A PRIMER ON SHEAF THEORY AND SHEAF COHOMOLOGY

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Abstract. These notes are intended to provide an introduction to basic sheaf theory and sheaf cohomology. The first few sections introduce some key definitions and technical tools leading to the definition of sheaf cohomology in the language of derived functors. The exposition is motivated whenever possible with ideas drawing from elementary manifold theory and algebraic topology. The last two sections initiate a discussion of ways in which the computation of sheaf cohomology groups can be simplified. Some familiarity with basic notions of category theory including (co)limits is assumed, as well as some degree of acquaintance with Abelian categories and chain complexes.

Contents

1. Introduction

"C'était comme si le bon vieux 'mètre cohomologique' standard dont on disposait jusqu'à présent pour 'arpenter' un espace, s'était soudain vu multiplier en une multitude inimaginablement grande de nouveaux 'mètres' de toutes les tailles, formes et substances imaginables, chacun intimement adapté à l'espace en question, et dont chacun nous livre à son sujet des informations d'une précision parfaite, et qu'il est seul à pouvoir nous donner."

\- *Grothendieck, Récoltes et Semailles*

Let X be a topological space, and denote by **Top**(X) the category with objects the open sets of X and morphisms given by inclusions (i.e. $\mathrm{Hom}_{\mathbf{Top}(X)}(U, V) = \{*\}$ if $U \subseteq V$, empty otherwise). As a rule of thumb, one can hope to learn more about a given geometric object by studying functions from or into that object. Sheaves are one way in which this practice can be formalized for topological spaces: a sheaf F on a space X can often be thought of as assigning to each open U in X a space of functions on U such that the assignment is compatible with taking restrictions and going from local to global sections. In fact, sheaves prove to be quite successful at realizing the rule of thumb suggested above: one can show further down the line that **Top**(X) can be identified as a subcategory of the category **Sh**(X) of sheaves of sets on X, and that this information is sufficient to recover X up to homeomorphism, provided precisely that X is a sober space (i.e. every irreducible subset of X is of the form $\overline{\{x\}}$ for a unique $x \in X$). This is the case, in particular, if X is just assumed to be Hausdorff. (For more details, see for instance [11].) Sheaves also offer an alternative framework in which to view familiar objects such as (smooth) manifolds; this new perspective will naturally lead us to introducing the notion of *schemes*. We shall elaborate on this viewpoint in Section 2, as soon as we have established the rudiments of the subject.

The purpose of this article is to provide an introduction to basic concepts in sheaf theory and sheaf cohomology, with some emphasis put in developing the homological algebra framework in which the latter can be naturally expressed. Sections 2 through 4 present elementary constructions that can be performed on sheaves. Sections 5 and 6 contain a streamlined treatment of relevant notions from category theory and homological algebra, with the aim to arrive at a definition of sheaf cohomology in Section 7. Section 8 presents an alternative way to compute sheaf cohomology groups, and hints at further constructions of a similar kind. Finally, Section 9 provides an introductory account to Čech cohomology, which often simplifies computations of sheaf cohomology groups.

As hinted at by the opening quote, sheaf cohomology theory appears to be a remarkable generalization of more conventional cohomology theories. For a given space X, any sheaf of abelian groups, i.e. any compatible assignment of abelian groups to elements of $\mathbf{Top}(X)$ in a sense to be made precise later, gives rise to a family of cohomology groups of X with coefficients in that sheaf. In particular, for a fixed abelian group A, one may consider the sheaf that comes as close as possible to asssigning A to every open in X, while still satisfying certain compatibility conditions. We call this sheaf the *constant sheaf* on X associated to A, and denote it by A_X. We may then restrict our attention to the sheaf cohomology $H^*(X, A_X)$ of X with coefficients in A_X. If X is locally contractible and paracompact, we are ensured the existence of a natural isomorphism $H^*(X, A_X) \cong H^*(X, A)$ between sheaf cohomology with coefficients in A_X and singular cohomology taking coefficients in A (see for instance [15]). So restricting one's attention to CW-complexes (which are paracompact and locally contractible), which is not an unreasonable thing to do due to the fact that all spaces are weakly equivalent to a CW-complex, one can recover all singular cohomology groups from sheaf cohomology. Immediate from the de Rham theorem and the above discussion is also the fact that we have, for any manifold X, an isomorphism with de Rham cohomology $H^*_{deR}(X) \cong H^*(X, \mathbb{R}_X)$.

Hence we have recovered standard cohomological constructions without even looking at sheaf cohomology with coefficients in non-constant sheaves. If one allows for any sheaf of coefficients, the situation rapidly takes a flavor of its own. For instance, sheaf cohomology may no longer be homotopy invariant. This tells us that homotopy theory has "limited leverage" on sheaf cohomology, whereas singular cohomology can be entirely introduced in homotopy-theoretic terms (via the use of Eilenberg-Maclane spectra).

2. Sheaves on a Topological Space

We formulate a definition for the notion of a sheaf on a space X with values in a fixed concrete category **C**, meaning that **C** can be thought of as a category of sets with some additional structure (so **C** may for instance denote the category of sets, abelian groups or commutative rings):

Definition 2.1. A *presheaf* F on a space X with values in a category **C** is a contravariant functor $F : \mathbf{Top}(\mathrm{X}) \to \mathbf{C}$. We write $\mathrm{res}^F_{V,U} : F(V) \to F(U)$ (where res stands for "restriction morphism") for the image of the inclusion morphism $U \hookrightarrow V$ under F, or simply $\mathrm{res}_{V,U}$ when F is clear from the context.
A *sheaf* F is a presheaf on X with values in a concrete category **C** satisfying the following local-to-global condition:

Given an open cover $\{U_i\}$ of $U \in \mathbf{Top}(\mathrm{X})$ and elements $\{f_i \in F(U_i)\}$ such that $\mathrm{res}_{U_i,U_{ij}} f_i = \mathrm{res}_{U_j,U_{ij}} f_j$ for all i, j, there exists a unique element $f \in F(U)$ satisfying $\mathrm{res}_{U,U_i} f = f_i$ for all i.

