

BUILDING UP TO THE PONTRYAGIN-THOM THEOREM & COMPUTATION OF $\pi_*(MO)$

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ABSTRACT. This expository paper is intended to provide a careful treatment of the Pontryagin-Thom theorem relating the unoriented cobordism ring to the stable homotopy groups $\pi_*(MO)$ of the Thom spectrum. We then exploit this connection to obtain geometric information about the classification of closed manifolds from a direct computation of $\pi_*(MO)$. Relevant notions from algebraic topology and differential topology such as vector bundles and classifying spaces, spectra, the Steenrod algebra, transversality and cobordism are introduced along the way.

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“Ainsi, le point de vue fécond n’est autre que cet “oeil” qui à la fois nous fait découvrir, et nous fait reconnaître l’unité dans la multiplicité de ce qui est découvert. Et cette unité est véritablement la vie même et le souffle qui relie et anime ces choses multiples.”

- Grothendieck, Récoltes et Semailles

1. INTRODUCTION

The main theorems which we set out to prove in this expository paper will allow us to connect the *unoriented cobordism ring* Ω_*^o , which carries geometric data about the classification of closed smooth manifolds up to the equivalence relation of cobordism, with the homotopy-theoretic *Thom spectrum* MO , whose stable homotopy groups $\pi_*(MO)$ will be shown to be isomorphic to Ω_*^o . Understanding the structure of Ω_*^o gives us some insight into the classification problem for closed manifolds. In particular, knowing the rank of Ω_n^o equates to knowing how many distinct

cobordism classes of closed n -manifolds fail to be realizable as the boundary of an $(n + 1)$ -manifold. For instance, we shall prove in particular that $\Omega_3^o = 0$, hence every closed 3-manifold can be realized as the boundary of some 4-manifold. The full picture will be displayed in Section 9, using precisely the passage into the world of homotopy theory enabled by the *Pontryagin-Thom theorem*, a careful treatment of which is given in Section 8. The two main characters involved, Ω_*^o and MO , are introduced in Section 7.

The essence of the Pontryagin-Thom theorem may be summarized by the following diagram:

$$\begin{array}{ccc}
 \text{Th}(\nu_M) & \longrightarrow & MO(k) \\
 \uparrow & & \uparrow \\
 \nu_M & \longrightarrow & \gamma_k^\infty \\
 \downarrow & & \downarrow \\
 M & \longrightarrow & \text{Gr}_k(\mathbb{R}^\infty),
 \end{array}$$

which delineates a route between an arbitrary closed n -manifold M and the Thom spectrum, a component of which appears at the top right of the ladder. In order to make sense of this diagram, we will need ideas from the theory of classifying spaces and vector bundles as well as some results from differential topology, and so we will take the time to introduce the relevant notions before moving on to the main construction. Section 2 is meant as a short introduction to the theory of principal G -bundles and classifying spaces, with a specialization to vector bundles. Next, Section 3 is devoted to an exposition of relevant results from differential topology. We state without proof standard theorems such as Sard's theorem and the Whitney embedding and approximation theorems, but attempt to give a careful treatment of the notion of transversality which is central to the Pontryagin-Thom construction.

The second objective of this paper is to put the Pontryagin-Thom theorem to use by explicitly computing the stable homotopy groups of the Thom spectrum. This can be done quite concisely, provided one is willing to learn about some modern elements of algebraic topology. Thus, we take this computation as a motivation for introducing the notion of spectra in Section 4 and for developing the basics of the Steenrod algebra in Section 5. This will allow us to carry out the computation of $\pi_*(MO)$ in Section 9. We conclude our exposition with a remarkable result stating that whether or not a smooth closed manifold M can be realized as a boundary is a property which solely depends on the homotopy type of M . Stating this result requires the language of Stiefel-Whitney classes, which we introduce in Section 6.

2. CLASSIFYING SPACES AND VECTOR BUNDLES

In this section, we give a streamlined treatment of the theory of principal G -bundles and classifying spaces. A more comprehensive treatment of the material of this section may be found in Mitchell [13]. Vector bundles are a special class of such objects which lie at the heart of the results that we set out to prove in this paper, and so later in this section we specialize our attention to these structures and provide a brief overview of standard constructions. For a classical treatment

of the theory of vector bundles, see Atiyah [1] or the more recent text by Hatcher [5].

Spaces equipped with a G -action by some topological group G are a common occurrence “in nature”. To mention a few such occurrences, deck transformations of the universal cover \tilde{X} of a connected space X correspond to the action of the fundamental group $\pi_1(X)$ on \tilde{X} ; any Lie group G canonically induces an action on the coset space G/H for any subgroup $H \leq G$; the Hopf fibration, which presents the 3-sphere as an S^1 -bundle over S^2 , each of whose fibers comes equipped with an action of the circle group, is yet another instance of this behavior. The notion of principal G -bundles allows for a systematic study of some such phenomena. In particular, we shall state a homotopy classification theorem for principal G -bundles and give a high-level argument as to why it must hold. This will lead us naturally to the notions of *classifying spaces* and *universal bundles*. We then explain how these results specialize to vector bundles.

First, we lay down some preliminary notions. Let G be a topological group. Recall that a map $f : X \rightarrow Y$ between two right G -spaces is said to be G -equivariant, or a G -map, if it commutes with the respective group actions, i.e. if $f(xg) = f(x)g$ for all $x \in X, g \in G$.

Definition 2.1. A *principal G -bundle* is a continuous map $\xi : P \rightarrow B$ such that every fiber $\xi^{-1}(b)$ comes equipped with a free and transitive right G -action, together with a family of *local trivializations* $\{U_\alpha\}$ forming an open cover of B , such that each U_α fits into a commutative triangle:

$$\begin{array}{ccc} \xi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times G \\ & \searrow \xi & \swarrow p_1 \\ & & U_\alpha \end{array}$$

where φ_α is a G -equivariant homeomorphism. We call E the *total space* and B the *base space* of the bundle ξ .

In particular, the existence of local trivializations tells us that the pre-image of each U_α under ξ decomposes into a trivial product of U_α with G , so that the fiber $\xi^{-1}(x)$ above each point $x \in U$ is mapped homeomorphically onto a copy of G . Principal G -bundles naturally fit into a category where a morphism between two principal G -bundles $\xi : P \rightarrow B$ and $\eta : P' \rightarrow B'$ is defined to be a pair of G -maps $(g : P \rightarrow P', f : B \rightarrow B')$ such that the associated square commutes:

$$\begin{array}{ccc} P & \xrightarrow{g} & P' \\ \xi \downarrow & & \downarrow \eta \\ B & \xrightarrow{f} & B'. \end{array}$$

Next, we introduce a key construction on principal G -bundles. We assume the reader is familiar with the categorical notion of pullbacks.

Definition 2.2. Let $\xi : P \rightarrow B$ be a principal G -bundle. Given a continuous map $C \rightarrow B$, we define the *pullback of ξ under f* to be the map $f^*\xi : f^*P \rightarrow C$ obtained as the categorical pullback in **Top** of the maps ξ and f .

Hence the pullback bundle $f^*\xi : f^*P \rightarrow C$ fits into a commutative square:

$$\begin{array}{ccc} f^*P & \longrightarrow & P \\ f^*\xi \downarrow & & \downarrow \xi \\ C & \xrightarrow{f} & D, \end{array}$$

and the total space f^*P is given concretely by the set:

$$f^*P = \{(x, a) \in C \times P \mid f(x) = \xi(a)\}.$$

It can be verified that $f^*\xi : f^*P \rightarrow C$ has the structure of a principal G -bundle, with local trivializations provided by open sets of the form $f^{-1}(U)$ where U is a local trivialization for the bundle $\xi : P \rightarrow D$. In particular, if $f : A \rightarrow X$ is an inclusion and $\xi : P \rightarrow X$ is a principal G -bundle over X , then $f^*\xi : f^*P \rightarrow A$ is simply the restriction of ξ to the subspace A .

We then obtain an assignment $\mathcal{P}_G : \mathbf{Top} \rightarrow \mathcal{S}et$ sending a space X to the set $\mathcal{P}_G(X)$ of isomorphism classes of principal G -bundles over X and a map $f : X \rightarrow Y$ to the induced pullback assignment $f^* : \mathcal{P}_G(Y) \rightarrow \mathcal{P}_G(X)$ at the level of bundles. Playing around with universal properties readily yields that \mathcal{P}_G is a valid contravariant functor, i.e. $(fg)^* = g^*f^*$ and $(id_X)^* = id_{\mathcal{P}_G(X)}$. This assignment turns out to be homotopy invariant. The following result may be established almost verbatim following [Theorem 1.6] of Hatcher [5].

Proposition 2.3. *Given a principal G -bundle $\xi : P \rightarrow C$ and homotopic maps $f \simeq g : B \rightarrow C$, the associated pullback bundles f^*P, g^*P over B are isomorphic, provided B is paracompact.*

Thus, we obtain a well-defined contravariant functor:

$$\mathcal{P}_G : \mathbf{hTop}^{op} \rightarrow \mathcal{S}et$$

where \mathbf{hTop} is the *homotopy category* of spaces and homotopy classes of maps.

Let us say a word about functors of this kind in the realm of pointed spaces. Say that a functor $F : \mathbf{hTop}_*^{op} \rightarrow \mathcal{S}et$ satisfies the *wedge axiom* if it commutes with coproducts, i.e. $F(X \vee Y) \cong F(X) \times F(Y)$ for all spaces X, Y . Next, say that F satisfies the *Mayer Vietoris* axiom if whenever a space X can be expressed as a union of opens $X = U \cup V$, the resulting diagram

$$\begin{array}{ccc} & F(U) & \\ l_{U \cap V, U}^* \swarrow & & \nwarrow l_{X, U}^* \\ F(U \cap V) & & F(X) \\ l_{U \cap V, V}^* \swarrow & & \nwarrow l_{X, V}^* \\ & F(V) & \end{array}$$

has the property that whenever two elements $x \in F(U), y \in F(V)$ are mapped to the same element of $F(U \cap V)$, there exists a $z \in F(X)$ such that $l_{X, U}^*(z) = x$ and $l_{X, V}^*(z) = y$. The following remarkable result was established by Brown [2] in his study of cohomology theories:

Theorem 2.4. (*Brown Representability Theorem*) Let $F : \mathbf{hTop}_*^{op} \rightarrow \mathcal{S}et$ be a contravariant functor on connected spaces which commutes with coproducts and satisfies the Mayer-Vietoris axiom. Then F is representable.

Now, the functor \mathcal{P}_G satisfies the Mayer-Vietoris axiom and the unpointed version of the wedge axiom (wedge sums becomes disjoint unions), and a sleight of hand makes it possible to infer representability of \mathcal{P}_G from the above statement. To say that \mathcal{P}_G is representable amounts to saying that there exists a space BG and an associated principal G -bundle $EG \rightarrow BG$ with base space BG such that for every space X , the set $[X, BG]$ of homotopy classes of maps from X into BG is in a natural bijection with the elements of $\mathcal{P}_G(X)$, with correspondence given by sending a map $f : X \rightarrow BG$ to the principal G -bundle $f^*EG \rightarrow X$ over X obtained by pullback of the universal principal G -bundle $EG \rightarrow BG$. In that case, we call BG a *classifying space* for principal G -bundles and refer to $EG \rightarrow BG$ as the associated *universal principal G -bundle*.

The previous paragraph is a concrete embodiment of a general feature of representable functors, whose behavior is always determined by some analog of the “pullback” of a universal class. We take an instant to flesh out the underlying abstract framework, which applies equally well to the study of generalized cohomology theories, to which we will devote more time in Section 4.

[Interlude] (Representable Functors) Let $F : \mathcal{C}^{op} \rightarrow \mathcal{S}et$ be an arbitrary contravariant functor taking values in the category of sets. Then the Yoneda lemma states that for any given object A in \mathcal{C} , there exists a natural set bijection $\text{Fun}(h_A, F) \cong F(A)$, where Fun denotes the category of functors and natural transformations between them, and $h_A = \mathcal{C}(-, A)$ denotes the represented contravariant functor on A acting on objects by sending a given $B \in \text{ob}\mathcal{C}$ to the morphism set $\mathcal{C}(B, A)$, and acting on morphisms by pre-composition. The above bijection is established by sending a given natural transformation $\eta : h_A \rightarrow F$ to the element of $F(A)$ given by $\eta_A(id_A)$.

Now, suppose that the functor $F : \mathcal{C}^{op} \rightarrow \mathcal{S}et$ is *representable*, meaning that there exists a natural isomorphism $\eta : h_A \xrightarrow{\sim} F$ for some fixed object A in \mathcal{C} . Then for every object $X \in \text{ob}\mathcal{C}$, we have a natural bijection:

$$(2.5) \quad \eta_X : \mathcal{C}(X, A) \xrightarrow{\sim} F(X).$$

We wish to give an explicit description of this correspondence. The Yoneda lemma associates to the given natural isomorphism $\eta : h_A \xrightarrow{\sim} F$ a designated element $u = \eta_A(id_A) \in F(A)$. Now, let $f : X \rightarrow A$ be an arbitrary morphism in \mathcal{C} . By naturality of η , we get a commutative square:

$$\begin{array}{ccc} \mathcal{C}(A, A) & \xrightarrow{f^*} & \mathcal{C}(X, A) \\ \eta_A \downarrow & & \downarrow \eta_X \\ F(A) & \xrightarrow{Ff} & F(X) \end{array}$$

In particular, we get that the element $\eta_X f^*(id_A) = \eta_X(f)$ must equal $Ff(\eta_A(id_A)) = Ff(u)$. Thus we find that the correspondence given in (2.5) *must* behave by sending

a morphism $f : X \rightarrow A$ to the evaluation of the universal element $u \in F(A)$ under the induced map $Ff : F(A) \rightarrow F(X)$. Setting $F = \mathcal{P}_G$ retrieves the case at hand.

