ has an initial object $1 \in ob\mathscr{C}$ (a dual argument applies when $\mathscr{C}$ has a terminal object). For all $x \in ob\mathscr{C}$, we have that $\mathscr{C}(1, x)$ is a one-point set. Now, let $\mathbf{1}$ denote the category with one object and one morphism (say with unique object z), and consider the functors $F : \mathscr{C} \to \mathbf{1}$, $l : \mathbf{1} \to \mathscr{C}$, where F projects all objects of $\mathscr{C}$ to z and all morphisms to id_z, and l is the inclusion functor sending z to $1 \in ob\mathscr{C}$ (and id_z to id_1). Since $\mathbf{1}(z, F(x)) = \mathbf{1}(z, z)$ is a one point set for all objects x of $\mathscr{C}$, we get a (trivially natural) bijection

$$\mathbf{1}(z, F(x)) \cong \mathscr{C}(l(z), x)$$

for all $x \in ob\mathscr{C}$.

Thus l and F form a pair of adjoint functors, and it follows that $|\mathcal{N}\mathscr{C}|$ is homotopy equivalent to $|\mathcal{N}\mathbf{1}|$, which is readily seen to be the one point space. Hence $|\mathcal{N}\mathscr{C}|$ is contractible. □

Coming back to our original context, the category $\mathscr{E}$ under consideration has the property that every hom set is a one point set, hence every object of $\mathscr{E}$ is both initial and final. It follows from the above that $|\mathcal{N}\mathscr{E}|$ is a contractible space.

A.5. **Construction at the Level of Spaces.** So far, we have seen that the functor $F : \mathscr{E} \to G$ described in section 3 induces a continuous map $|\mathcal{N}F| : |\mathcal{N}\mathscr{E}| \to |\mathcal{N}G|$ whose domain is contractible. Further, this map is obtained by application of the geometric realization functor from a simplicial map $\mathcal{N}F : \mathcal{N}\mathscr{E} \to \mathcal{N}G$ with the property that each fiber is a free G-space. It remains to exploit the latter fact to verify that the map $|\mathcal{N}F|$ is indeed a principal G-bundle, after which the sought-after result will follow from Theorem 1.9.

Given an element $[x]$ of $|\mathcal{N}G|$, find the unique representative x which lies in the interior of a non-degenerate topological n-simplex, labeled by some element $k \in \mathcal{N}G_n$. Because $\mathcal{N}F$ takes non-degenerate simplices to non-degenerate simplices by Lemma A.6 and restricts to a homeomorphism on the interior of topological n-simplices, every element in the preimage of x under $|\mathcal{N}F|$ must also lie in the interior of a non-degenerate n-simplex. The distinct n-simplices forming a copy of G in the preimage of k under $\mathcal{N}F$ then pick out as many distinct points in the corresponding topological n-simplices mapping down to x, on which G acts freely on the right. Hence each fiber under the map $|\mathcal{N}F|$ appears to be a free right G-space. The verification of local triviality is more delicate, and we refer to Theorem 8.2 of May [15] for a proof.

Finally, we show that if G is an abelian group, then $|\mathcal{N}G|$ can be given the structure of a topological group. Rather than exhibiting a topological group structure at the level of spaces, we resort to the following observation:

Lemma A.10. *Let $\mathcal{C}, \mathscr{D}$ be categories with finite products and terminal objects, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor which commutes with taking finite products and terminal objects. Then F takes group objects in $\mathscr{C}$ to group objects in $\mathscr{D}$.*

The verification is just a matter of running through the definitions. Now, it can be shown that the geometric realization functor satisfies the hypothesis of the above lemma (see for instance [14] for details). Thus, if we can equip $\mathcal{N}G$ with the structure of a group object at the level of simplicial sets, it will follow that $|\mathcal{N}G|$ is a group object in **Top**, i.e. that it is a topological group.

Construction A.11. Let $\mathcal{N}G$ denote the nerve of the one object groupoid category associated to the abelian group G, as before. Define maps

$$\mu : \mathcal{N}G \times \mathcal{N}G \to \mathcal{N}G$$
$$i : \mathcal{N}G \to \mathcal{N}G$$

given on each degree n by the assignments:

$$\mu_n : ((g_1|...|g_{n+1}), (h_1|...|h_{n+1})) \mapsto (g_1h_1|...|g_{n+1}h_{n+1})$$
$$i_n : (g_1|...|g_{n+1}) \mapsto (g_1^{-1}|...|g_{n+1}^{-1}).$$

We claim that μ and i are valid simplicial maps. It suffices to show that they both commute with faces and degeneracies in every degree. In both cases, commutativity with degeneracies is immediate. However, the fact that G is abelian is essential in showing that either map commutes with face maps. For instance, for a given $n \geq 2$ and a given $1 \leq i \leq n-1$, we get that

$$\begin{aligned}\mu_n d_i((g_1|...|g_{n+1}), (h_1|...|h_{n+1})) &= \mu_n((g_1|...|g_{i+1}g_i|...|g_{n+1}), (h_1|...|h_{i+1}h_i|...|h_{n+1})) \\ &= (g_1h_1|...|g_{i+1}g_ih_{i+1}h_i|...|g_nh_n),\end{aligned}$$

while

$$\begin{aligned}d_i\mu_n((g_1|...|g_{n+1}), (h_1|...|h_{n+1})) &= d_i((g_1h_1|...|g_{n+1}h_{n+1})) \\ &= (g_1h_1|...|g_{i+1}h_{i+1}g_ih_i|...|g_nh_n),\end{aligned}$$

so that since G is abelian the two equations coincide. A similar argument applies to showing that the map $i : \mathcal{N}G \to \mathcal{N}G$ commutes with face maps. It now follows immediately from the underlying group structure of G that μ and i endow $\mathcal{N}G$ with the structure of a group object in **sSet**, as needed.

Remark A.12. The fact that G is abelian is essential to obtaining a topological group structure on $BG = |\mathcal{N}G|$. Indeed, we saw in Section 1.2 that $\pi_1(BG) = G$. Further, the fundamental group functor $\pi_1 : \mathbf{Top} \to \mathbf{Grp}$ preserves finite products and final objects, hence it takes topological groups to group objects in the category **Grp** by Lemma A.10. But group objects in **Grp** are precisely abelian groups, as a result of requiring the multiplication map $\mu : G \times G \to G$ to be a group homomorphism. Thus we see that BG being a topological group implies that $G = \pi_1(BG)$ is an abelian group.

Putting everything together, we have proven the following theorem:

Theorem A.13. *Let G be a discrete group. Then a model for the classifying space BG is provided by the space $|\mathcal{N}G|$, together with the map:*

$$|\mathcal{N}F| : |\mathcal{N}\mathscr{E}| \to |\mathcal{N}G|,$$

where G is the one object groupoid category with hom set G and composition induced by the group structure of G, $\mathscr{E}$ is the category with $ob\mathscr{E} = G$ and hom sets $\mathscr{E}(g,h) = \{*\}$ for all $g, h \in G$, and $F : \mathscr{E} \to G$ is the functor given by the one point projection at the level of objects and the assignment $(g \to h) \mapsto hg^{-1}$ at the level of morphisms. Further, the classifying space BG can be given the structure of a topological group if and only if G is abelian.

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