We follow the convention that a presheaf is required to map the empty set to the terminal object in **C**, if there is one.
Notice that the definition of a presheaf makes sense with values in an arbitrary category, and that a presheaf on X with values in **C** may be alternatively defined as a covariant functor $F : \mathbf{Top}(\mathrm{X})^{\mathrm{op}} \to \mathbf{C}$, where $\mathbf{Top}(\mathrm{X})^{\mathrm{op}}$ denotes the opposite category (reversed arrows) of $\mathbf{Top}(\mathrm{X})$. Hence, presheaves naturally fit in the category $\mathrm{Fun}(\mathbf{Top}(\mathrm{X})^{\mathrm{op}}, \mathbf{C})$ with objects: functors $F : \mathbf{Top}(\mathrm{X})^{\mathrm{op}} \to \mathbf{C}$ and morphisms: natural transformations of such functors. We extend this terminology to sheaves, and give the corresponding categories names in two important cases:

Definition 2.2. A *sheaf morphism* between two sheaves F, G on X with values in **C** is a natural transformation $\eta : F \to G$ of the underlying presheaves; namely, a sheaf morphism consists of a collection of morphisms $\eta_U : F(U) \to G(U)$ in **C** such that for each inclusion $U \hookrightarrow V$ we have a commutative square

$$\begin{array}{ccc} F(V) & \xrightarrow{\mathrm{res}^F_{V,U}} & F(U) \\ \downarrow{\scriptstyle \eta_V} & & \downarrow{\scriptstyle \eta_U} \\ G(V) & \xrightarrow[\mathrm{res}^G_{V,U}]{} & G(U) \end{array}$$

Sheaves on X with values in a concrete category **C** then fit into a category (properly speaking, the full subcategory of the category of **C**-valued preshaves satisfying the sheaf axiom). We write $\mathbf{Ab}(\mathrm{X})$ (resp. $\mathbf{Sh}(\mathrm{X})$) to denote the resulting category of sheaves of abelian groups on X (resp. sheaves of sets on X). We shall sometimes refer to elements of $\mathbf{Ab}(\mathrm{X})$ as *abelian sheaves*.

Noting that we have essentially created a way of associating "algebraic data" to a given topological space, a reader with some background in Algebraic Topology may be wondering whether there is a natural way to associate to any continuous map $f : X \to Y$ a corresponding "functor" between the associated sheaf categories, in the same way for instance that a map $f : X \to Y$ induces (graded) group homomorphisms $f_* : H_*(X) \to H_*(Y)$ on homology and $f^* : H^*(Y) \to H^*(X)$ on cohomology, with the assignment being functorial in both these instances.

Such assignments may indeed be carried out in the settings of, say, sheaves of abelian groups:

Definition 2.3. Let $f : X \to Y$ be a continuous map of spaces. Define the *pushforward* of f to be the map:

$$f_* : \mathbf{Ab}(\mathrm{X}) \to \mathbf{Ab}(\mathrm{Y}),$$
$$\mathrm{F} \mapsto (f_*\mathrm{F} : V \mapsto \mathrm{F}(f^{-1}(V)), \text{for any } V \in \mathbf{Top}(\mathrm{Y})),$$

with behavior with respect to inclusions inherited from F. We refer to $f_*\mathrm{F} \in \mathbf{Ab}(\mathrm{Y})$ as the pushforward of F under f.

Proposition 2.4. *Given a sheaf $F \in \boldsymbol{Ab}(X)$, the pushforward of F under f as defined above is a valid abelian sheaf on Y. The assignment induces a functor $f_* : \boldsymbol{Ab}(X) \to \boldsymbol{Ab}(Y)$ between sheaf categories.*

The verification of the above proposition is straightforward. One may further construct an assignment $f^{-1} : \mathbf{Ab}(\mathrm{Y}) \to \mathbf{Ab}(\mathrm{X})$ sending in a functorial way an abelian sheaf G on Y to a sheaf $f^{-1}G \in \mathbf{Ab}(\mathrm{X})$, the so-called *pullback* of G under f. These two constructions are closely linked to one another, as the pair f_*, f^{-1} turns out to be a pair of adjoint functors. As the definition of the pullback of f is slightly more involved, we hold off this discussion until a later section. For now, we proceed to list a few important examples that bring the above definitions to life.

Examples 2.5. (i) Let A be a fixed abelian group, and let x be a point in X. Define the *skyscraper sheaf* $A^x \in \mathbf{Ab}(\mathrm{X})$ via the assignment:

$$A^x : U \to \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases},$$

with $\mathrm{res}_{V,U}$ acting as id_A if $x \in U \cap V$ and as the unique zero morphism in $\mathbf{Ab}$, the category of abelian groups, otherwise.

More generally, let $j : \{x\} \hookrightarrow X$ denote the natural inclusion map, and let $\mathrm{A}^x \in \mathbf{Ab}(\{x\})$ denote the skyscraper sheaf over $\{x\}$ (so that A^x sends $\{x\}$ to A and the empty set to 0). We can then take the pushforward $j_*\mathrm{A}^x$ of each A^x under their respective inclusion maps, and we may further consider the *direct product sheaf* $F := \prod_{x\in X}(j_*A^x)$ on X, given by the assignment:

$$U \mapsto \prod_{x\in X}(j_*A^x)(U) \tag{2.6}$$

where the usual product is taken in $\mathbf{Ab}$, and restriction maps are defined component wise. By definition, $(j_*A^x)(U) = A^x(j^{-1}(U)))$, so that the above expression simplifies to:

$$\mathrm{F} : U \mapsto \prod_{x\in U} A \tag{2.7}$$

While the above generalization may seem a bit far-fetched for now, it will allow us to put the adjunction of j_* and j^{-1} to good use at a later time.

(ii) Let M be a smooth manifold. We may consider the presheaf of rings $\mathcal{O}_M$ sending an open U in M to the ring $C^\infty(U)$ of smooth real-valued functions on U, with $\text{res}_{V,U} : C^\infty(V) \to C^\infty(U)$ acting as the usual restriction map. The gluing lemma for smooth functions on a smooth manifold ensures that $\mathcal{O}_M$ is indeed a sheaf of rings on M (see [9] for a proof).

(iii) Let A be a fixed abelian group. We may consider the *constant presheaf* $\underline{\text{A}}$ of abelian groups sending every non-empty open in X to A, and sending $U \hookrightarrow V$ to id_A (or to the zero morphism if U is the empty set). Observe that $\underline{A}$ may fail to be a sheaf - assume A has at least two elements and consider for instance the constant presheaf on the discrete two-point space $\{p, q\} = \{p\} \cup \{q\}$, with local sections of $\underline{\text{A}}(p), \underline{\text{A}}(q)$ given by distinct elements $a, b \in A$ resp., so that no global section can possibly exist. We shall define the natural sheaf analog of $\underline{A}$ as soon as we have introduced the tools necessary to do so.

Construction 2.8. Given a sheaf morphism $\varphi : F \to G$, where F and G are sheaves with values in some abelian category, we may define an associated *kernel* (resp. *cokernel*) *presheaf* denoted $\ker\varphi$ (resp. $\text{coker}\varphi$), sending any open $U \in \mathbf{Top}(\text{X})$ to the object $\ker(\varphi_U)$ (resp. $\text{coker}(\varphi_U)$) provided by $\mathcal{A}$ as an abelian category.
The action of $\ker\varphi$ on inclusions $U \hookrightarrow V$ can be understood via the following diagram, where $l_U : \ker(\varphi_U) \to F(U)$ is the canonical morphism in $\mathcal{A}$ associated to $\ker(\varphi_U)$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(\varphi_V) & \xrightarrow{l_V} & F(V) & \xrightarrow{\varphi_V} & G(V) \\
 & & \Big\downarrow{\scriptstyle \exists!\rho_{V,U}} & & \Big\downarrow{\scriptstyle \text{res}^{\text{F}}_{V,U}} & & \Big\downarrow{\scriptstyle \text{res}^{\text{G}}_{V,U}} \\
0 & \longrightarrow & \ker(\varphi_U) & \xrightarrow[l_U]{} & F(U) & \xrightarrow[\varphi_U]{} & G(U)
\end{array}$$

The existence of a unique $\rho_{V,U} : \ker(\varphi_V) \to \ker(\varphi_U)$ making the diagram commute is ensured by the universal property of $(\ker(\varphi_U), l_U)$, invoked via the fact that $\varphi_U(\text{res}^F_{V,U} l_V) = 0$ by commutativity of the right square. Uniqueness of $\rho_{V,U}$ ensures that this construction makes $\ker\varphi$ into a well-defined presheaf on X with values in $\mathcal{A}$.