We state the classification theorem in its traditional form:

Theorem 2.6. (*Classification Theorem for Principal G -Bundles*) *Let G be a topological group. Then there exists a classifying space BG for principal G -bundles, unique up to homotopy equivalence, such that for every space X , there is a natural bijection:*

$$[X, BG] \cong \mathcal{P}_G(X),$$

obtained by sending a map $f : X \rightarrow BG$ to the pullback under f of a fixed universal bundle $EG \rightarrow BG$.

Of particular interest to us are *vector bundles* and their corresponding classifying spaces. We start by specializing to principal $O(n)$ -bundles, then describe their close ties with vector bundles. Let $V_n^o(\mathbb{R}^\infty)$ be the *Stiefel manifold* of orthonormal n -frames in \mathbb{R}^∞ , and $\text{Gr}_n(\mathbb{R}^\infty)$ be the *Grassmanian manifold* of n -planes in \mathbb{R}^∞ . These spaces may be topologized as the direct limits $\varinjlim_k V_n^o(\mathbb{R}^{n+k})$, resp. $\varinjlim_k \text{Gr}_n(\mathbb{R}^{n+k})$ taken under inclusions, each of whose constituent spaces is readily seen to be compact Hausdorff, the latter as the image of the former under the canonical projection map sending an orthonormal n -frame to the n -plane it spans. The same projection map on the direct limits:

$$\xi : V_n^o(\mathbb{R}^\infty) \rightarrow \text{Gr}_n(\mathbb{R}^\infty)$$

may be verified to have the structure of a principal $O(n)$ -bundle, with the $O(n)$ action on the fibers given by matrix multiplication. We claim that ξ is a universal principal $O(n)$ -bundle. The verification of this fact is greatly simplified by the following theorem, whose proof may be found in Mitchell [13]:

Theorem 2.7. *Let $\xi : E \rightarrow B$ be a principal G -bundle whose total space E is contractible. Then ξ is a universal principal G -bundle and B is a model for the classifying space BG .*

We may then prove the following:

Proposition 2.8. *The Stiefel manifold $V_n^o(\mathbb{R}^\infty)$ is contractible, hence $\xi : V_n^o(\mathbb{R}^\infty) \rightarrow \text{Gr}_n(\mathbb{R}^\infty)$ is a universal principal $O(n)$ -bundle and $\text{Gr}_n(\mathbb{R}^\infty)$ is a model for $BO(n)$.*

Proof. Denote by $\{e_i\}_{i \geq 1}$ the standard basis for \mathbb{R}^∞ . We construct a map $f : V_n^o(\mathbb{R}^\infty) \rightarrow V_n^o(\mathbb{R}^\infty)$ such that $\text{id}_{V_n^o(\mathbb{R}^\infty)} \simeq f \simeq c_x$, where $x := (\frac{1}{\sqrt{n}}e_1, \dots, \frac{1}{\sqrt{n}}e_n)$ and c_x is the constant map at $x \in V_n^o(\mathbb{R}^\infty)$. Given an element $v \in \mathbb{R}^\infty - \{0\}$, we may always find a finite number corresponding to the last non-zero entry in the expression of v as a linear combination over $\{e_i\}$. Denote this assignment by $\sigma : \mathbb{R}^\infty - \{0\} \rightarrow \mathbb{N}$. Next, let $T : V_n^o(\mathbb{R}^\infty) \rightarrow V_n^o(\mathbb{R}^\infty)$ be the linear operator sending (v_1, \dots, v_n) to (v'_1, \dots, v'_n) , where each v'_i is obtained from v_i by shifting all entries in the basis expression of v_i by one unit to the right. Finally, define

$$f : V_n(\mathbb{R}^\infty) \rightarrow V_n(\mathbb{R}^\infty)$$

$$(v_1, \dots, v_n) \mapsto T^{\max\{\sigma(v_1), \dots, \sigma(v_n)\}}(v_1, \dots, v_n).$$

By construction, we may then use straight line homotopies normalized to live in $S^\infty(\sqrt{n})$ to get that $\text{id}_{V_n^o(\mathbb{R}^\infty)} \simeq f \simeq c_x$, as needed. Thus $V_n^o(\mathbb{R}^\infty)$ is contractible, and the second part of the proposition follows from Theorem 2.7 \square

Now, recall that a *rank n real vector bundle* over a space B consists of a continuous map $\xi : E \rightarrow B$ such that the fiber over each point has the structure of an n -dimensional real vector space, together with an open cover $\{U_\alpha\}$ of B admitting fiber-preserving homeomorphisms $\varphi_\alpha : \xi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ which restrict to linear isomorphisms on the fibers. There exists a general process known as the *Borel construction* which specializes to a natural correspondence between principal $O(n)$ -bundles and rank n vector bundles. The Borel construction takes a principal $O(n)$ -bundle $E \rightarrow B$ to an associated rank n vector bundle $E \times_{O(n)} \mathbb{R}^n \rightarrow B$. This assignment preserves classifying spaces and universal bundles. By investigating the structure of the balanced product $\gamma_n^\infty := V_n^\sigma(\mathbb{R}^\infty) \times_{O(n)} \mathbb{R}^n$, we thus find that a model for the universal rank n -vector bundle is given by the *tautological vector bundle* $\gamma_n^\infty \rightarrow \text{Gr}_n(\mathbb{R}^\infty)$, where

$$\gamma_n^\infty := \{(p, v) \in \text{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty \mid v \in p\}.$$

There are many ways to exploit existing vector bundles in order to create new ones. See Section 3.f of Milnor [12] for an elegant functorial procedure allowing one to carry out this process in great generality. The construction boils down to one key idea, which is to construct a “strapped-up” total space to which we give the *final topology* with respect to the appropriate collection of local trivializations so as to obtain a valid vector bundle structure. For our purposes, it will be enough to consider *Whitney sums* and *external products* of vector bundles.

Construction 2.9. Let $\xi : E \rightarrow B$ and $\eta : E' \rightarrow B'$ be two vector bundles of rank n , resp. m over B , resp. B' . Then we may define the *external product bundle* associated to ξ and η to be the rank $(n + m)$ vector bundle over $B \times B'$ given by simply considering the product map $\xi \times \eta : E \times E' \rightarrow B \times B'$, so that the fiber over each point $(x, y) \in B \times B'$ is the vector space direct sum $\xi^{-1}(x) \times \eta^{-1}(y)$.

Next, given two vector bundles $\xi : E_1 \rightarrow B$ and $\eta : E_2 \rightarrow B$ over the same base space B , of rank n , resp. m , we may form the *Whitney sum* $\xi \oplus \eta : E_1 \oplus_B E_2 \rightarrow B$, a rank $(n + m)$ vector bundle over B with total space given by

$$E_1 \oplus_B E_2 := \{(v_1, v_2) \in E_1 \times E_2 \mid \xi(v_1) = \eta(v_2)\},$$

in such a way that the fiber of $\xi \oplus \eta$ over each point $x \in B$ is given by the vector space direct sum $\xi^{-1}(x) \oplus \eta^{-1}(x)$. The Whitney sum may be expressed more concisely as the pullback of the external product bundle $\xi \times \eta$ over $B \times B$ under the diagonal map $\Delta : B \rightarrow B \times B$.

It will also be useful to know that any vector bundle $\xi : E \rightarrow B$ over a paracompact base space admits a Euclidian metric, i.e. a continuous choice of positive-definite inner product on the fibers of E (see [1.4.10] [1] for a proof). We may then define the *closed disk bundle* $D(E)$, resp. the *unit sphere bundle* $S(E)$ associated with ξ to be the subspaces of the total space given fiberwise by vectors of length at most 1, resp. unit length vectors. Then, define the *Thom space* of ξ to be the quotient space:

$$\text{Th}(E) := D(E)/S(E).$$

In the following sections, we shall use $\text{Th}(E)$ and $\text{Th}(\xi)$ interchangeably to refer to the Thom space of the vector bundle $\xi : E \rightarrow B$.

There is an equivalent characterization of the Thom space of ξ . Starting with the total space E , consider the associated *sphere bundle* $\text{Sph}(E)$ obtained from E

by taking the fiberwise one-point compactification at infinity, so that each fiber becomes a sphere of dimension $n = \text{rk}\xi$. There is a canonical section $s : B \rightarrow \text{Sph}(E)$ given by sending each point $b \in B$ to the point at infinity of its associated fiber. Then, one may readily verify that we have a homeomorphism:

$$\text{Th}(\xi) \cong \text{Sph}(E)/s(B).$$

This interpretation of the Thom space of a bundle readily yields the identity $\text{Th}(\xi \oplus \eta) \cong \text{Th}(\xi) \wedge \text{Th}(\eta)$. Indeed, the left hand side may be obtained by first taking the one-point compactification of the subspaces $\xi^{-1}(x), \eta^{-1}(x) \subseteq \xi^{-1}(x) \oplus \eta^{-1}(x)$ fiberwise, identifying the resulting points at infinity separately and then identifying the last two points at infinity, which precisely describes the right hand side.

Lastly, we state a key result which will be needed in future work; a more extensive discussion may be found in chapter 10 of Milnor [12]. Given a rank n vector bundle $\xi : E \rightarrow B$, we may ask for an element $\mu \in \tilde{H}^n(\text{Th}(E); \mathbb{F}_2)$ with the property that the restriction homomorphism $\tilde{H}^n(\text{Th}(E); \mathbb{F}_2) \rightarrow \tilde{H}^n(S^n; \mathbb{F}_2)$ induced by the inclusion of any fiber $S^n \rightarrow \text{Th}(E)$ takes μ to the non-zero element in $\mathbb{F}_2 \cong \tilde{H}^n(S^n; \mathbb{F}_2)$. It is a fact that such an element always exists and is unique; we call μ the *Thom class* associated with the bundle ξ . We then have the following:

Theorem 2.10. (*Thom Isomorphism Theorem*) *Let $\xi : E \rightarrow B$ be a rank n vector bundle. Then the map*

$$\theta : H^i(B; \mathbb{F}_2) \rightarrow \tilde{H}^{n+i}(\text{Th}(E); \mathbb{F}_2)$$

obtained by sending a class $x \in H^i(B; \mathbb{F}_2)$ to its cup product with the Thom class $\mu \in \tilde{H}^n(\text{Th}(E); \mathbb{F}_2)$ is an isomorphism.

3. SOME ELEMENTS OF DIFFERENTIAL TOPOLOGY

We start by recalling some classical results from differential topology, to which we will frequently resort during our main construction in Section 7. The arguments given in this section loosely follow chapter 6 of Lee [8]. The following four theorems are carefully proven in chapters 5 and 6 of the same book.

Theorem 3.1. (*Sard's Theorem*) *Let $f : M \rightarrow N$ be a smooth map between manifolds. Then almost every point in N is a regular value of f .*

By “almost every point”, we mean that the set of points of N that fail to satisfy this property has measure zero with respect to a suitable measure on N , but we shall only exploit the resulting fact that the property of being a regular value of f is a *generic property*, meaning that it is satisfied by a dense subset of N .

Theorem 3.2. (*Whitney Approximation Theorem*) *Let $f : M \rightarrow N$ be a continuous map between manifolds. Then we may find a smooth map $g : M \rightarrow N$ homotopic to f . Further, if the restriction of f to some closed subset $A \subset M$ is already smooth, then the homotopy $f \simeq g$ may be achieved relative to A .*

Theorem 3.3. (*Whitney Embedding Theorem*) *Let M be a smooth n -manifold. Then there exists a smooth embedding $e : M \rightarrow \mathbb{R}^{2n}$.*

An analog of the Whitney embedding theorem for immersions into Euclidian space posits the existence of a smooth immersion $e : M \rightarrow \mathbb{R}^{2n-1}$ for any smooth n -manifold. We will show in Section 5 that this bound is sharp, as an early demonstration of the power of characteristic classes.

Theorem 3.4. (*Tubular Neighborhood Theorem*) Let $e : M \rightarrow \mathbb{R}^k$ be an embedding of a smooth compact n -manifold M with associated normal bundle ν_M . Then e extends to an embedding of an ϵ -disk neighborhood $D_\epsilon(\nu_M)$ for some $\epsilon > 0$.

Next, we introduce in more details the notion of *transversality*, which is an essential ingredient of the translation process from geometric data to homotopy-theoretic data which we will describe in Section 7.

Definition 3.5. Let $f : M \rightarrow N$ be a smooth map between manifolds, and suppose $X \subset N$ is an embedded submanifold. Say f is *transverse* to X , and write $f \pitchfork X$, if for every $x \in f^{-1}(X)$, we have:

$$df_x(T_x M) + T_{f(x)} X = T_{f(x)} N.$$

That is, we say that $f \pitchfork X$ if at every point of intersection of X with the image of f , the tangent space of the ambient manifold N is spanned by the sum of the tangent subspace of X and the image of the corresponding tangent space of M under the differential of f . In particular, if the image of f does not intersect X or if f is a smooth submersion, then the transversality condition is vacuously satisfied.

This seemingly innocuous concept turns out to be a powerful tool in detecting when certain objects are embedded submanifolds and in constructing new ones. Indeed, given $f : M \rightarrow N$ and $X \subset N$ as above, knowing that $f \pitchfork X$ ensures that $f^{-1}(X)$ is a valid embedded submanifold of M . Further, transversality is a *generic* property, in the sense that every non-transverse map is homotopic to a transverse one. We start by establishing the first of these facts.

Theorem 3.6. Let $f : M \rightarrow N$ be a smooth map between manifolds, and let $X \subset N$ be an embedded submanifold of N . If $f \pitchfork X$, then $f^{-1}(X)$ is an embedded submanifold of M . Further, we have that $\text{codim}_N X = \text{codim}_M f^{-1}(X)$

Proof. Write m, n for the dimensions of M , resp. N , and suppose $X \subset N$ is an embedded submanifold of codimension k . Recall that to prove that $f^{-1}(X) \subset M$ is an embedded submanifold of codimension k , it suffices to exhibit a *local defining function* about each point in $f^{-1}(X)$. That is, it is enough to display, for each point $a \in f^{-1}(X)$, an open $U \ni a$ of M and a smooth map $\eta : U \rightarrow \mathbb{R}^k$ such that $U \cap f^{-1}(X)$ appears as the level set $\eta^{-1}(c)$ of some regular value $c \in \mathbb{R}^k$ of η .

So look at an arbitrary point $a \in f^{-1}(X)$, and consider its image $f(a)$ in N . Since $X \subset N$ is an embedded submanifold, we can find an open $V \ni f(a)$ in N and an associated coordinate chart $\psi : V \xrightarrow{\sim} \mathbb{R}^n$ such that the first $(n - k)$ coordinates parametrize $V \cap X$. Composing this chart with the projection $\mathbb{R}^n \rightarrow \mathbb{R}^k$ onto the last k -coordinates yields a smooth map $\varphi : V \rightarrow \mathbb{R}^k$ whose differential has full rank everywhere and such that $\varphi^{-1}(0) = V \cap X$. Then, consider the smooth map:

$$\varphi f : f^{-1}(V) \rightarrow \mathbb{R}^k.$$

We claim that this is a valid local defining function about $a \in f^{-1}(X)$, with $f^{-1}(V) \cap f^{-1}(X)$ appearing as the regular level set of 0 associated to φf . To see this, choose an arbitrary element $u \in T_{\varphi f(a)} \mathbb{R}^k$. Since $d\varphi_{f(a)}$ has full rank, we have $z = d\varphi_{f(a)}(y)$ for some vector $y \in T_{f(a)} N$. By transversality of f , we have:

$$df_a(T_a M) + T_{f(a)} X = T_{f(a)} N,$$

hence we may write $y = df_a(u) + v$ for some choice of $u \in T_a M$, $v \in T_{f(a)} X$. By construction, we have that $d\varphi_{f(a)}$ is zero on v , hence by the chain rule we get that:

$$d(\varphi f)_a(u) = d\varphi_{f(a)}(y - v) = z,$$

so that $d(\varphi f)_a$ has full rank at every element of $(\varphi f)^{-1}(0)$, as needed. \square

We next come to a technical lemma, from which we will be able to deduce the genericity theorem for transversality, which together with Theorem 3.6 above lie at the heart of the Pontryagin-Thom construction.

Lemma 3.7. (*Parametric Transversality Theorem*) *Let $F : M \times S \rightarrow N$ be a smooth map, and let $X \subset N$ be an embedded submanifold such that $F \pitchfork X$. Then $F_s \pitchfork X$ for almost all $s \in S$, where $F_s := F|_{N \times \{s\}}$*

Proof. As a result of Theorem 3.6, we know that $F^{-1}(X)$ is an embedded submanifold of $M \times S$. Consider the projection map $\pi : M \times S \rightarrow S$. In particular, look at its restriction $\pi|_{F^{-1}(X)}$. We claim that if $s \in S$ is a regular value of $\pi|_{F^{-1}(X)}$, then $F_s \pitchfork X$. Applying Sard's theorem to the smooth map $\pi|_{F^{-1}(X)}$ will then suffice to prove the proposition.

So suppose $s \in S$ is a regular value of $\pi|_{F^{-1}(X)}$, and pick an element $x \in F_s^{-1}(X)$. Let $a \in T_{F_s(x)}N = T_{F(x,s)}N$ be arbitrary. By transversality of F , we have that:

$$dF_{(x,s)}(T_{(x,s)}(M \times S)) + T_{F(x,s)}X = T_{F(x,s)}N,$$

and hence that:

$$(3.8) \quad dF_{(x,s)}(u, t) - a \in T_{F(x,s)}X,$$

for some element $(u, t) \in T_{(x,s)}(M \times S) \cong T_xM \oplus T_sS$. To establish transversality of F_s , it is enough to show that:

$$(3.9) \quad d(F_s)_x(v) - a \in T_{F(x,s)}X$$

for some element $v \in T_xM$. Observe that we may write $F_s = F \circ l_s$, where $l_s : M \rightarrow M \times S$ denotes the inclusion $x \mapsto (x, 0)$, so that by the chain rule we see that $d(F_s)_x(v) = dF_{(x,s)}d(l_s)_x(v) = dF_{(x,s)}(v, 0)$. Thus if $t = 0$ in (3.8) above, we get that $dF_{(x,s)}(u, 0) = d(F_s)_x(u)$, so the element $v = u \in T_xM$ satisfies equation (3.9).

Otherwise, since s is a regular value and (x, s) lies in $F^{-1}(X)$ by hypothesis, we are ensured surjectivity of the map:

$$d(\pi|_{F^{-1}(X)})_{(x,s)} : T_{(x,s)}(F^{-1}(X)) \rightarrow T_sS.$$

Hence there exists some element of the form $(q, t) \in T_{(x,s)}M$ such that the image of (q, t) under the differential $dF_{(x,s)}(M \times S)$ lies in $T_{F(x,s)}X$, by construction. Consequently, we find that:

$$\begin{aligned} d(F_s)_x(u - q) - a &= dF_{(x,s)}(u - q, 0) - a \\ &= (dF_{(x,s)}(u, t) - a) - dF_{(x,s)}(q, t) \in T_{F(x,s)}X, \end{aligned}$$

by construction. Thus $v = u - q$ is a satisfactory solution for equation (3.9), completing the proof. \square

Theorem 3.10. (*Genericity Theorem for Transversality*) *Let $f : M \rightarrow N$ be a smooth map, and let $X \subset N$ be an embedded submanifold. Then $f \simeq g$ for some smooth map $g : M \rightarrow N$ such that $g \pitchfork X$.*

Proof. We set out to achieve a set-up analogous to the hypothesis of the parametric transversality theorem, i.e. we aim to construct a smooth map $F : M \times S \rightarrow N$ where $S = D^k$ for some unit disk $D^k \subset \mathbb{R}^k$ and $F|_{M \times \{0\}} = f$, whence the result will quickly follow. Using Theorems (3.2) and (3.3), pick an embedding $e : N \rightarrow \mathbb{R}^k$

of N and extend it to an embedding of some tubular neighborhood $T \supset e(N)$. Identifying T with an open disk of the normal bundle of N in \mathbb{R}^k containing the zero section, one readily sees that the restriction of the canonical projection map $\pi : \nu_N \rightarrow N$ to T yields a smooth retract $r : T \rightarrow N$ of $e(N)$ such that r is also a smooth submersion. Next, define a map $\delta : N \rightarrow \mathbb{R}_+$ by the assignment:

$$\delta(p) := \sup_{0 \leq \epsilon \leq 1} \{\epsilon \mid B_\epsilon(e(p)) \subset T\}.$$

This is a continuous function which we may precompose with f to obtain a positive continuous function $\delta f : M \rightarrow \mathbb{R}_+$, and a variant of Whitney's approximation theorem for functions allows us to find a smooth positive function $\sigma : M \rightarrow \mathbb{R}_+$ such that

$$0 < \sigma(p) < \delta f(p)$$

for all points $p \in M$.

We may then define the map $F : M \times D^k \rightarrow N$ as follows:

$$F(p, s) := r(e f(p) + \sigma(p)s).$$

This map is well-defined by construction of $\sigma : M \rightarrow \mathbb{R}_+$. We readily notice that F is smooth, and since r is a retract we have that $F(p, 0) = r e(f(p)) = f(p)$ for all $p \in M$, i.e. $F|_{M \times 0} = f$. The situation can be summarized in the following diagram:

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{e} & \mathbb{R}^k \\ i_0 \downarrow & \nearrow F & \nwarrow r & & \uparrow \\ M \times D^k & & & & T. \end{array}$$

Observe that for fixed $p \in M$, the restriction $F_p = F|_{\{p\} \times D^k}$ can be expressed as the composite of a local diffeomorphism onto a subset of T with the smooth submersion r . It follows that F has full rank everywhere, hence in particular that it is transverse to $X \subset N$. From Lemma (3.7), we then get that $F_s \pitchfork X$ for some $s \in D^k$. Letting $L \subset D^k$ be the straight line in D^k from 0 to s , we see that the restriction of F to $M \times L \cong M \times I$ yields a homotopy of $f = F|_{M \times \{0\}}$ with a smooth map F_s that is transverse to X , as needed. \square

4. A PRIMER ON SPECTRA

Let us come back to the Brown representability theorem introduced in Section 2. Suppose we are given a reduced generalized cohomology theory, i.e. a sequence of functors $(\tilde{h}^n : \mathbf{hTop}_* \rightarrow \mathcal{A}b)_n$ satisfying the Eilenberg-Steenrod axioms in reduced form save for the dimension axiom. In more details, we require each \tilde{h}^n to induce an isomorphism on coproducts:

$$\tilde{h}^n(\bigvee_k X_k) \xrightarrow{\sim} \bigoplus_k \tilde{h}^n(X_k),$$

to come equipped with a natural suspension isomorphism for all spaces X :

$$\tilde{h}^n(X) \xrightarrow{\sim} \tilde{h}^{n+1}(\Sigma X),$$

and to associate to every subspace inclusion $A \subset X$ an exact sequence:

$$\tilde{h}^n(Ci) \longrightarrow \tilde{h}^n(X) \longrightarrow \tilde{h}^n(A).$$

where Ci denotes the mapping cone associated to the inclusion $i : A \hookrightarrow X$. The above three properties are referred to as the *wedge*, *suspension* and *exactness axioms*, respectively.

Now, we may view each \tilde{h}^n as taking values in the category of sets. By definition, \tilde{h}^n is then a homotopy invariant, contravariant functor which commutes with coproducts. It can further be checked that the Mayer-Vietoris axiom is satisfied as a result of the above axioms. Hence by the Brown representability theorem, each functor \tilde{h}^n is representable, i.e. admits a natural isomorphism:

$$[X, T_n] \xrightarrow{\sim} \tilde{h}^n(X),$$

for some space T_n unique up to homotopy equivalence and equipped with a designated element $u_n \in \tilde{h}^n(T_n)$ such that the above bijection is given by sending a map $f : X \rightarrow T_n$ to the element $\tilde{h}^n(f)(u_n) \in \tilde{h}^n(X)$ (as explained in the interlude in Section 2).

Now, recall the key adjunction of functors

$$(4.1) \quad [\Sigma X, Y] \cong [X, \Omega Y],$$

where $\Sigma X = S^1 \wedge X$ denotes the reduced suspension of X and ΩY denotes the loop space of Y . Using this adjunction together with the suspension isomorphism above, we find that:

$$[X, \Omega T_{n+1}] \cong [\Sigma X, T_{n+1}] \cong \tilde{h}^{n+1}(\Sigma X) \cong \tilde{h}^n(X).$$

Hence the functor \tilde{h}^n is also represented by the space ΩT_{n+1} . It follows by uniqueness that we must have a homotopy equivalence:

$$(4.2) \quad T_n \xrightarrow{\sim} \Omega T_{n+1},$$

corresponding under the adjunction (4.1) to a unique map (up to homotopy) $\Sigma T_n \rightarrow T_{n+1}$. Hence the family of spaces representing the component functors of a generalized cohomology theory is inextricably linked by a collection of structure maps $\sigma_n : \Sigma T_n \rightarrow T_{n+1}$. This leads us to the following definition:

Definition 4.3. A *spectrum* E is defined to be a sequence of pointed spaces $\{E_n\}_{n \in \mathbb{Z}_{\geq 0}}$ indexed by the non-negative integers, together with *structure maps* $\sigma_n : \Sigma E_n \rightarrow E_{n+1}$ for each $n \in \mathbb{Z}$. A *map of spectra* $f : E \rightarrow E'$ is a collection of continuous maps $f_n : E_n \rightarrow E'_n$ which is compatible with the respective structure maps, i.e. the following square commutes for every n :

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma f_n} & \Sigma E'_n \\ \sigma_n \downarrow & & \downarrow \sigma'_n \\ E_{n+1} & \xrightarrow{f_{n+1}} & E'_{n+1}. \end{array}$$

Remark 4.4. In general, spectra need not satisfy the property exhibited in (4.2) above, namely that the adjoints $\tilde{\sigma}_n : E_n \rightarrow \Omega E_{n+1}$ of the structure maps are all weak homotopy equivalences. If that is the case, we call $\{E_n\}$ an *Ω -spectrum*.

A converse for the statement made above can be checked directly. Namely, the sequence of represented functors $[-, E_n] : \mathbf{hTop}_*^{op} \rightarrow \mathcal{S}et$ associated with any given Ω -spectrum can be shown to give a generalized cohomology theory. Thus, the

Brown representability theorem posits the existence of a bijective correspondence between Ω -spectra and generalized cohomology theories given by the assignment:

$$\{E_n\}_{n \in \mathbb{Z}} \mapsto \left(\tilde{h}^n : \mathbf{hTop}_*^{op} \rightarrow \mathcal{A}b, X \mapsto [X, E_n] \right)_{n \geq 0}.$$

This initial result gives legitimacy to the study of Ω -spectra as “concrete representations” of generalized cohomology theories, and by extension to spectra as a natural generalization of the latter.