Proposition 2.9. *Given a morphism $\varphi : F \to G$ of sheaves valued in an abelian category $\mathcal{A}$, the presheaf* $\ker\varphi$ *defined above is a valid sheaf on X with values in $\mathcal{A}$.*

Proof. Let $\{U_i\}$ be an open cover of U, and let $(f_i \in \ker(\varphi_{U_i}))$ be a compatible family of local sections. We can then look at the corresponding local sections $(l_{U_i}(f_i) \in F(U_i))$, compatible by commutativity of the diagram associated with each pair U_i, U_j:

$$\begin{array}{ccc}
\ker(\varphi_{U_i}) & \xrightarrow{l_{U_i}} & F(U_i) \\
{\scriptstyle \rho_{U_i,U_{ij}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{res}_{U_i,U_{ij}}} \\
\ker(\varphi_{U_{ij}}) & \xrightarrow{l_{U_{ij}}} & F(U_{ij}) \\
{\scriptstyle \rho_{U_j,U_{ij}}}\Big\uparrow & & \Big\uparrow{\scriptstyle \text{res}_{U_j,U_{ij}}} \\
\ker(\varphi_{U_j}) & \xrightarrow{l_{U_j}} & F(U_j)
\end{array}$$

Since F is a sheaf, this gives us a unique global section f in $F(U)$ compatible with each f_i. Finally, to check that f is indeed an element of $\ker(\varphi_U)$, we use uniqueness of global sections on G together with the fact that $\text{res}^G_{U,U_i}(\varphi_U(f)) = \varphi_{U_i}(f_i) = 0$ for each i, so that we must have $\varphi_U(f) = 0$. $\square$

(We admit to using element-theoretic language in the last sentence of the above argument; this can usually be justified when working in an Abelian Category due to the Freyd-Mitchell Embedding Theorem. We refer the interested reader to a discussion of this result in [17], for instance.)

On the other hand, we note that while a dual argument allows us to define $\text{coker}\varphi$ as a valid presheaf with values in $\mathcal{A}$, this construction might fail to satisfy the sheaf condition. Roughly speaking, $\text{coker}\varphi$ may be "missing" global sections. For instance, consider the presheaf F on $S^1 \subset \mathbb{C}$ with values in **Ab** sending an open U in S^1 to the group of holomorphic functions $f : U \to \mathbb{C}$ which admit a holomorphic logarithm. It turns out that F can be presented as the cokernel of the sheaf morphism $\varphi : \mathbb{Z}_{S^1} \to \mathcal{O}_{S^1}$, where $\mathbb{Z}_{S^1}$ sends U to the abelian group of locally-constant $\mathbb{Z}$-valued functions on U, and $\varphi_U : \mathbb{Z}_{S^1}(U) \to \mathcal{O}_{S^1}(U)$ takes an element $f : U \to \mathbb{Z}$ of $\mathbb{Z}_{S^1}$ to its inclusion in the group $\mathcal{O}_{S^1}(U)$ of holomorphic functions on U. Then, we may consider for instance a cover (U_i) of S^1 by proper open subsets, each of which is assigned the local section $f_i : z \mapsto z$. These local sections are trivially compatible, each of them admits a logarithm, but they cannot be glued into a global section admitting a holomorphic logarithm on the entire unit circle.

Examples (i) and (ii) above will naturally lead us to the notion of stalks associated to a sheaf at each point of the underlying space X; example (iii) and the above construction for the cokernel presheaf will take us to the somewhat more delicate concept of sheafification of a presheaf.

3. Interlude: Smooth Manifolds as Ringed Spaces

Before continuing our exposition of basic sheaf theory, we take a moment to further discuss example (ii) above, and argue that the notion of a (Hausdorff, 2nd countable) topological space M equipped with an appropriate sheaf of rings is sufficient to fully describe the structure of a smooth manifold, as promised in the introductory paragraph. This short interlude is based on an enriching remark in [16], and provides a natural doorway into the world of schemes. The material in this section is isolated from the rest of these notes, and may safely be skipped on a first reading.

So let M be a smooth n-manifold, which we may view as an element of the category **Diff** of (finite-dimensional) smooth manifolds and smooth maps. We may then consider the pair $(\text{M}, \mathcal{O}_M)$, where M is the underlying topological space and $\mathcal{O}_M$ the sheaf of rings on M whose local sections are smooth real-valued functions on opens in M, as in example (ii). We refer to any such pair $(X, \mathcal{O}_X)$, where X is a space and $\mathcal{O}_X$ a sheaf of rings on X, as a *ringed space*, and we call $\mathcal{O}_X$ the *structure sheaf* of X.

Now, one can show that a continuous map $f : \text{M} \to \text{N}$ between manifolds is smooth if and only if for any local section $h \in C^\infty(V)$ on some open $V \subset N$, the corresponding composite hf is a valid local section on M, i.e. a well-defined element of $C^\infty(f^{-1}(V))$. Hence, if we define a *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to consist of a pair $(f, f^\#)$, where $f : X \to Y$ is a continuous map and

$f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves from $\mathcal{O}_Y$ to the pushforward $f_*\mathcal{O}_X$ of $\mathcal{O}_X$ under f, then smooth maps between manifolds are in one-to-one correspondence with morphisms between the associated ringed spaces.

We further observe that the smooth structure of a given smooth manifold M can be completely captured in the language of ringed spaces. Indeed, the requirement that for each $p \in M$ there is an open $U_\alpha \subset M$ containing p and a diffeomorphism $\varphi_\alpha : U_\alpha \to \mathbb{B}^n$ onto an open ball in $\mathbb{R}^n$ is equivalent to requiring local isomorphisms of ringed spaces $(U_\alpha, \mathcal{O}_{U_\alpha}) \to (\mathbb{B}^n, \mathcal{O}_{\mathbb{B}^n})$, where the sheaf of rings $\mathcal{O}_{U_\alpha}$ on U_α is obtained by natural restriction of the original sheaf $\mathcal{O}_X$ on X.

We have thus constructed a category $\mathcal{RS}$ of ringed spaces and ringed space morphisms in which we can view **Diff** as a full subcategory under the assignment described above. The argument can be further formalized to show that one can define a fully faithful functor from **Diff** onto a full subcategory of $\mathcal{RS}$. We refer the interested reader to [2] for more details on this identification, and to [6] for a more advanced discussion of the case in which M is not assumed to be finite-dimensional.