A few operations can readily be defined on spectra. Given a spectrum E , we can construct the k th suspension of E to be the spectrum $\Sigma^k E$ given in degree n by $(\Sigma^k E)_n := E_{n+k}$, with structure maps given by the appropriate shifts of the structure maps of E . We may also define the wedge sum of two spectra by taking component-wise wedge sums.

Example 4.5. For any space X , we may define the *suspension spectrum* $\Sigma^\infty X$ associated with X , given in degree n by the n -fold iterated suspension $\Sigma^n X = \Sigma(\dots(\Sigma X)\dots)$, with structure maps $\sigma_n : \Sigma \Sigma^n X \rightarrow \Sigma^{n+1} X$ induced by the homeomorphisms $\Sigma \Sigma^n X \cong S^1 \wedge S^n \wedge X \cong S^{n+1} \wedge X \cong \Sigma^{n+1} X$. Of particular interest is the suspension spectrum $\mathbb{S} := \Sigma^\infty S^0$, with n th component space $\Sigma^n S^0 \cong S^n$, which is referred to as the *sphere spectrum*.

In fact, the study of spectra was initially propelled by a result known as the *Freudenthal suspension theorem*, which states as a special case that the suspension homomorphism $\pi_{n+k}(S^k) \rightarrow \pi_{n+k+1}(S^{k+1})$ becomes an isomorphism for large enough values of k (more precisely, for $k > n + 1$). The computation of the homotopy groups of spheres is of great interest in algebraic topology, and spectra are tailor-made for the computation of groups belonging to this “stable range”.

Thus we are led to define the n th *stable homotopy group* of a spectrum E via the formula:

$$\pi_n(E) := \varinjlim_k \pi_{n+k}(E_k),$$

where the direct system structure is given in degree k by the composite:

$$\pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma E_k) \xrightarrow{\sigma_*} \pi_{n+k+1}(E_{k+1}).$$

In the above sequence of morphisms, the first map denotes the suspension homomorphism sending an element $[\alpha] : S^{n+k} \rightarrow X$ to its image under the suspension functor

$$[\Sigma \alpha] : \Sigma S^{n+k} \cong S^{n+k+1} \rightarrow \Sigma X,$$

and the second map is induced by the structure map $\sigma_n : \Sigma E_n \rightarrow E_{n+1}$ at the level of homotopy groups.

The stable homotopy groups $\pi_n(E)$ can be given a group structure by relying on the underlying group structure of the standard homotopy groups $\pi_{n+k}(E_k)$ and making use of the structure maps of E : given two elements $[\alpha], [\beta] \in \pi_n(E)$, pick representatives $\alpha \in \pi_{n+k}(E_k)$, $\beta \in \pi_{n+l}(E_l)$, and assume without loss of generality that $k \leq l$. By taking iterated suspensions in the directed system $(\dots \rightarrow \pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+k+1}(E_{k+1}) \rightarrow \dots)$, map α to the element $\Sigma^{l-k} \alpha \in \pi_{n+l}(E_l)$,

and define the sum of $[\alpha]$ and $[\beta]$ to be the equivalence class in $\pi_n(E)$ of the element $[\Sigma^{l-k}\alpha + \beta] \in \pi_{n+l}(E_l)$ obtained via the composite:

$$S^{n+l} \rightarrow S^{n+l} \vee S^{n+l} \xrightarrow{(\Sigma^{l-k}\alpha) \vee \beta} E_l,$$

where the first map is the *pinch map* sending the equator of S^{n+l} to a point.

Example 4.6. (Eilenberg-MacLane Spectra) Let us apply the Brown representability theorem to reduced singular cohomology with coefficients in a given abelian group G . We get a sequence of spaces $\{K(G, n)\}_{n \in \mathbb{Z}}$, each of which is unique up to homotopy equivalence and fits into a natural isomorphism:

$$\tilde{H}^n(X; G) \cong [X, K(G, n)].$$

In particular, for a given integer n , letting $X = S^q$ yields:

$$\pi_q(K(G, n)) = [S^q, K(G, n)] = \tilde{H}^n(S^q; G) = \begin{cases} G & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}$$

The spaces $K(G, n)$ are known as the *Eilenberg-MacLane spaces* associated with the group G . By the above discussion, the family of spaces $\{K(G, n)\}_n$ has the structure of an Ω -spectrum, which we call the *Eilenberg-MacLane spectrum* associated to the group G and denote by HG . The stable homotopy groups of HG are directly computable. Indeed, the direct limit

$$\pi_n(HG) := \varinjlim_m \pi_{n+m}(K(G, m))$$

behaves particularly nicely in this case: for $n = 0$, the directed system is constant equal to G , and for n non-zero, the difference in indices between π_{n+m} and $K(G, m)$ implies that all the elements of the directed system are zero. It follows that:

$$\pi_n(HG) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

With the Eilenberg-MacLane spectra in hand, we may proceed to define homology and cohomology groups of an arbitrary spectrum E with coefficients in a group G as follows:

$$H_n(E; G) := \varinjlim_k \tilde{H}_{n+k}(E(k); G),$$

$$H^n(E; G) := \varprojlim_k \tilde{H}^{n+k}(E(k); G).$$

Remark 4.7. The last two definitions could be given on more conceptual grounds as $H_n(E; G) := \pi_n(E \wedge HG)$, resp. $H^n(E; G) := [E, \Sigma^n HG]$ and shown to coincide with the above under reasonable conditions which are satisfied by the spectra considered in this paper. A modern treatment may be found in May [10].

We conclude this section by formalizing the notion of a ring structure on a spectrum, which is exhibited by Eilenberg-MacLane spectra and will also be present in the Thom spectrum we introduce in the next section. Say $E = \{E_k\}$ is a *ring spectrum* if it comes equipped with a unit map $\eta : S^0 \rightarrow E_0$ and product maps $p_{n,m} : E_n \wedge E_m \rightarrow E_{n+m}$ which fit into commutative diagrams (up to homotopy) corresponding to the associativity and unital nature of the product structure. The definition naturally extends to that of a *commutative ring spectrum* in the graded sense. See Section 25.2 of May [9] for more details.

If E is a ring spectrum, then $\pi_*(E)$ may be given a graded ring structure (resp. graded commutative if E itself is) by defining the product of elements $[\alpha] \in \pi_n(E)$, $[\beta] \in \pi_m(E)$ with representatives $\alpha \in \pi_{n+k}(E_k)$, $\beta \in \pi_{m+l}(E_l)$ to be the homotopy class of the composite:

$$S^{n+m+k+l} \cong S^{n+k} \wedge S^{m+l} \xrightarrow{\alpha \wedge \beta} E_k \wedge E_l \xrightarrow{p_{k,l}} E_{k+l}.$$

5. A CRASH COURSE ON THE STEENROD ALGEBRA

In order to present the computation of the stable homotopy groups $\pi_*(MO)$, we need to introduce a powerful algebraic object known as the *Steenrod algebra*. A treatment in full details would go against the expository nature of this paper; rather, we attempt to provide a coherent and motivated introduction to the ingredients necessary to the computation of $\pi_*(MO)$. The reader wishing to supplement the material for this section may wish to consult the classical reference [14], or the more recent treatments in chapter 1 of Miller [11] and chapter 10 of Davis and Kirk [3].

Remark 5.1. In the following discussion, cohomology of X will always mean singular cohomology with coefficients in the field \mathbb{F}_2 , and we shall often write $H^*(X)$ as shorthand for the corresponding cohomology ring $H^*(X; \mathbb{F}_2)$. An analogous construction may be done over \mathbb{F}_p for odd primes; however, we shall only be concerned with the case $p = 2$ in this paper.

One motivation behind the construction of the Steenrod algebra is that of giving the cohomology ring $H^*(X; \mathbb{F}_2)$ of a space X the structure of a graded R -module for some meaningful choice of graded ring R . Recall that the action of R on a graded R -module $M = \bigoplus_{n \geq 0} M_n$ is required to satisfy $R_k M_n \subset M_{k+n}$, where R_k denotes the degree k component of R . In attempting to make a “universal” choice of such a graded ring (applicable to arbitrary spaces), we are led to consider natural transformations of functors $H^*(-; \mathbb{F}_2) \rightarrow H^{*+k}(-; \mathbb{F}_2)$ as representing the action of an element of degree k in our ring. The right notion for an element in such a ring turns out to be that of a collection of natural transformations $\theta_n : H^n \rightarrow H^{n+k}$ of a fixed degree k , one for each $n \in \mathbb{Z}$, on which we impose a further structural condition in order to obtain a valid module structure. In the following definition, we write $O(n, k)$ for the set of natural transformations $H^n \rightarrow H^k$, to which we refer as *cohomology operations*.

Definition 5.2. A *stable cohomology operation of degree k* is a family of cohomology operations $\theta_n \in O(n, n+k)$ for $n \geq 0$ which are compatible with the suspension isomorphisms in the sense that, for any n and for any space X , the following square commutes:

$$\begin{array}{ccc} H^{n+1}(\Sigma X) & \xrightarrow{\theta_{n+1}} & H^{n+k+1}(\Sigma X) \\ \uparrow & & \uparrow \\ H^n(X) & \xrightarrow{\theta_n} & H^{n+k}(X). \end{array}$$

Letting $\sigma^* \theta_{n+1}$ be the composite $H^n(X) \rightarrow H^{n+1}(\Sigma X) \rightarrow H^{n+k+1}(\Sigma X) \rightarrow H^{n+k}(X)$, this is equivalent to requiring that $\sigma^* \theta_{n+1} = \theta_n$ for all n .

Remark 5.3. In particular, starting with an arbitrary cohomology operation $\theta \in O(n, n+k)$, we may obtain an associated stable cohomology operation of degree k

by constructing the sequence:

$$\dots, (\tau^*)^2\theta, \tau^*\theta, \theta, \sigma^*\theta, (\sigma^*)^2\theta, \dots$$

where τ^* is the reverse operation to σ^* obtained by traversing the above commutative square the other way around.

Stable cohomology operations can be described equivalently as elements of a direct limit of cohomology groups of Eilenberg-MacLane spaces:

Construction 5.4. To simplify notation, write K_n for the n th Eilenberg-MacLane space $K(\mathbb{F}_2, n)$. Using the Yoneda lemma applied to the contravariant functor $h_{K_n} = [-, K_n]$, together with the fact that Eilenberg-MacLane spaces represent singular cohomology with \mathbb{F}_2 coefficients, we find that:

$$\text{Fun}(H^n, H^k) \cong \text{Fun}(h_{K_n}, H^k) \cong H^k(K_n) \cong [K_n, K_k].$$

Hence the k th cohomology group of the Eilenberg-MacLane space $K(\mathbb{F}_2, n)$ provides an equivalent way of characterizing cohomology operations of type $O(n, k)$.

In particular, observe that each map $\sigma^* : O(n, n+k) \rightarrow O(n-1, n+k-1)$ as defined above may be reinterpreted as a map $\sigma^* : H^{n+k}(K_n) \rightarrow H^{n+k-1}(K_{n-1})$, giving rise to a directed system:

$$\left(\dots \longrightarrow H^{n+k}(K_n) \xrightarrow{\sigma^*} H^{n+k-1}(K_{n-1}) \longrightarrow \dots \right),$$

whose inverse limit is given concretely by:

$$\varprojlim_n H^{n+k}(K_n) = \prod_n H^{n+k}(K_n) / (\theta_{n-1} \sim \sigma^*\theta_n).$$

Hence under this viewpoint, stable cohomology operations of degree k correspond precisely to elements of $\varprojlim_n H^{n+k}(K_n)$.

Stable cohomology operations are also closed under composition: given stable cohomology operations θ, φ of degree r , resp. s , we may construct a stable operation $\varphi\theta$ of degree $(r+s)$ by setting $(\varphi\theta)_n$ to be the composite $\varphi_{n+r} \circ \theta_n$. Letting $\eta : H^n(X) \rightarrow H^{n+1}(\Sigma X)$ denote the suspension isomorphism, we readily see that:

$$\sigma^*(\varphi\theta)_{n+1} := \eta^{-1}(\varphi\theta)_{n+1}\eta = (\eta^{-1}\varphi_{n+r+1}\eta)(\eta^{-1}\theta_{n+1}\eta) = \varphi_{n+r} \circ \theta_n = (\varphi\theta)_n,$$

as needed. Hence we may formulate the following definition:

Definition 5.5. The *Steenrod algebra*, denoted by \mathcal{A} , is the graded associative \mathbb{F}_2 -algebra given in degree r by the collection of all stable cohomology operations of degree r under addition and composition.

Following up on Construction (5.4) above, and in light of the definition of the cohomology groups of a spectrum given in Section 4, we have a correspondence:

$$\mathcal{A}^k \cong H^k(H\mathbb{F}_2)$$

between the degree k elements of the Steenrod algebra and the k th cohomology group of the Eilenberg-MacLane spectrum $H\mathbb{F}_2$.

We may now equip the cohomology ring of any space X with a left action of \mathcal{A} , which gives $H^*(X)$ a valid \mathcal{A} -module structure by the above discussion. The following two results combine to make the computation of the \mathcal{A} -module structure much more tractable by providing a designated collection of well-behaved stable cohomology operations which generate the entire Steenrod algebra.

Theorem 5.6. For each $i \geq 0$, there exists a stable cohomology operation $Sq^i = \{Sq_n^i\}_{n \geq 0}$, called the i th Steenrod square such that the collection of Steenrod squares $\{Sq^0, Sq^1, \dots\}$ is the unique family satisfying the following properties:

- The 0th Steenrod square acts as the identity for all $n \geq 0$: $Sq_n^0 = Id_{H^n}$;
- The i th Steenrod square acts on elements of degree i as the cup product squaring map: $Sq_i^i(x) = x^2$;
- The i th Steenrod square acts trivially in degrees $n > i$: $Sq_n^i = 0$;
- (Cartan formula) Given elements x, y in degree n , resp. m , the i th Steenrod square satisfies:

$$Sq_{n+m}^i(x \cup y) = \sum_{p+q=i} Sq_n^p(x) \cup Sq_m^q(y);$$

- (Adem relations) The Steenrod squares are subject to the multiplicative relations (for $i < 2j$):

$$Sq^i Sq^j = \sum_k \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k,$$

where the binomial coefficients are taken mod 2.

We shall not concern ourselves with the Adem relations in practice; however, they are essential to formulating the following result:

Theorem 5.7. The Steenrod algebra \mathcal{A} is isomorphic to the \mathbb{F}_2 -algebra generated by the Steenrod squares $\{Sq^i\}_{i \geq 0}$ modulo the Adem relations.

Notation 5.8. We sometimes abuse notation and write $Sq = Sq^0 + Sq^1 + Sq^2 + \dots$ for the *total Steenrod square* given by the formal sum of all Steenrod squares (in some common degree). This is always a well-defined operation by the 3rd property mentioned in Theorem 5.7 above, and allows us to rephrase the Cartan formula more concisely as:

$$(5.9) \quad Sq(x \cup y) = Sq(x) \cup Sq(y).$$

Further, it is possible to exhibit a basis for \mathcal{A} known as the *Cartan-Serre basis*:

$$\{Sq^{i_1} \dots Sq^{i_k} \mid k \geq 1, i_{j-1} \geq 2i_j\},$$

whose elements are called *admissible squares*. Of particular interest in the above collection are those admissible squares for which consecutive exponents differ by the sharpest bound, namely those of the form:

$$(5.10) \quad Sq^{I_r} := Sq^{2^{r-1}} Sq^{2^{r-2}} \dots Sq^2 Sq^1,$$

so that Sq^{I_r} has degree $2^r - 1$.

Finally, we observe that \mathcal{A} may be given a coproduct structure via the assignment

$$\begin{aligned} \mu : \mathcal{A} &\longrightarrow \mathcal{A} \otimes \mathcal{A}, \\ Sq^k &\mapsto \sum_{i+j=k} Sq^i \otimes Sq^j, \end{aligned}$$

which may be checked to be cocommutative. Further, μ is a homomorphism of \mathbb{F}_2 -algebras, so that \mathcal{A} has the structure of a *Hopf algebra*. We invite the reader unfamiliar with this notion to consult Section 1.10 of Miller [11].

Example 5.11. We compute the \mathcal{A} -module structure on $H^*(\mathbb{R}P^\infty) = H^*(BO(1))$. By Theorem 8.8 above, we need only understand the action of the Steenrod squares. Recall that $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x]$ is a polynomial ring with one generator x in degree 1. We obtain directly from the axioms that the action of the total Steenrod square on x is given by:

$$Sq(x) = Sq^0(x) + Sq^1(x) + Sq^2(x) + \dots = x + x^2.$$

Next, using the Cartan formula as in equation (8.10), we see that:

$$\begin{aligned} (Sq^0 + Sq^1 + Sq^2 + \dots)(x^n) &= Sq(x^n) \\ &= (Sq(x))^n = x^n(1 + x^2)^n = \sum_{i=0}^n \binom{n}{i} x^{n+i}, \end{aligned}$$

whence it follows by matching degrees that $Sq^i(x^n) = \binom{n}{i} x^{n+i}$, with binomial coefficients taken once again mod 2. This computation fully determines the action of \mathcal{A} on $H^*(\mathbb{R}P^\infty)$. In particular, note that each element Sq^{I_r} as defined in (5.11) takes x to x^{2^r} , and that every other admissible square Sq^I sends x to 0.

We may now form the *dual Steenrod algebra* $\mathcal{A}_* := \text{Hom}(\mathcal{A}, \mathbb{F}_2)$, which inherits a commutative Hopf algebra structure from \mathcal{A} where the algebra, resp. coalgebra structure on \mathcal{A}_* is induced from the coalgebra, resp. algebra structure on \mathcal{A} . The elements Sq^{I_r} then come into play in the following remarkable result:

Theorem 5.12. (Milnor) *The dual Steenrod algebra is isomorphic to the polynomial algebra:*

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_r \mid |\xi_r| = 2^r - 1],$$

where each generator ξ_r is dual to Sq^{I_r} .

The coproduct structure on \mathcal{A}_* is given explicitly by the map $\psi : \mathcal{A}_* \rightarrow \mathcal{A}_* \otimes \mathcal{A}_*$ acting on the above generators via:

$$\psi : \xi_r \mapsto \sum_{i+j=r} \xi_i^{2^j} \otimes \xi_j.$$

6. STIEFEL-WHITNEY CLASSES

We take a moment to introduce Stiefel-Whitney classes and Stiefel-Whitney numbers, which will be featured in the remarkable result with which we choose to end this paper (Theorem 9.3). Fortunately, we have introduced enough tools in the previous sections to be able to concisely define Stiefel-Whitney classes. The proper study of characteristic classes (of which Stiefel-Whitney classes are one instance) deserves much more space than we can afford in this brief section, and we refer the interested reader to the classical book by Milnor and Stasheff [12] for a wonderful treatment.

Let $\xi : E \rightarrow B$ be a given rank n vector bundle. Recall from Section 4 that we may construct the Thom isomorphism:

$$\theta : H^i(B; \mathbb{F}_2) \xrightarrow{\sim} \tilde{H}^{n+i}(\text{Th}(\xi); \mathbb{F}_2).$$

We may then define the *ith Stiefel-Whitney class* of the bundle ξ to be the element of $H^i(B; \mathbb{F}_2)$ given by:

$$w_i(\xi) := \theta^{-1} Sq^i(\mu),$$

where $\mu \in H^n(\text{Th}(\xi); \mathbb{F}_2)$ denotes the Thom class of ξ . The following properties of Stiefel-Whitney classes may be derived from the naturality of the Thom isomorphism and the properties of Steenrod squares listed in Theorem 5.6 above; see chapter 8 of Milnor [12] for a detailed verification.

Theorem 6.1. *Given a rank n vector bundle $\xi : E \rightarrow B$, the Stiefel-Whitney classes $w_i(\xi) \in H^i(B; \mathbb{F}_2)$ are the unique family of cohomology classes satisfying the following properties:*

- $w_0(\xi) = 1$, the unit in $H^*(B; \mathbb{F}_2)$, and $w_i(\xi) = 0$ for $i > n$;
- (Naturality) Stiefel-Whitney classes commute with pullbacks: given a map $f : B' \rightarrow B$ and a bundle ξ over B , we have that $w_i(f^*\xi) = f^*(w_i(\xi))$;
- (Whitney product formula) Given vector bundles ξ and η over the same base space, we have:

$$w_i(\xi \oplus \eta) = \sum_{p+q=i} w_p(\xi) \cup w_q(\eta);$$

- Letting γ_1^1 denote the tautological rank 1 bundle over $\mathbb{R}P^1$, we have that $w_1(\gamma_1^1) \neq 0$.

As with Steenrod squares, write $w(\xi) := w_0(\xi) + w_1(\xi) + \dots$ for the total Stiefel-Whitney class of ξ . Then we may view $w(\xi)$ as an element in the group of units of the direct product of cohomology groups $\prod_i H^i(B; \mathbb{F}_2)$, whose inverse $\bar{w}(\xi) = \bar{w}_0(\xi) + \bar{w}_1(\xi) + \dots$ is given inductively by the formula (dropping the parameter ξ):

$$\bar{w}_0 = 1, \quad \bar{w}_n = w_1 \bar{w}_{n-1} + w_2 \bar{w}_{n-2} + \dots + w_{n-1} \bar{w}_1.$$

In particular, if two vector bundles ξ and η over the same base space sum to a trivial bundle ϵ^n , the Whitney product formula indicates that:

$$(6.2) \quad w(\xi)w(\eta) = w(\xi \oplus \eta) = w(\epsilon^n) = 1,$$

where the fact that $w(\epsilon^n) = 1$ follows from naturality combined with the fact that trivial bundles admit a bundle map to the one-point space.

Of immediate interest to us are the Stiefel-Whitney classes associated to the tangent bundle τ_M of a smooth n -manifold M , which we denote by $w_i(M) := w_i(\tau_M)$. Remarkably, the Stiefel-Whitney classes of a smooth closed manifold M only depend on the *homotopy type* of M . This follows from the Wu formula expressing $w(M)$ in terms of Steenrod operations, which themselves are homotopy invariant. See Section 23.3 of May [9] for more details on this result.

In later sections, we will be led to looking at a collection of numbers obtained through a natural choice of homology-cohomology pairing:

Definition 6.3. Define the *Stiefel-Whitney numbers* of M to be the collection of \mathbb{F}_2 values obtained by evaluating degree n monomials of the form $w_{i_1}(M) \dots w_{i_k}(M) \in H^n(M; \mathbb{F}_2)$ on the fundamental class of M .

As an early indication of the power of Stiefel-Whitney classes, we conclude this interlude by following-up on the promise made in Section 3, and give a concise argument proving that the bound provided by the immersion version of the Whitney embedding theorem is indeed sharp. Let M be an n -manifold, and suppose we are given an immersion $i : M \rightarrow \mathbb{R}^{n+k}$. Then the Whitney sum $\tau_M \oplus \nu_M$ is isomorphic

to the trivial bundle ϵ^{n+k} . Hence by the above discussion, we see that $w(M)$ and $w(\nu_M)$ are inverses of one another. In particular, we have that:

$$w_i(\nu_M) = \bar{w}_i(M),$$

so that if r denotes the highest degree of a non-zero element in $\bar{w}_i(M)$, then by the first axiom of Theorem 5.6 the normal bundle ν_M must be of rank at least r , which puts a lower bound on the dimension of the euclidian space in which M can be immersed. In particular, look at $\mathbb{R}P^n$ for n of the form 2^s . Then it can be computed that $w(\mathbb{R}P^n) = 1 + x + x^n$, where x denotes the generator of the cohomology ring $H^*(\mathbb{R}P^n; \mathbb{F}_2)$. Hence we get that

$$\bar{w}(\mathbb{R}P^n) = 1 + x + \dots + x^{n-1},$$

which implies by the above discussion that the smallest k for which there can exist an immersion $i : \mathbb{R}P^n \rightarrow \mathbb{R}^{n+k}$ has to be $k = n - 1$, as needed.

7. THE UNORIENTED COBORDISM RING AND THE THOM SPECTRUM

Cobordism is a coarser equivalence relation than diffeomorphism, but it has the advantage of being more tractable. In this section, we only introduce the notion of *unoriented cobordism*, with which the Pontryagin-Thom theorem is concerned. However, this is only one of the many flavors of cobordism which have been the subject of intense study following Thom's initial breakthrough [15] in 1954. Alternative cobordism theories such as oriented cobordism, complex cobordism, framed cobordism and spin cobordism may be defined, and a generalized version of the Pontryagin-Thom theorem using the notion of a (\mathcal{B}, f) -structure shows that all these theories may be put in relation with the homotopy groups of an appropriate spectrum. A treatment at this level of generality may be found in Weston [16].

The underlying idea of cobordism is simple: we call two closed n -manifolds M and N *cobordant*, and write $[M] = [N]$, if there exists a compact $(n + 1)$ -manifold W whose boundary is diffeomorphic to the disjoint union $M \amalg N$. We shall write $\partial W|_M$, resp. $\partial W|_N$ for the restriction of the boundary of W to the region diffeomorphic to M , resp. N . Unoriented cobordism is easily seen to produce an equivalence relation on closed n -manifolds: any manifold M is cobordant to itself, as witnessed by the compact $(n + 1)$ -manifold $M \times I$; a cobordism between M and N can also be interpreted as a cobordism between N and M , hence the relation is symmetric. Finally, suppose M is cobordant to N and N is cobordant to P , as witnessed by compact $(n + 1)$ -manifolds W , resp. Z . We can obtain a diffeomorphism $f : \partial W|_N \rightarrow \partial Z|_N$ defined via the composite $f : \partial W|_N \xrightarrow{\sim} N \xrightarrow{\sim} \partial Z|_N$. The adjunction space $W \cup_f Z$ then provides a cobordism between M and P . Hence for each $n \geq 0$, we may consider the following object, to which we refer as the *n th unoriented cobordism group*:

$$\Omega_n^o := \left\{ \begin{array}{l} \text{Smooth closed} \\ n\text{-manifolds} \end{array} \right\} / (\text{Cobordism}).$$

Then, we call the associated graded object $\Omega_*^o = \bigoplus_{n \geq 0} \Omega_n^o$ the *unoriented cobordism ring*. A justification for these names is in order:

Lemma 7.1. *The sets $\{\Omega_n^o\}_{n \geq 0}$ can be given the structure of a graded commutative ring Ω_*^o as defined above under disjoint unions and cartesian products of manifolds. Further, every element is 2-torsion, so Ω_*^o has the structure of an \mathbb{F}_2 -algebra.*

Proof. Viewing the empty set \emptyset as a closed n -manifold in every dimension, it is readily verified that the operation of disjoint union gives Ω_n^* a well-defined abelian group structure with additive unit the cobordism class $[\emptyset]$. The inverse of a class $[M]$ is given by $[M]$ itself, as the compact $(n+1)$ -manifold $M \times I$ determines a cobordism between $M \amalg M$ and the empty set, so that $[M \amalg M] = [M] + [M] = 0$, whence it follows that every element is 2-torsion. Likewise, we readily see that cartesian product of manifolds gives a well-defined graded commutative ring structure to Ω_*^o , with unit the one-point space $[*]$ viewed as a closed 0-manifold. \square

Remark 7.2. More generally, it is possible to parametrize cobordism with respect to a given topological space X . That is, we could consider the *unoriented cobordism ring over X* , with n th component given by:

$$\Omega_n^o(X) := \left\{ \begin{array}{l} (M, g : M \rightarrow X), \\ \text{where } M: \text{ smooth closed } n\text{-manifold} \\ \text{and } g : M \rightarrow X : \text{ continuous map} \end{array} \right\} / (\text{Compatible cobordism}),$$

where cobordisms are pairs $(W, f : W \rightarrow X)$ subject to the obvious compatibility condition. In particular, our original construction of the unoriented cobordism ring reduces to the special case $\Omega_*^o = \Omega_*^o(*)$ where $X = \{*\}$ is the one-point space.