The definition of a scheme, a central notion in Algebraic Geometry which captures the "most general" settings in which one may study spaces defined as the vanishing sets of polynomial functions, is now but a couple of steps away. The key idea is to shift one's focus away from a Euclidian local structure, and to ask instead for local isomorphisms of ringed spaces $(U, \mathcal{O}_U) \to (\text{Spec(R)}, \mathcal{O}_{Spec(R)})$, where Spec(R) is the prime spectrum of some commutative ring R, given the Zariski Topology and equipped with a canonical sheaf of rings. Then, we define a *scheme* to be a *locally ringed space* $(X, \mathcal{O}_X)$ such that for each $x \in X$ there is an open $U \ni x$ admitting an isomorphism of locally ringed spaces $(U, \mathcal{O}_U) \to (\text{Spec(R)}, \mathcal{O}_{Spec(R)})$ for some commutative ring R. Hence the only missing notion is that of a *locally* ringed space and their associated morphisms, a further specification on ringed spaces which has to do with the concept of stalks to be discussed in the next section. We do not, however, plan to discuss the construction of the ringed space $(\text{Spec(R)}, \mathcal{O}_{Spec(R)})$, as that would take us too far afield. A traditional treatment of scheme theory may be found in chapter II of [7].

4. Stalks and Sheafification

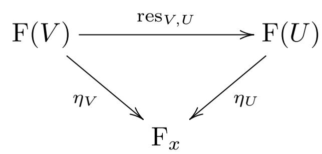
In the following, unless otherwise specified, we let F denote a sheaf of abelian groups on a topological space X, keeping in mind that most of the following constructions may be carried out in other concrete categories.

Definition 4.1. Let F be a presheaf on X. For any point $x \in$ X, define the *stalk* F_x of F at x to be the abelian group given as the direct limit

$$\mathrm{F}_x := \varinjlim_{V \ni x, V \in \mathbf{Top}(X)} \mathrm{F}(V). \tag{4.2}$$

Hence the stalk of F at x comes equipped with a family of compatible maps $(\eta_U : \mathrm{F}(U) \to \mathrm{F}_x)_{U \in N_x}$, where $N_x = \{U \subset X \text{ open} \mid U \ni x\}$ is the neighborhood system of x in X, with compatibility taken in the sense that for every inclusion pair

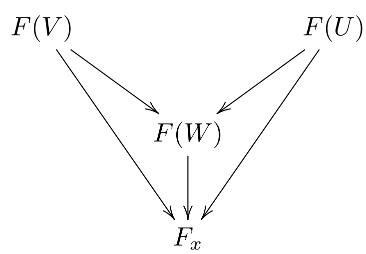
$U \subset V$ in N_x, the associated triangle



commutes. Further, Fx is defined to be the "upper limit", i.e. the initial object, in the collection of all objects C in **Ab** which have this property. That is, for any such C, there exists a unique map of abelian groups $\mathrm{F}_x \to C$ "down" into C making everything commute.
While the definition is quite succinct, the concept of a direct limit can seem quite intractable at times, and a few clarifying remarks are in order. F_x can be expressed concretely as the abelian group given by:

$$F_x = \coprod_{V \ni x} F(V) \big/ \sim \tag{4.3}$$

where $(f, U) \sim (g, V)$ iff there is an open $W \subset U \cap V$ containing x such that $\mathrm{res}_{U,W} f = \mathrm{res}_{V,W} g$, i.e. two local sections above x are identified whenever they agree on some small enough neighborhood of x. Whenever $U \ni x, f \in F(U)$, we call the natural image f_x of f in F_x the *germ* of f in F_x. As a final comment on the definition, the origin of the equivalence relation presented above is captured by the existence, by definition of F_x, of a commutative diagram:



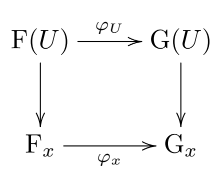
whenever $W \subset U \cap V$ are elements of N_x.

Given a sheaf morphism $\varphi : F \to G$, we naturally obtain an induced group homomorphism $\varphi_x : F_x \to G_x$ at the level of stalks for every $x \in X$, by simply taking a representative (f, U) and defining $\varphi_x[(f, U)] := (\varphi_U(f))_x$. The fact that φ_x is well defined is readily verified by applying the natural transformation φ to the above diagram, so that whenever $(f, U) \sim (g, V)$ via $\mathrm{res}^{\mathrm{F}}_{U,W} f = \mathrm{res}^{\mathrm{F}}_{V,W} g$ for some $W \ni x$, we have by naturality of φ:

$$\mathrm{res}^{\mathrm{G}}_{U,W}(\varphi_U f) = \varphi_W(\mathrm{res}^{\mathrm{F}}_{U,W} f) = \varphi_W(\mathrm{res}^{\mathrm{F}}_{V,W} g) = \mathrm{res}^{\mathrm{G}}_{V,W}(\varphi_V g)$$

and thus $(\varphi_U(f), U) \sim (\varphi_V(g), V)$ in G_x. This assignment is clearly functorial as a map ${}_x : \mathbf{Ab}(\mathrm{X}) \to \mathbf{Ab}$, and for any $U \ni x$ open, φ_U and φ_x fit into the commutative square:

(4.4)



Lemma 4.5. *Let $\varphi : F \to G$, $\psi : F \to G$ be sheaf morphisms. Then $\varphi = \psi$ iff $\varphi_x = \psi_x$ for all $x \in X$.*

Proof. Only the reverse direction needs to be checked. So suppose $\varphi_x = \psi_x$ for all $x \in X$, and pick $s \in F(U)$. We show that in fact $\varphi_U(s) = \psi_U(s)$. Letting s_x denote the image of s in F_x, we have by the commutative square (3.4) above, for each $x \in U$:

$$(\varphi_U(s))_x = \varphi_x(s_x) = \psi_x(s_x) = (\psi_U(s))_x$$

Hence we can find a representative $(t^{W_x}, W_x), t^{W_x} \in G(W_x)$ such that $\text{res}_{U,W_x}(\varphi_U(s))$ $= \text{res}_{U,W_x}(\psi_U(s)) = t^{W_x}$ for each $x \in U$. The resulting family $(t^{W_x} \in G(W_x))$ forms a compatible collection of local sections for U, and it follows by uniqueness of gluings in G that we must have $\varphi_U(s) = \psi_U(s)$, as needed. □

Lemma 4.6. *Let $\varphi : F \to G$ be a sheaf morphism. Then for all $x \in X$, $(\ker\varphi)_x = \ker(\varphi_x)$, where we identify $(\ker\varphi)_x$ with its image under the canonical inclusion $l_x : (\ker\varphi)_x \hookrightarrow F_x$.*

Proof. By functoriality, we immediately have $\varphi_x \circ l_x = (\varphi \circ l)_x = 0$, so that $(\ker\varphi)_x \subseteq \ker(\varphi_x)$. Next, let s_x be a given element of $\ker(\varphi_x)$, with representative $s \in F(U)$ for some open $U \ni x$. By the above discussion, we have a commutative diagram:

$$\begin{array}{ccccc} \ker(\varphi_U) & \xrightarrow{l} & F(U) & \xrightarrow{\varphi_U} & G(U) \\ \downarrow & & \downarrow & & \downarrow \\ \ker(\varphi)_x & \xrightarrow[l_x]{} & F_x & \xrightarrow[\varphi_x]{} & G_x \end{array}$$

Hence $(\varphi_U(s))_x = \varphi_x(s_x) = 0$, so that for some open $W \subseteq U$ about x, we have $\text{res}^G_{U,W}\varphi_U(s) = 0$, thus $\varphi_W(\text{res}^F_{U,W}s) = 0$ by naturality of φ. It follows that $(\text{res}^F_{U,W}s)_x = s_x$ lies in $(\ker\varphi)_x$, as needed. □

Examples 4.7. (i) For a fixed abelian group A, and given $p \in X$, let p_*F denote the skyscraper sheaf on X at p as defined earlier. Then it is clear that $(p_*F)_p = A$, and, as long as X is assumed to be at least T1 (say X is Hausdorff), any point q distinct from p will admit a neighborhood $V \ni q$ such that $p \notin V$, which in turn ensures that $(p_*F)_q = 0$.