This generalization is a meaningful one, as it can be verified that the sequence of functors $\Omega_n^o(-) : \mathbf{Top} \rightarrow \mathcal{Ab}$ defines a valid generalized homology theory and that the Pontryagin-Thom theorem extends to a natural isomorphism of homology theories. In practice, working with cobordism over a space X mainly boils down to keeping track of additional structure along the way, and we choose to omit it from our exposition for the sake of clarity. We invite the interested reader to consult Chapter 18 of Kupers [7] for more details.

We now define the *Thom spectrum MO* , whose construction relies on the ideas introduced in Section 2. Recall that a model for the classifying space of rank k vector bundles is given by the infinite Grassmanian of k -planes $BO(k) = \text{Gr}_k(\mathbb{R}^\infty)$, with associated universal bundle given by the tautological bundle $\gamma_k^\infty \rightarrow BO(k)$. Consider the rank $(k+1)$ -bundle $\epsilon^1 \oplus \gamma_k^\infty$ over $BO(k)$ given by Whitney sum of the trivial line bundle with the universal rank k bundle over $BO(k)$. By the classification theorem, this bundle may be obtained as the pullback of the universal rank $(k+1)$ bundle under a map (unique up to homotopy) $f : BO(k) \rightarrow BO(k+1)$. Hence we obtain a bundle map $\epsilon^1 \oplus \gamma_k^\infty \rightarrow \gamma_{k+1}^\infty$. Precomposing the associated map at the level of Thom spaces with the homeomorphism $\Sigma\text{Th}(\gamma_k^\infty) \cong \text{Th}(\epsilon^1 \oplus \gamma_k^\infty)$ yields a map $\Sigma\text{Th}(\gamma_k^\infty) \rightarrow \text{Th}(\gamma_{k+1}^\infty)$.

Definition 7.3. The *Thom spectrum MO* is the spectrum with component spaces $MO(k) := \text{Th}(\gamma_k^\infty)$ and structure maps given by the composites:

$$\Sigma MO(k) = \Sigma\text{Th}(\gamma_k^\infty) \xrightarrow{\sim} \text{Th}(\epsilon^1 \oplus \gamma_k^\infty) \rightarrow \text{Th}(\gamma_{k+1}^\infty) = MO(k+1).$$

Of particular interest to us will be the stable homotopy groups associated to the Thom spectrum, defined as in Section 5 by:

$$\pi_n(MO) := \varinjlim_k \pi_{n+k}(MO(k)).$$

We further observe that MO can be given the structure of a commutative ring spectrum, so that $\pi_*(MO)$ inherits a graded commutative ring structure. The construction is as follows: from the natural product map $BO(n) \times BO(m) \rightarrow$

$BO(n+m)$ at the level of Grassmanians, we obtain an associated bundle map $\gamma_n^\infty \oplus \gamma_m^\infty \rightarrow \gamma_{n+m}^\infty$ with domain the external direct sum of γ_n^∞ and γ_m^∞ . We then use the homeomorphism $\text{Th}(\gamma_n^\infty \oplus \gamma_m^\infty) \cong \text{Th}(\gamma_n^\infty) \wedge \text{Th}(\gamma_m^\infty)$ described in Section 2 to obtain a product map $p_{n,m} : MO(n) \wedge MO(m) \rightarrow MO(n+m)$ at the level of Thom spaces. The unit map is the obvious one.

Remark 7.4. As with cobordism, it is possible to take a space X as parameter and define the abelian groups:

$$\pi_n(MO \wedge X_+) := \varinjlim_k \pi_{n+k}(MO(k) \wedge X_+),$$

with appropriate direct system maps obtained using suspension homomorphisms. It can be proven that the functors $MO_n(-) := \pi_n(MO \wedge -)$ then form a valid generalized homology theory. See Section 22.1 of May [9] for more details.

We conclude this section by mentioning two constructions from traditional algebraic topology which carry over to the settings of spectra, and will be used in the computation of $\pi_*(MO)$. First, for any spectrum E , the Hurewicz homomorphisms $h : \pi_{n+k}(E_k) \rightarrow \tilde{H}_{n+k}(E_k)$ strapped together induce a map on the direct limit $h : \pi_n(E) \rightarrow H_n(E)$, to which we refer as the *stable Hurewicz homomorphism* associated to E in degree n . Next, the Thom isomorphisms $\theta : H^i(BO(k); \mathbb{F}_2) \rightarrow \tilde{H}^{n+i}(MO(k); \mathbb{F}_2)$ may be shown to induce a *stable Thom isomorphism*:

$$\Phi : H^i(BO) \rightarrow H^i(MO),$$

where the shift in degree was cancelled by passage to inverse limits. Dually, we may also obtain a homology version of the stable Thom isomorphism:

$$\Phi : H_i(MO) \rightarrow H_i(BO).$$

We refer to Section 25.3 of May [9] for the details of this construction.

8. THE PONTRYAGIN-THOM THEOREM

Having introduced the unoriented cobordism ring and the Thom spectrum, we now set out to describe the Pontryagin-Thom construction. The crux of the argument will be to make use of transversality and classifying maps to move data back and forth between cobordism groups and stable homotopy groups.

Construction 8.1. We first construct a map $\alpha : \Omega_n^o \rightarrow \pi_n(MO)$. Given a cobordism class $[M] \in \Omega_n^o$, pick a closed n -manifold representative M . Using the Whitney embedding theorem and the tubular neighborhood theorem, find an embedding $e : M \rightarrow \mathbb{R}^{n+k}$ of M into euclidian space that extends to an embedding of a tubular neighborhood T of $e(M)$. By compactness of M , we may assume that T is an ϵ -disk bundle $D_\epsilon(\nu_M)$ of the normal bundle ν_M for some $\epsilon > 0$. Hence up to a rescaling, the pointed space $\bar{T}/\partial\bar{T}$ is homeomorphic to the Thom space $\text{Th}(\nu_M)$ of the normal bundle of M with respect to the chosen embedding. Letting $(\mathbb{R}^{n+k})^+$ denote the one-point compactification of \mathbb{R}^{n+k} , we may define a map:

$$\bar{\eta} : (\mathbb{R}^{n+k})^+ \rightarrow \bar{T}/\partial\bar{T}$$

given by the identity on $T \subset \mathbb{R}^{n+k}$ and sending everything else to the basepoint $\partial\bar{T}$ of the target. We readily see that $\bar{\eta}$ is continuous by noting that either a closed set $A \subset \bar{T}/\partial\bar{T}$ does not contain the basepoint, hence is sent to itself in $\mathbb{R}^{n+k} \subset (\mathbb{R}^{n+k})^+$, or A does contain the basepoint, in which case we have that $\bar{\eta}^{-1}(A) =$

$q^{-1}(A) \cup (\partial\bar{T} \cup ((\mathbb{R}^{n+k})^+ - T))$, which is closed (here $q : \bar{T} \rightarrow \bar{T}/\partial\bar{T}$ denotes the quotient map). Composing $\bar{\eta}$ with the homeomorphisms $S^{n+k} \cong (\mathbb{R}^{n+k})^+$ and $\bar{T}/\partial\bar{T} \cong \text{Th}(\nu_M)$, we obtain a map $\eta : S^{n+k} \rightarrow \text{Th}(\nu_M)$, which we call the *collapse map* associated with this embedding.

Now, consider the *Gauss map* $M \rightarrow \text{Gr}_k(\mathbb{R}^{n+k})$ sending a point $p \in M$ to the k -dimensional normal space $(T_p M)^\perp$ at p associated with the embedding e . This assignment may be composed with the canonical inclusion $\text{Gr}_k(\mathbb{R}^{n+k}) \hookrightarrow \text{Gr}_k(\mathbb{R}^\infty)$ to obtain a map $M \rightarrow \text{Gr}_k(\mathbb{R}^\infty)$. The pullback of the universal $O(k)$ bundle $\gamma_k^\infty \rightarrow \text{Gr}_k(\mathbb{R}^\infty)$ along this map corresponds precisely to the normal bundle $\nu_M \rightarrow M$. Hence, from the bundle map $\nu_M \rightarrow \gamma_k^\infty$ appearing in the pullback square:

$$\begin{array}{ccc} \nu_M & \longrightarrow & \gamma_k^\infty \\ \downarrow & & \downarrow \\ M & \longrightarrow & \text{Gr}(\mathbb{R}^{n+k}) \longrightarrow \text{Gr}(\mathbb{R}^\infty) \end{array}$$

we get an associated map at the level of Thom spaces $f : \text{Th}(\nu_M) \rightarrow \text{Th}(\gamma_k^\infty)$. Finally, the above discussion leads to the diagram:

$$\begin{array}{ccccc} (\mathbb{R}^{n+k})^+ & \xrightarrow{\bar{\eta}} & \bar{T}/\partial\bar{T} & & \\ \uparrow & & \downarrow & & \\ S^{n+k} & \xrightarrow{\eta} & \text{Th}(\nu_M) & \xrightarrow{f} & \text{Th}(\gamma_k^\infty) = MO(k). \\ & \searrow & \alpha[M] \in \pi_{n+k}(MO(k)) & \nearrow & \end{array}$$

We define $\alpha[M]$ to be the resulting element $l[f\eta]$ of $\pi_n(MO)$, where l denotes the canonical inclusion $l : \pi_{n+k}(MO(k)) \hookrightarrow \varinjlim_k \pi_{n+k}(MO(k)) = \pi_n(MO)$.

Next, we describe the map $\beta : \pi_n(MO) \rightarrow \Omega_n^o$. Start with an element $[\varphi] \in \pi_n(MO)$ and choose a representative $S^{n+k} \rightarrow MO(k)$. By compactness of S^{n+k} , this map factors through a map $\varphi : S^{n+k} \rightarrow \text{Th}(\gamma_k^{k+m})$ for some large enough $m > 0$. Let $\xi : \text{Gr}_k(\mathbb{R}^{m+k}) \rightarrow \text{Th}(\gamma_k^{m+k})$ denote the zero section associated to the bundle $\gamma_k^{m+k} \rightarrow \text{Gr}(\mathbb{R}^{m+k})$, viewed inside the Thom space of γ_k^{m+k} . Then $\xi(\text{Gr}(\mathbb{R}^{m+k}))$ appears as an embedded submanifold of $\text{Th}(\gamma_k^{m+k})$ of codimension k . By the Whitney approximation theorem and the genericity theorem for transversality, we may choose a map $f : S^{n+k} \rightarrow \text{Th}(\gamma_k^{m+k})$ which is homotopic to φ , such that f is smooth and transverse to $\xi(\text{Gr}(\mathbb{R}^{m+k}))$. Hence by Theorem 3.6, $X = f^{-1}\xi(\text{Gr}(\mathbb{R}^{m+k}))$ is an embedded submanifold of S^{n+k} of codimension k , hence of dimension n .

Observe that $\xi(\text{Gr}_k(\mathbb{R}^{k+m}))$ is a compact subset of the Hausdorff space $\text{Th}(\gamma_k^{k+m})$, hence it is closed, so that its preimage X under f is a closed subset of the compact space S^{n+k} , thus X is compact. We define the image of $[\varphi]$ under $\beta : \pi_n(MO) \rightarrow \Omega_n^o$

to be the cobordism class of the closed n -manifold X defined above. The construction is summarized in the following diagram:

$$\begin{array}{ccccc}
 S^{n+k} & \xrightarrow{\varphi} & \text{Th}(\gamma_k^{k+m}) & \xrightarrow{l} & \text{Th}(\gamma_k^\infty) = MO(k) \\
 & \xrightarrow[\simeq]{f} & & & \\
 \uparrow l & & \uparrow \xi & & \\
 \beta[\varphi] := X & \xleftarrow[f^{-1}\xi]{} & \text{Gr}_k(\mathbb{R}^{k+m}) & &
 \end{array}$$

Having described the maps $\alpha : \Omega_n^o \rightarrow \pi_n(MO)$ and $\beta : \pi_n(MO) \rightarrow \Omega_n^o$, we have yet to verify that they are indeed well-defined, i.e. that the resulting assignments is independent of any choices made along the way. We prove that both maps are independent of the choice of initial representatives, but omit the verification of other details to keep the exposition digestible. A full verification may be found in Chapter 19 of Kupers [7], further taking into account a space X as parameter as mentioned in Remarks 7.2 and 7.4.

Lemma 8.2. *The assignments $[M] \mapsto \alpha[M]$ and $[\varphi] \mapsto \beta[\varphi]$ are independent of the choice of representatives.*

Proof. First suppose that we are given homotopic elements $\varphi \simeq \psi$ of $\pi_n(MO)$. Then φ, ψ are both elements of $\pi_{n+k}(MO(k))$ for some k . Choose a large enough m so that φ and ψ both factor through $\text{Th}(\gamma_k^{m+k})$, and let $f \simeq \varphi$ and $g \simeq \psi$ be the homotopic smooth maps chosen as in construction (8.1) to be transverse to the zero section $\xi(\text{Gr}_k(\mathbb{R}^{m+k}))$ for some large enough m . Then we may find a homotopy $h : S^{n+k} \times I \rightarrow MO(k)$ from f to g . In turn, by compactness, h factors through $\text{Th}(\gamma_k^{k+N})$ for some large enough N .

Now, by the relative Whitney approximation theorem and the genericity theorem for transversality, we can find a smooth map $\mu : S^{n+k} \times I \rightarrow \text{Th}(\gamma_k^{k+N})$ homotopic to h relative to $S^{n+k} \times \partial I$ such that μ is transverse to $\xi(\text{Gr}_k(\mathbb{R}^{n+k}))$. Thus by Theorem 3.6, $\mu^{-1}(\xi(\text{Gr}_k(\mathbb{R}^{n+k})))$ is a smooth embedded submanifold of codimension k , hence of dimension $(n+1)$ in $S^{n+k} \times I$. By construction, μ restricts to f , resp g on $S^{n+k} \times \{0\}$, resp. $S^{n+k} \times \{1\}$, so that $\mu^{-1}(\xi(\text{Gr}_k(\mathbb{R}^{n+k})))$ appears to be a valid cobordism between $\beta[\varphi]$ and $\beta[\psi]$, as needed. Next, we show that if M is null-bordant, then $\alpha[M]$ is null-homotopic. This will suffice to prove that $\alpha : \Omega_n^o \rightarrow \pi_n(MO)$ respects cobordism classes provided it is a group homomorphism, which we show in Lemma (7.4). So suppose $M = \partial W$ can be realized as the boundary of an $(n+1)$ -manifold W . Let $e : T \rightarrow S^{n+k}$ be the chosen tubular neighborhood embedding in the construction of $\alpha[M]$, viewed as a map with codomain $S^{n+k} \cong (\mathbb{R}^{n+k})^+$. Then e extends to an embedding $E : L \rightarrow D^{n+k+1}$ of a tubular neighborhood L of W into the unit ball in \mathbb{R}^{n+k+1} , with the property that $E|_{S^{n+k}} = e$. Observe that by construction, the corresponding maps at the level of Thom spaces $S^{n+k} \rightarrow \text{Th}(\nu_M)$ and $D^{n+k+1} \rightarrow \text{Th}(\nu_W)$ fit into a commutative square with respect to the inclusion maps:

$$(8.3) \quad \begin{array}{ccc}
 S^{n+k} & \longrightarrow & \text{Th}(\nu_M) \\
 \downarrow & & \downarrow \\
 D^{n+k+1} & \longrightarrow & \text{Th}(\nu_W).
 \end{array}$$

Further, observe that the classifying maps $M, W \rightarrow \text{Gr}_k(\mathbb{R}^\infty) = BO(k)$ used in the construction of the map α commute with the inclusion map $M \rightarrow W$. The natural isomorphism $\text{Vect}_{\mathbb{R}}^k \cong h_{BO(k)}$ allows us to infer commutativity of the triangle:

$$\begin{array}{ccc} \nu_M & \xrightarrow{\quad} & \nu_W \\ & \searrow & \swarrow \\ & \gamma_k^\infty & \end{array}$$

and hence commutativity of the corresponding diagram at the level of Thom spaces. Putting this fact together with diagram (8.3), we see that the map $\alpha[M] : S^{n+k} \rightarrow \text{Th}(\nu_M) \rightarrow MO(k)$ factors through the composite:

$$S^{n+k} \rightarrow D^{n+k+1} \rightarrow \text{Th}(\nu_W) \rightarrow MO(k),$$

and hence in particular that it factors through the contractible space D^{n+k+1} , which readily implies that $\alpha[M]$ is contractible, as claimed. \square

We now show that the map α is a group homomorphism and that the maps α and β are mutually inverse. It will then follow that β is also a group homomorphism and that together these two maps provide the sought-after group isomorphism.

Lemma 8.4. *The map $\alpha : \Omega_n^o \rightarrow \pi_n(MO)$ is a group homomorphism.*

Proof. Let M, N be representatives of two cobordism classes in Ω_n^o . We show that the classes $\alpha[M \amalg N]$ and $\alpha[M] + \alpha[N]$ in $\pi_n(MO)$ are equal, where $\alpha[M] + \alpha[N]$ refers to the composite of two elements of $\pi_n(MO)$ as defined in Section 5.

Choose embeddings $e : M \rightarrow \mathbb{R}^{n+k}$, $i : N \rightarrow \mathbb{R}^{n+k}$ of M , resp. N into euclidian spaces of the same dimension. By compactness, we may compose these embeddings with translations so that they each lie in a distinct hemisphere of the sphere $S^{n+k} \cong (\mathbb{R}^{n+k})^+$ and extend to embeddings of the tubular neighborhoods we would have chosen for M and N individually. Now, recall that the associated Thom space $\text{Th}(\nu_{M \amalg N})$ is obtained by fiberwise one-point compactification followed by an identification of all the resulting points at infinity. This process may be broken down by identifying all points at infinity associated to M , resp. N separately, then further identifying the two remaining points at infinity, so that we get a canonical identification $\text{Th}(\nu_{M \amalg N}) \cong \text{Th}(\nu_M) \vee \text{Th}(\nu_N)$.

Now, by construction, the collapse map $S^{n+k} \rightarrow \bar{T}/\partial\bar{T}$ associated to the embedding of $M \amalg N$ described above restricts on each hemisphere to the collapse map of the corresponding manifold. Thus we may describe $\alpha[M \amalg N]$ as the composite:

$$S^{n+k} \rightarrow S^{n+k} \vee S^{n+k} \xrightarrow{f \vee g} \text{Th}(\nu_M) \vee \text{Th}(\nu_N) \cong \text{Th}(\nu_{M \amalg N}) \rightarrow MO(k),$$

where f , resp. g are the collapse maps associated to the embeddings $e : M \rightarrow \mathbb{R}^{n+k}$, resp. $i : N \rightarrow \mathbb{R}^{n+k}$. This agrees with the behavior of $\alpha[M] + \alpha[N]$, as needed. \square

Lemma 8.5. *The maps $\alpha : \Omega_n^o \rightarrow \pi_n(MO)$ and $\beta : \pi_n(MO) \rightarrow \Omega_n^o$ are mutually inverse.*

Proof. We first show that the composite $\beta \circ \alpha$ equals the identity on Ω_n^o . Start with a representative M of an arbitrary element of Ω_n^o . Recall that the associated map $\alpha[M] : S^{n+k} \rightarrow MO(k)$ (for some k) is given by the composite $f\eta$, where $\eta : S^{n+k} \rightarrow \text{Th}(\nu_M)$ is the collapse map and $f : \text{Th}(\nu_M) \rightarrow MO(k)$ is the bundle map associated to the Gauss map $M \rightarrow \text{Gr}_k(\mathbb{R}^\infty)$ viewed at the level of Thom

spaces. By compactness, the latter map factors through $\text{Th}(\gamma_k^{m+k})$ for some large enough m , and we find that the composite $\alpha[M]$ is already smooth and transverse to the zero section of $\text{Th}(\gamma_k^{m+k}) \rightarrow \text{Gr}_k(\mathbb{R}^{m+k})$. Since only M was mapped to the image of the zero section of $\text{Th}(\gamma_k^{m+k})$ under the composite $f\eta$, it follows that the pre-image of the zero section under this composite, i.e. the image of $\alpha[M]$ under β , is precisely M , as needed.

Next, we verify that $\alpha \circ \beta = id_{\pi_n(MO)}$. As before, starting with a homotopy class $[\varphi] : S^{n+k} \rightarrow MO(k)$, choose m large enough so that the map factors through $\text{Th}(\gamma_k^{m+k})$ by compactness, and choose a smooth representative φ which is transverse to the image of the zero section $\xi : \text{Gr}_k(\mathbb{R}^{m+k}) \rightarrow \text{Th}(\gamma_k^{m+k})$. Write M for the resulting embedded submanifold of dimension n in S^{n+k} , so that $[M] = \beta[\varphi]$. Now, observe that $\text{Th}(\gamma_k^{k+m}) - \{\infty\}$ provides a valid tubular neighborhood around $\xi(\text{Gr}_k(\mathbb{R}^{m+k}))$, which pulls back under φ to a tubular neighborhood of M in S^{n+k} avoiding the basepoint. Then, we see that the map $\varphi : S^{n+k} \rightarrow \text{Th}(\gamma_k^{m+k})$ factors through the collapse map $\eta : S^{n+k} \rightarrow \text{Th}(\nu_M)$ associated to the embedding of M obtained above, i.e. the following triangle commutes:

$$\begin{array}{ccc} S^{n+k} & \xrightarrow{\varphi} & \text{Th}(\gamma_k^{m+k}) \\ & \searrow \eta & \nearrow \bar{f} \\ & & \text{Th}(\nu_M). \end{array}$$

Composing on the right with the inclusion into $MO(k)$, the bottom composite is precisely $\alpha[M]$, whence we see that $[\varphi] = \alpha[M] = \alpha \circ \beta[\varphi]$, as claimed. \square

Putting everything together, we have proven the following theorem:

Theorem 8.6. (*Pontryagin-Thom Theorem*) *For every $n \geq 0$, the maps*

$$\Omega_n^o \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} \pi_n(MO)$$

provide group isomorphisms between the n th unoriented cobordism group and the n th stable homotopy group of the Thom spectrum.

While we have decided to restrict our attention to the above theorem for clarity of exposition, it is possible to keep track of more structure along the way in order to establish the following result (in the terminology of Remarks 7.2 and 7.4). For instance, the ring structure on the Thom spectrum briefly mentioned in Section 7 induces a ring structure on $\pi_*(MO)$, which can be shown to be compatible with the ring structure on Ω_*^o given by Cartesian products. In greater generality:

Theorem 8.7. *For any space X , there exists a ring isomorphism:*

$$\Omega_*^o(X) \xrightarrow{\sim} \pi_*(MO \wedge X_+)$$

which induces a natural isomorphism of generalized homology theories.

9. COMPUTATION OF $\pi_*(MO)$

We now come to the computation of the stable homotopy groups of the Thom spectrum, which by the Pontryagin-Thom theorem will provide insight into the classification of closed manifolds up to cobordism. The following discussion relies

on the background notions laid out in Section 5, and follows the proof given in Chapter 25 of May [9]; an emphasis is put on the overall structure of the argument. We conclude this paper with a striking result characterizing cobordism using only Stiefel-Whitney numbers.

Recall that Stiefel-Whitney classes are natural transformations $\text{Vect}_{\mathbb{R}}^n \rightarrow H^i$. By representability of $\text{Vect}_{\mathbb{R}}^n \cong h_{BO(n)}$ and the Yoneda lemma, we find that

$$\text{Fun}(\text{Vect}_{\mathbb{R}}^n, H^i) \cong \text{Fun}(h_{BO(n)}, H^i) \cong H^i(BO(n); \mathbb{F}_2),$$

hence each w_i corresponds to a unique element of $H^i(BO(n); \mathbb{F}_2)$. One may then use the Serre spectral sequence to show that $H^*(BO(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, \dots, w_n]$ as polynomial algebras, where $|w_i| = i$ for all i - see [Prop. 2.3.7] of Kochman [6] for a proof. Thus, passing to direct limits, we find that $H^*(BO) \cong \mathbb{F}_2[w_i \mid |w_i| = i]$. Dualizing gives that $H_*(BO) \cong \mathbb{F}_2[b_i \mid |b_i| = i]$ as \mathbb{F}_2 -vector spaces. This isomorphism can be shown to extend to an isomorphism of \mathbb{F}_2 -algebras. By the homology version of the stable Thom isomorphism, it follows that $H_*(MO) \cong \mathbb{F}_2[a_i \mid |a_i| = i]$, where each a_i is the unique preimage of b_i under $\Phi : H_i(MO) \rightarrow H_i(BO)$.