(ii) A typical illustration is that of the stalk $\mathcal{O}_{M,p}$ of germs of smooth real-valued functions at a point p in some smooth manifold M. Thinking of the tangent space T_pM of M at p as a set of derivations at p $\{X_p : C^\infty(M) \to \mathbb{R}\}$, one can capture the local behavior of tangent vectors $X_p \in TpM$ by showing that $X_p(f) = X_p(g)$ provided the functions f, g agree on some small enough neighborhood W of p. In other words, a tangent vector X_p is a well-defined operator on the stalk $\mathcal{O}_{M,p}$ of $\mathcal{O}_M$ at p.

Coming back to our interpretation of a smooth manifold as a ringed space $(M, \mathcal{O}_M)$, one can rather easily prove that the stalks $\mathcal{O}_{M,p}$, which are then taken to be commutative rings, are in fact local rings. Indeed, fix $p \in M$, and consider the evaluation morphism $\text{ev}_p : \mathcal{O}_{M,p} \to \mathbb{R}, f \to f(p)$. Then ev_p is clearly surjective, so that, letting $\mathfrak{m}_p := \ker \text{ev}_p$, the quotient $\mathcal{O}_{M,p}/\mathfrak{m}_\mathfrak{p}$ is isomorphic to the field $\mathbb{R}$, and therefore $\mathfrak{m}_\mathfrak{p}$ is maximal in $\mathcal{O}_{M,p}$. We can just as easily verify that every element

$f \in \mathcal{O}_{M,p} - \mathfrak{m}_p$ is a unit, since for any such f, $f(p)$ is non-zero and by continuity we can locally construct a well-defined multiplicative inverse for f in $\mathcal{O}_{M,p}$. It follows by elementary commutative algebra that $\mathcal{O}_{M,p}$ has a unique maximal ideal, i.e. the stalk of $\mathcal{O}_M$ at p is a local ring for every $p \in M$.

We saw in the previous section that smooth maps $\pi : M \to N$ between manifolds can be put in one-to-one correspondence with morphisms of ringed spaces $(\pi, \pi^{\#}) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$. If we let $\mathfrak{m}_p$ denote the unique maximal ideal in $\mathcal{O}_{M,p}$ for each $p \in M$, and likewise for N, we observe that for any $q \in N$, the induced ring homomorphism on stalks $\pi^{\#}_q : \mathcal{O}_{N,q} \to \mathcal{O}_{M,\pi^{-1}(q)}$ satisfies the relation $\pi^{\#}_q(\mathfrak{m}_\mathfrak{q}) \subseteq \mathfrak{m}_{\pi^{-1}(q)}$. We call such a ring homomorphism a *local ring map.*

We can now make sense of the definition of schemes presented at the end of the previous section. Define a *locally ringed space* to be a ringed space $(X, \mathcal{O}_X)$ such that the stalk of $\mathcal{O}_X$ at each point $x \in X$ is a local ring. A *morphism of locally ringed spaces* is then a morphism of ringed spaces such that the induced ring homomorphism on stalks is a local ring map at each point $x \in X$.

We are now in a position to rectify our previous attempt to construct a natural notion of cokernel associated to a sheaf morphism and that of a constant sheaf associated to a fixed abelian group.

Construction 4.8. Let F be a presheaf of on X. Define the *sheafification* of F to be the functor $\mathrm{F}^+ : \mathbf{Top}(\mathrm{X})^{op} \to \mathbf{Ab}$ given by:

$$F^+ : U \longmapsto \left\{ \; f : U \to \textstyle\coprod_{x \in X} F_x \;\middle|\; \begin{array}{r} \forall x \in X, f(x) \in F_x, \text{ and} \\ \exists\, W \ni x \text{ open }, \exists\, g \in F(W) \text{ s.t} \\ f(y) = g_y \text{ in } F_y \text{ for all } y \in W \end{array} \right\} \tag{4.9}$$

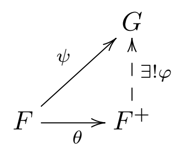
In words, F^+ admits as local sections over U the group formed by all functions from U into the disjoint union of the stalks of X which respect stalks and locally agree with (possibly different) local sections of F at the level of stalks. By defining the behavior of F^+ with respect to inclusions $U \subset V$ in the natural way (as an actual restriction map on functions, i.e. $\mathrm{res}^{F^+}_{V,U} f := f|_U$), we obtain a valid presheaf on X. Further, we can readily see that F^+ satisfies the sheaf condition, so that we have in fact that $\mathrm{F}^+ \in \mathbf{Ab}(\mathrm{X})$. Indeed, given a cover (U_i) of $U \in \mathbf{Top}(\mathrm{X})$ and compatible sections $(f_i \in F^+(U_i))$, simply define $f \in F^+(U)$ to map x to $f_i(x)$ where $x \in U_i$ for some i. Then f is a well-defined element of $F^+(U)$, and it is the unique such element satisfying $\mathrm{res}^{F^+}_{U,U_i} f = f_i$ for all i by construction.

By construction, presheaves and their sheafification are indistinguishable at the level of stalks; the verification of the following lemma is straightforward.

Lemma 4.10. *Let F be a presheaf on X with sheafification F^+. Then for all $x \in X$, $F_x \cong F^+_x$.*

As a substitute for the rather involved explicit construction above, it is possible to characterize the sheafification of a given presheaf uniquely up to isomorphism as an object in $\mathbf{Ab}(\mathrm{X})$ satisfying a universal property which should be familiar to many readers:

Proposition 4.11. *Let F be a presheaf on X with sheafification F^+ as defined above. Then there exists a presheaf morphism $\theta : F \to F^+$, and the pair $(F^+, \theta : F \to F^+)$ is initial in **Ab**(X) with respect to this property, i.e. given any sheaf G equipped with* *a presheaf morphism* $\psi : F \to G$*, there exists a unique sheaf morphism* $\varphi : F^+ \to G$ *making the following triangle commute:*



Proof. Define $\theta : F \to F^+$ by $\theta_U : F(U) \to F^+(U), s \mapsto (\theta_U(s) : x \mapsto s_x)$, where s_x is the image of $s \in F(U)$ under the canonical inclusion $F(U) \to F_x$. One can verify that the collection $(\theta_u)_{U \in \mathbf{Top}(X)}$ is compatible with restriction maps and hence yields a well-defined presheaf morphism.

Now suppose we are given a presheaf morphism $\psi : F \to G$. To construct $\varphi : F^+ \to G$, we define each $\varphi_U : F^+(U) \to G(U)$ as follows: let $f \in F^+(U)$, with associated cover of U given by (U_x), i.e. for each U_x, there is a $s^{U_x} \in F(U_x)$ such that we have $f(y) = (s^{U_x})_y$ in F_y for all $y \in U_x$. Look at the induced collection $(\psi_{U_x}(s^{U_x}) \in G(U_x))$, compatible by naturality of ψ, hence by the sheaf property of G, there exists a unique $t \in G(U)$ such that $\mathrm{res}_{U,U_x} t = \psi_{U_x}(s^{U_x})$ for all U_x, and we set $\varphi_U(f) := t$, well-defined by uniqueness (assuming we fix a cover (U_x) of U associated to $f \in F^+(U)$ once and for all).

We omit the details of checking the naturality of φ, and verify uniqueness. Let $\varphi' : F^+ \to G$ be any presheaf morphism also making the above triangle commute. By (Lemma 4.5) above, equality can be tested at the level of stalks, where we find that $\theta_x = \mathrm{id}_{F_x}$, giving us immediately from the above triangle $\varphi_x = \psi_x = \varphi'_x$ for all x by functoriality, hence we are done. □

Remark 4.12. One could call the sheafification of a presheaf F the sheaf "generated by" the presheaf F, as this construction is identical in spirit to that of the free abelian group $\mathbb{Z} < S >$ or more generally the free R-module $R^{\oplus S}$ on a given set S of generators. In the latter case, the corresponding version of the above triangle ensures that a map out of $R^{\oplus S}$ in the category **R-Mod** of R-modules is fully determined by its behavior on the generating set, in that we have a bijection for any given R-module M:

$$\mathrm{Hom}_{\mathbf{Set}}(S, M) \cong \mathrm{Hom}_{\mathbf{R\text{-}Mod}}(R^{\oplus S}, M) \tag{4.13}$$

where M is interpreted as its underlying set in **Set**. We will have more to say about that in the next section. For now, we proceed to resolve quite a few open ends relative to constructions encountered earlier.

Construction 4.14. (i) We are now ready to present the dual construction of the pushforward $f_* : \mathbf{Ab}(\mathrm{X}) \to \mathbf{Ab}(\mathrm{Y})$ induced by a map of spaces f. Define the *pullback* of f to be the map $f^{-1} : \mathbf{Ab}(\mathrm{Y}) \to \mathbf{Ab}(\mathrm{X})$ (sometimes denotes f^*) sending a given sheaf G to the *sheafification* f^{-1}G of the presheaf given by:

$$U \mapsto \varinjlim_{V \supset f(U), V \in \mathbf{Top}(Y)} \mathrm{G}(V)$$

While this construction is substantially more involved than that of the pushforward, it will appear in the next section that the two functors are closely linked to one another, in that sheaf morphisms $F \to f_*G$ in $\mathbf{Ab}(\mathrm{Y})$ are in a natural bijection with sheaf morphisms $f^{-1}F \to G$ in $\mathbf{Ab}(\mathrm{X})$. For now, we simply highlight that the construction of stalks could have been presented as the pullback of an appropriate continuous map. Indeed, fix $x \in$ X, and let $j : \{x\} \hookrightarrow$ X denote the natural inclusion map. Observe that a sheaf on $\{x\}$ is just a choice of abelian group, so that $\mathbf{Ab}(\{x\})$ can be canonically identified with $\mathbf{Ab}$ (in particular, the sheaf condition is vacuous and sheaves and presheaves on $\{x\}$ are the same thing). Hence, the pullback of j can be viewed as a functor $j^{-1} : \mathbf{Ab}(\mathrm{X}) \to \mathbf{Ab}$ given by:

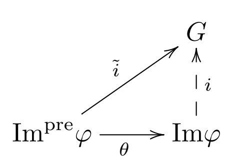
$$j^{-1} : F \mapsto \varinjlim_{U \ni j(\{x\}), U \in \mathbf{Top}(X)} \mathrm{F}(U) =: F_x$$

(ii) Let A be an abelian group. We can now formalize the relationship between the constant presheaf $\underline{A}$ (sending every non-empty open to A) and the *constant sheaf* A_X on X associated to A, by defining the latter as the sheafification of the constant presheaf on X. By (Lemma 4.10) above, the stalk of A_X at every point is isomorphic to that of $\underline{A}$ and hence is "constant" equal to A at every point. Slightly more generally, we say that an abelian sheaf F on X is a *constant sheaf* if it has constant stalks, i.e. if there is a fixed abelian group A such that $F_x = A$ for all $x \in X$. Further, a closer look at the sheafification construction yields a rather concise description for the behavior of A_X. Namely, A_X sends an open U in X to the abelian group of locally constant functions $f : U \to A$, equivalently to the abelian group $\mathrm{Hom}_{\mathbf{Top}}(U, A)$ where the underlying set A is given the discrete topology.

(iii) Recall from (Construction 2.8) that the naive way of defining a cokernel presheaf associated to a sheaf morphism $\varphi : F \to G$ may fail to yield a valid sheaf; the same can be said of image presheaves. We define the *sheaf cokernel* (resp. the *image sheaf*) associated to $\varphi : F \to G$ to be the sheafification of the presheaf given by $\mathrm{coker}^{\mathrm{pre}}\varphi : U \mapsto \mathrm{coker}(\varphi_U)$ (resp. $\mathrm{Im}^{\mathrm{pre}}\varphi : U \mapsto \mathrm{im}(\varphi_U)$), and denote it by $\mathrm{coker}\varphi$ (resp. $\mathrm{Im}\varphi$). In particular, one can verify that the obstruction to the sheaf property encountered in (Construction 2.8) is automatically dealt with by the sheafification process, in that the compatible local sections *themselves* can be used to define an appropriate global section.

Further, (Proposition 4.11) gives us a natural identification of $\mathrm{Im}\varphi$ as a subobject of G, as the presheaf morphism $\tilde{i} : \mathrm{Im}^{\mathrm{pre}}\varphi \to G$ given by inclusion corresponds to a unique sheaf morphism $i : \mathrm{Im}\varphi \to G$, which we know is a monomorphism by agreement of $\mathrm{Im}^{\mathrm{pre}}\varphi$ and $\mathrm{Im}\varphi$ at the level of stalks:

(4.15)



One can then prove the analog of (Lemma 4.6) for image sheaves, namely that for all $x \in X$, we have $(\mathrm{Im}\varphi)_x = \mathrm{im}(\varphi_x)$. This allows us to establish an important local-to-global principle, with which we close this section:

Lemma 4.16. *A cochain complex $F^\bullet$ of sheaves in $\mathbf{Ab}(X)$ is exact iff for each $x \in X$, the associated complex of stalks $... \to F_x^{i-1} \to F_x^i \to F_x^{i+1} \to ...$ is exact.*

Proof. It suffices to prove that a composite of sheaf morphisms $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ is exact at G if and only if the associated composite of abelian group homomorphisms

$F_x \xrightarrow{\varphi_x} G_x \xrightarrow{\psi_x} H_x$ is exact at G_x for all $x \in X$.