Next, we need to understand the \mathcal{A}_* -comodule structure of $H_*(MO)$. This will be facilitated by the following observation: given any spectrum E and a space X , a map $X \rightarrow E_k$ may be used to induce a map $\Sigma^{-k}(\Sigma^\infty X) \rightarrow E$ given in each degree by the composite $\Sigma^{n-k} X \rightarrow \Sigma^{n-k} E_k \rightarrow E_n$, where the second map is an iteration of structure maps. In particular, it's not too difficult to see that there is a homotopy equivalence $\mathbb{R}P^\infty \rightarrow MO(1)$, which we may use to induce a map $i : \Sigma^{-1}(\Sigma^\infty \mathbb{R}P^\infty) \rightarrow MO$. By naturality of the \mathcal{A}_* -coaction, we get an associated commutative square:

$$\begin{array}{ccc} H_{*+1}(\mathbb{R}P^\infty) & \xrightarrow{i_*} & H_*(MO) \\ \gamma \downarrow & & \downarrow \gamma \\ \mathcal{A}_* \otimes H_{*+1}(\mathbb{R}P^\infty) & \xrightarrow{i_*} & \mathcal{A}_* \otimes H_*(MO), \end{array}$$

where it can be verified that i_* takes generators to generators via $i_* : \bar{x}^{n+1} \mapsto a_n$, where \bar{x}^n denotes the dual basis element to $x^n \in H^*(\mathbb{R}P^\infty)$. Hence it suffices to understand the coaction of \mathcal{A}_* on $\mathbb{R}P^\infty$. Recall from earlier work in Example 5.12 that admissible square of the form $Sq^{I_r} := Sq^{2^{r-1}} Sq^{2^{r-2}} \dots Sq^1$ are the only ones acting non-trivially on x , via $Sq^{I_r}(x) = x^{2^r}$. We may then deduce that the \mathcal{A}_* -coaction on generators of $H_*(\mathbb{R}P^\infty)$ is given modulo decomposables by:

$$\gamma(\bar{x}^n) = \begin{cases} 1 \otimes \bar{x}^n & + (\text{decomposables}) \text{ if } n \text{ is not of the form } 2^r, \\ 1 \otimes \bar{x}^n + \xi_j \otimes \bar{x} & + (\text{decomposables}) \text{ if } n = 2^r \text{ for some } r. \end{cases}$$

The \mathcal{A}_* -coaction on generators $a_n \in H_*(MO)$ then follows by naturality:

$$\begin{aligned} \gamma(a_n) &= \gamma \circ i_*(\bar{x}^{n+1}) = i_* \circ \gamma(\bar{x}^{n+1}) \\ &= \begin{cases} 1 \otimes a_n & + (\text{decomposables}) \text{ if } n+1 \text{ is not of the form } 2^r, \\ \xi_r \otimes 1 + 1 \otimes a_n & + (\text{decomposables}) \text{ if } n+1 = 2^r \text{ for some } r. \end{cases} \end{aligned}$$

Now, define the abstract \mathbb{F}_2 -algebra $N_* := \mathbb{F}_2[u_i \mid |u_i| = i, i \neq 2^r - 1]$. We may then consider the following sequence of morphisms:

$$(9.1) \quad H_*(MO) \xrightarrow{\gamma} \mathcal{A}_* \otimes H_*(MO) \xrightarrow{1 \otimes f} \mathcal{A}_* \otimes N_*,$$

where the map $f : H_*(MO) \rightarrow N_*$ is given on generators as follows:

$$f : a_i \mapsto \begin{cases} u_i & \text{if } i \text{ is not of the form } 2^r - 1, \\ 0 & \text{if } i = 2^r - 1 \text{ for some } r. \end{cases}$$

The composite in (9.1) is a homomorphism of \mathbb{F}_2 -algebras and of \mathcal{A}_* -comodules, and it is in fact an isomorphism. To see this, note that generators of $H_*(MO)$ are taken precisely to the generators of $\mathcal{A}_* \otimes N_*$:

$$a_i \mapsto \begin{cases} 1 \otimes a_i & + \text{ (decomposables)} \\ \xi_r \otimes 1 + 1 \otimes a_i & + \text{ (decomposables)} \end{cases} \mapsto \begin{cases} 1 \otimes u_i & \text{if } i \text{ not of the form } 2^r - 1, \\ \xi_r \otimes 1 & \text{if } i = 2^r - 1 \text{ for some } r. \end{cases}$$

Hence upon dualizing, denoting the dual of N_* by N^* , we get a map

$$\mathcal{A} \otimes N^* \rightarrow H^*(MO)$$

which is an isomorphism of \mathcal{A} -modules and of \mathbb{F}_2 -coalgebras. In particular, it appears that $H^*(MO)$ is a free \mathcal{A} -module with generators given by a basis $\{v_\alpha\}$ for N^* . Recall from Section 4 that the cohomology of $H^*(MO)$ in degree i may be represented via:

$$H^i(MO) \cong [MO, \Sigma^i H\mathbb{F}_2],$$

where $\Sigma^i H\mathbb{F}_2$ denotes the i th suspension of the Eilenberg-MacLane spectrum $H\mathbb{F}_2$. Letting α denote the degree of each v_α , we therefore get maps $f_\alpha : MO \rightarrow \Sigma^\alpha H\mathbb{F}_2$ corresponding to each v_α . Taking the wedge of these maps yields a map

$$\bigvee_{\alpha} f_{\alpha} := f : MO \rightarrow \bigvee_{\alpha} \Sigma^{\alpha} H\mathbb{F}_2.$$

But as we observed in Section 4, homotopy groups of suspended Eilenberg MacLane spectra can be immediately computed to be

$$\pi_k(\Sigma^\alpha H\mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & k = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Hence by distributivity of taking homotopy groups over wedge products, we see that $\pi_*(\bigvee_{\alpha} \Sigma^{\alpha} H\mathbb{F}_2) \cong N^*$, as \mathbb{F}_2 -algebras. We now claim that f induces an isomorphism on stable homotopy groups.

Lemma 9.2. *The map $f : MO \rightarrow \bigvee_{\alpha} \Sigma^{\alpha} H\mathbb{F}_2$ is a weak homotopy equivalence.*

Proof. For clarity, we switch notation and write $\mathbb{Z}/2 \equiv \mathbb{F}_2$. Observe that the map f induces an isomorphism on $\mathbb{Z}/2$ -cohomology by construction. Hence it also induces an isomorphism on $\mathbb{Z}/2$ homology. From there, we may inductively contemplate the short exact sequence of coefficients:

$$0 \longrightarrow \mathbb{Z}/2^k \longrightarrow \mathbb{Z}/2^{k+1} \longrightarrow \mathbb{Z}/2^k \longrightarrow 0.$$

In particular, we may use naturality of the associated homology long exact sequence together with the 5-lemma to see that f must induce an isomorphism on $\mathbb{Z}/2^k$ for all $k \geq 1$, and hence on homology with coefficients in the direct limit $\mathbb{Z}/2^\infty := \varinjlim_k \mathbb{Z}/2^k$. Next, observe that the fact that $\pi_*(MO)$ is 2-torsion (from the

Pontryagin-Thom isomorphism) implies that both $H_*(MO; \mathbb{Q})$ and $H_*(MO; \mathbb{Z}/p^\infty)$ for p an odd prime must be trivial. Now, consider the short exact sequence of coefficients:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}/p^\infty \longrightarrow 0.$$

Using naturality of the associated homology long exact sequence and relying on the above discussion, we may apply the 5-lemma once again to see that f must induce an isomorphism on integral homology. It then follows from the spectrum version of the dual Whitehead theorem that f must in fact be a weak homotopy equivalence. \square

Thus the map f induces an isomorphism between $\pi_*(MO)$ and $\pi_*(\bigvee_\alpha \Sigma^\alpha H\mathbb{F}_2)$ as \mathbb{F}_2 -vector spaces. Finally, we check that this is in fact an isomorphism of \mathbb{F}_2 -algebras by considering the sequence of morphisms:

$$\pi_*(MO) \xrightarrow{h} H_*(MO) \xrightarrow{\gamma} \mathcal{A}_* \otimes H_*(MO) \xrightarrow{1 \otimes f} \mathcal{A}_* \otimes N_*,$$

where h denotes the stable Hurewicz homomorphism. First observe that for any $[y] \in \pi_n(MO)$, we have that $\gamma \circ h(y) = 1 \otimes h(y)$. Indeed, picking a representative $y \in \pi_{n+k}(MO(k))$ for the class $[y]$, recall that the Hurewicz homomorphism is given by taking the image of $1 \in H_{n+k}(S^{n+k})$ under the induced map by y on homology. Now, the coaction of \mathcal{A}_* is trivial on S^{n+k} , since its cohomology is concentrated in degree $(n+k)$. Hence by naturality of the coaction of \mathcal{A}_* , γ must also act trivially on $h(y)$, as claimed. Using Lemma (9.2) and naturality with respect to the map $f_* : \pi_*(MO) \rightarrow \pi_*(\bigvee_\alpha \Sigma^\alpha H\mathbb{F}_2)$ tells us that h must be a monomorphism of \mathbb{F}_2 -algebras. Thus the full composite is a monomorphism from $\pi_*(MO)$ to $\mathbb{F}_2 \otimes N_* \cong N_*$. The latter have equal dimension in every degree as \mathbb{F}_2 -vector spaces by previous work, so h be an isomorphism. We have thus proven:

Theorem 9.3. *The stable homotopy groups of MO are given as a graded ring by the polynomial algebra:*

$$\pi_*(MO) \cong \mathbb{F}_2[u_i \mid |u_i| = i, i \neq 2^r - 1].$$

Combining the above theorem with the Pontryagin-Thom theorem gives a complete classification of closed manifolds up to unoriented cobordism. It remains to find explicit generators for the non-trivial cobordism classes corresponding to the generators u_i in the above. In his original 1954 paper [15], Thom observed that for even values of i , the cobordism class corresponding to the element u_i could be represented by $\mathbb{R}P^i$. The same cannot be said of the case where i is odd, in which case $\mathbb{R}P^i$ is null-bordant (by Theorem 9.4 below together with the observation that all the associated Stiefel-Whitney numbers vanish). It took two years for Dold [4] to produce valid representatives for the non-trivial cobordism classes u_i for i odd, which may be described as follows: for $n > m$, define the *Dold manifold* $H_{n,m}$ to be the codimension 1 submanifold of $\mathbb{R}P^n \times \mathbb{R}P^m$ given by:

$$H_{n,m} := \{([x_0 : \dots : x_n], [y_0 : \dots : y_m]) \mid \sum_{j=1}^m x_j y_j = 0\}.$$

Now, if i is odd and not of the form $2^r - 1$, we may express $i + 1$ in the form $2^p(2q + 1)$ for some positive integers $p, q \geq 1$. Dold proved that we may then take $H_{2^{p+1}q, 2^p}$ as a valid representative of the cobordism class in Ω_i^O corresponding to u_i .

We conclude this final section with a remarkable criterion for determining when two closed manifolds are cobordant, using only knowledge of their Stiefel-Whitney numbers. The homotopy invariance of Stiefel-Whitney numbers mentioned in Section 6 indicates that the existence of a cobordism between two manifolds only depends on their respective homotopy type, hence requires no knowledge of the underlying smooth structures.

Theorem 9.4. *Two closed n -manifolds are cobordant if and only if they have the same Stiefel-Whitney numbers. In particular, a closed n -manifold is null-bordant if and only if all of its Stiefel-Whitney numbers vanish.*

Proof. (Sketch) We first observe that the first part of the theorem follows from the second. Indeed, two closed n -manifolds M and N are cobordant if and only if $[M \amalg N] = 0$. Now, the Stiefel-Whitney numbers of $M \amalg N$ are given by the sum of the Stiefel-Whitney numbers of M and N , so they all vanish if and only if the Stiefel-Whitney numbers of M and N coincide, since we are working over \mathbb{F}_2 .

Hence it suffices to prove that a manifold M is null-bordant if and only if all of its Stiefel-Whitney numbers vanish. Recall that the cohomology of BO is the polynomial algebra generated by the Stiefel-Whitney classes:

$$H^*(BO) = \mathbb{F}_2[w_i \mid |w_i| = i].$$

Hence in particular, $H^n(BO)$ is generated as a vector space by all degree n products of Stiefel-Whitney classes, and we may consider the assignment

$$H^n(BO) \otimes \Omega_n^o \xrightarrow{\#} \mathbb{F}_2$$

acting on generators by sending $w \otimes [M]$ to the associated Stiefel-Whitney number $\langle w, \mu_M \rangle$, where μ_M denotes the fundamental class of M . The crux of the proof lies in the verification that the following diagram commutes, which is detailed in Section 25.5 of May [9]:

$$\begin{array}{ccccc} H^n(BO) \otimes \Omega_n^o & \xrightarrow{id \otimes \alpha} & H^n(BO) \otimes \pi_n(MO) & \xrightarrow{id \otimes h} & H^n(BO) \otimes H_n(MO) \\ \# \downarrow & & & & \downarrow id \otimes \Phi \\ \mathbb{F}_2 & \xleftarrow{\langle, \rangle} & & & H^n(BO) \otimes H_n(BO). \end{array}$$

In this diagram, $\alpha : \Omega_n^o \rightarrow H^n(BO)$ is the Pontryagin-Thom isomorphism described in Section 7, $h : \pi_n(MO) \rightarrow H_n(MO)$ is the stable Hurewicz homomorphism, and $\Phi : H_n(MO) \rightarrow H_n(BO)$ is the homology version of the stable Thom isomorphism. Thus to say that all of the Stiefel-Whitney numbers of M vanish is to say that $w \# [M] = 0$ for all elements $w \in H^n(BO)$, which by commutativity of the above diagram is equivalent to saying that

$$\langle w, \Phi h \alpha [M] \rangle = 0,$$

for all $w \in H^n(BO)$. But the bottom horizontal map is a pairing of dual vector spaces, hence this is equivalent to saying that $\Phi h \alpha [M] = 0$. Now, α and Φ are isomorphisms and h is a monomorphism, so this amounts precisely to saying that $[M] = 0$, i.e. that M is null-bordant, concluding the proof. \square

Acknowledgments. I would like to thank Peter May for pointing me towards the study of this beautiful topic, and for sharing his invaluable insights with me during the writing process. I would also like to thank Colin Aitken for offering his unwavering support and unparalleled enthusiasm throughout the REU. Finally, I am indebted to Mark Behrens for a fruitful conversation which was absolutely essential to the writing of the last section of this paper.

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