First assume $\ker\psi = \mathrm{Im}\varphi$ when viewed as subobjects of G. By diagram (4.15) above, for each open $U \subset X$, $\mathrm{im}\varphi_U$ is taken to its inclusion in $G(U)$ by $i\theta = \tilde{i}$, so that $\psi_U \circ \varphi_U = 0$, hence by functoriality we have that $\psi_x \circ \varphi_x = (\psi \circ \varphi)_x = 0$, so that $\mathrm{im}\varphi_x \subseteq \ker\psi_x$. Now, let s_x be an element of $\ker\psi_x$. From $\psi_x(s_x) = (\psi_U(s))_x = 0$, we get that $\mathrm{res}_{U,W}\psi_U(s) = \psi_W(\mathrm{res}_{U,W}s)$ for some $W \subset U$ about x. Hence $\mathrm{res}_{U,W}s \in \ker\psi_W = (\mathrm{Im}\varphi)_W$, so that by the above observation we get that $(\mathrm{res}_{U,W}s)_x = s_x \in (\mathrm{Im}\varphi)_x = \mathrm{im}(\varphi_x)$, as needed.

Next, suppose that for all $x \in X$, we have that $\ker(\psi_x) = \mathrm{im}(\varphi_x)$. Suppose $s \in (\mathrm{Im}\varphi)(U)$. Then we can write $s = (s_x)_{x \in U}$ as an element of $\prod_{x \in U}(\mathrm{Im}^{\mathrm{pre}}\varphi)_x$, so that each $s_x \in (\mathrm{Im}^{\mathrm{pre}}\varphi)_x = \mathrm{im}(\varphi_x) = \ker(\psi_x)$. Hence $\psi(s) = (\psi_x(s_x)) = 0$ and $s \in (\ker\psi)(U)$, and therefore $(\mathrm{Im}\varphi)(U) \subseteq (\ker\psi)(U)$. Now let $s \in (\ker\psi)(U)$. Then for each $x \in U$, we have $s_x \in (\ker\psi)_x = \ker(\psi_x) = \mathrm{im}(\varphi_x)$, and we may write $s_x = \varphi_x(t_x) = (\varphi(t))_x$ for some $t \in F(W)$ corresponding to a locally determined open $W \ni x$. Together, these form an element of $\prod_{x \in U}(\mathrm{im}\varphi)_x \subset \prod_{x \in U} G_x$ locally agreeing with $\mathrm{im}\varphi_U$ at the level of stalks, hence we have that $(s_x)_{x \in U} = ((\varphi(t))_x)_{x \in U} \in (\mathrm{Im}\varphi)(U)$, completing the proof. □

In particular, whenever we have a sheaf inclusion $l : F \to G$, we may define the *quotient sheaf* $G/H \in \mathbf{Ab}(\mathrm{X})$ to be the sheafification of the assignment $U \mapsto G(U)/l(F(U))$. First focusing on the presheaf assignment: the fact that direct limits of abelian groups preserve exactness yields short exact sequences $0 \to F_x \to G_x \to (G/F)_x \to 0$ at the level of stalks for each $x \in X$. Since stalks remain unchanged by sheafification, this immediately results in a short exact sequence of abelian sheaves $0 \to F \xrightarrow{l} G \to G/H \to 0$ by the above lemma.

5. A Few Notable Adjunctions

Adjoint functors lie at the heart of Category Theory, and due to its rather categorical nature, sheaf theory turns out to naturally involve a rich source of examples on the topic. We briefly recall the definition of an adjunction pair, and state some of the results that make the existence of such a pair a useful thing to know. For more details, see for instance the wonderful treatment in [1] [Section VIII.1].

Given a pair of functors $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, say that F, G form an *adjunction pair*, that F is *left-adjoint* to G, and that G is *right-adjoint* to F, if for every pair of objects $A \in \mathcal{A}, B \in \mathcal{B}$, we have a natural bijection:

$$\varphi_{A,B} : \mathrm{Hom}_{\mathcal{B}}(FA, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, GB) \tag{5.1}$$

where naturality is taken in the sense that for any morphisms $f : A' \to A$ in $\mathcal{A}$, $g : B \to B'$ in $\mathcal{B}$, we have commutative squares:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}(FA, B) & \xrightarrow{\varphi_{A,B}} & \mathrm{Hom}_{\mathcal{A}}(A, GB) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \mathrm{Hom}(A, Gg)} \\ \mathrm{Hom}_{\mathcal{B}}(FA, B') & \xrightarrow[\varphi_{A,B'}]{} & \mathrm{Hom}_{\mathcal{A}}(A, GB') \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}(FA, B) & \xrightarrow{\varphi_{A,B}} & \mathrm{Hom}_{\mathcal{A}}(A, GB) \\ \downarrow{\scriptstyle \mathrm{Hom}(Ff, B)} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Hom}_{\mathcal{B}}(FA', B) & \xrightarrow[\varphi_{A',B}]{} & \mathrm{Hom}_{\mathcal{A}}(A', GB) \end{array}$$

If $F : \mathcal{A} \to \mathcal{B}$ is left-adjoint to $G : \mathcal{B} \to \mathcal{A}$, then G is the unique functor satisfying such a property up to natural isomorphism. Further, it can be shown that as a left-adjoint functor, F is compatible with the operation of taking colimits in $\mathcal{A}$, in the sense that given any functor $L : \mathcal{I} \to \mathcal{A}$ from some shape category $\mathcal{I}$, we have a canonical isomorphism in $\mathcal{B}$:

$$F(\varinjlim L) \xrightarrow{\sim} \varinjlim (F \circ L),$$

and, as is often the case, the dual statement is true: right-adjoint functors commute with (inverse) limits.

Now suppose F, G as above are *additive* functors between *abelian* categories $\mathcal{A}$ and $\mathcal{B}$, meaning that they induce abelian group homomorphisms on the hom-sets for any given pair of objects (so that for instance, for any object A in $\mathcal{A}$, F induces a ring homomorphism $F : \mathrm{Hom}_{\mathcal{A}}(A, A) \to \mathrm{Hom}_{\mathcal{B}}(FA, FA)$, where composition of maps is interpreted as multiplication in the respective rings). Then provided F is left-adjoint to G, one can show that F is *right-exact*, meaning that it takes any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$ to an exact sequence

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$$

in $\mathcal{B}$, the reason being chiefly that F preserves cokernels, which are colimits; in particular, F preserves epimorphisms. Dually, provided G is right-adjoint to F, we are ensured it is left-exact (and preserves monomorphisms). This is quite an important fact in our settings, as (wlog) left-exactness of a functor $F : \mathbf{Ab}(X) \to \mathbf{Ab}$ is what we need in order to be able to develop the theory of right-derived functors and to define some notion of cohomology with coefficients in a given sheaf. This will be done in the next section.

Now is a good time to observe that $\mathbf{Ab}$(X) forms a valid abelian category. One can check that the kernel and cokernel sheaf associated to a sheaf morphism $\varphi : F \to G$, the zero sheaf $(0_U : U \mapsto 0)$ and the direct product sheaf $F_1 \oplus F_2$ given on opens via $((F_1 \oplus F_2)_U : U \mapsto F_1(U) \oplus F_2(U))$ and on inclusions as $(U \hookrightarrow V) \mapsto (\mathrm{res}^{F_1 \oplus F_2}_{VU} : (s,t) \mapsto (\mathrm{res}^{F_1}_{VU} s, \mathrm{res}^{F_2}_{V,U} t))$ all satisfy the defining universal properties of the respective objects in $\mathbf{Ab}$(X). $\mathrm{Hom}_{\mathbf{Ab}(X)}(F, G)$ naturally inherits the structure of an abelian group by defining the sum of sheaf morphisms component-wise in $\mathbf{Ab}$, and the technical conditions linking kernels to cokernels are verified.

We now come back to our story and present adjunction pairs that naturally arise in the study of sheafs. The first pair comes from fixing a space X and looking at functors between the categories $\mathbf{Ab}$ and $\mathbf{Ab}$(X). Consider the pair of functors:

$$\Gamma(X, \_) : \mathbf{Ab}(\mathrm{X}) \to \mathbf{Ab},\ F \mapsto F(X)$$
$$\mathcal{S}_X : \mathbf{Ab} \to \mathbf{Ab}(\mathrm{X}),\ A \mapsto A_X$$

where $\Gamma(X, \_)$ is the *global section functor* sending a given sheaf F on X to the abelian group of global sections of F, and $\mathcal{S}_X$ is the *constant sheaf functor* which sends an abelian group A to the sheafification of the constant presheaf $\underline{A}$ on X.

Theorem 5.2. *Given any space X, the global section functor* $\Gamma(X, \_)$ *is right-adjoint to the constant sheaf functor* $\mathcal{S}_X$.

Proof. The content of (Proposition 4.11) tells us that the sheafification functor can be viewed as the "free object" functor in **Ab**(X) via to the universal property it satisfies. Hence it can be thought of as the (unique) left-adjoint functor to the "forgetful" functor $U : \mathbf{Ab}(X) \to \mathcal{P}\mathbf{Ab}(X)$, in that given any presheaf F and any sheaf G on X, we have natural bijections

$$\mathrm{Hom}_{\mathcal{P}\mathbf{Ab}(X)}(F, UG) \cong \mathrm{Hom}_{\mathbf{Ab}(X)}(F^+, G). \tag{5.3}$$

So to verify the statement of the theorem, it is enough to produce natural bijections, for any $A \in \mathbf{Ab}$ and any $F \in \mathbf{Ab}(X)$:

$$\mathrm{Hom}_{\mathbf{Ab}}(A, F(X)) \cong \mathrm{Hom}_{\mathcal{P}\mathbf{Ab}(X)}(\underline{A}, UF). \tag{5.4}$$

where $\underline{A}$ is the constant presheaf associated to A on X, and the result follows by composition. Consider the following maps (where we identify F with UF):

$$\begin{aligned} \gamma_{A,F} : \mathrm{Hom}_{\mathbf{Ab}}(A, F(X)) &\to \mathrm{Hom}_{\mathcal{P}\mathbf{Ab}(X)}(\underline{A}, F), \\ (f : A \to F(X)) &\mapsto (\psi : \underline{A} \to F, \psi_U := \mathrm{res}^F_{X,U} \circ f) \\ \eta_{A,F} : \mathrm{Hom}_{\mathcal{P}\mathbf{Ab}(X)}(\underline{A}, F) &\to \mathrm{Hom}_{\mathbf{Ab}}(A, F(X)), \\ (\varphi : \underline{A} \to F) &\mapsto (\varphi_X : A \to F(X)) \end{aligned}$$

Then one can check that these maps are natural in A and F and are inverses of each other. □

The above discussion implies the following:

Corollary 5.5. *Given any space X, the global section functor* $\Gamma(X, \_)$ *is left-exact.*

Remark 5.6. The adjunction mentioned "in passing" in the above proof merits some further consideration. In particular, it tells us that the forgetful functor $U : \mathbf{Ab}(X) \to \mathcal{P}\mathbf{Ab}(X)$ is right-adjoint to the sheafification functor, hence left-exact.

We now come to another adjunction pair hinted at in previous sections, arising from fixing a continuous map between spaces $f : X \to Y$ and considering the pullback and pushforward functors induced by f:

Theorem 5.7. *Let* $f : X \to Y$ *be a continuous map between spaces. Then the pushforward functor* $f_* : \boldsymbol{Ab}(X) \to \boldsymbol{Ab}(Y)$ *is right adjoint to the pullback functor* $f^{-1} : \boldsymbol{Ab}(Y) \to \boldsymbol{Ab}(X)$.

Proof. Given sheaves F on X and G on Y, we are required to produce natural isomorphisms:

$$\mathrm{Hom}_{\mathbf{Ab}(Y)}(G, f_*F) \cong \mathrm{Hom}_{\mathbf{Ab}(X)}(f^{-1}G, F). \tag{5.8}$$

We describe the maps and omit the details of the naturality check.

Let $(\psi_U : \varinjlim_{W \supset f(U)} G(W) \to F(U))_{U \in \mathbf{Top}(X)}$ be the data of a sheaf morphism in $\mathrm{Hom}_{\mathbf{Ab}(X)}(f^{-1}G, F)$. Given $V \in \mathbf{Top}(Y)$, we wish to construct a map $G(V) \to F(f^{-1}(V))$. Since $V \supset f(f^{-1}1(V))$, we have a canonical inclusion:

$$\eta_V : G(V) \to \varinjlim_{W \supset f(f^{-1}(V))} G(W),$$

which we may then compose with

$$\psi_{f^{-1}(V)} : \varinjlim_{W \supset f(f^{-1}(V))} G(W) \to F(f^{-1}(V))$$

to get a map $\psi_{f^{-1}(V)}\eta_V : G(V) \to F(f^{-1}(V))$. Once can check compatibility with restriction maps using the commutative squares associated with ψ and η:

$$\begin{array}{ccc} \varinjlim_{W \supset f(f^{-1}(V))} G(W) & \xrightarrow{\psi_{f^{-1}(V)}} & F(f^{-1}(V)) \\ \downarrow{\scriptstyle \mathrm{res}^G_{f^{-1}V, f^{-1}U}} & & \downarrow{\scriptstyle \mathrm{res}^F_{f^{-1}V, f^{-1}U}} \\ \varinjlim_{W \supset f(f^{-1}(U))} G(W) & \xrightarrow[\psi_{f^{-1}(U)}]{} & F(f^{-1}(U)) \end{array} \qquad \begin{array}{ccc} G(V) & \xrightarrow{\eta_V} & \varinjlim_{W \supset f(f^{-1}(V))} G(W) \\ \downarrow{\scriptstyle \mathrm{res}^G_{V,U}} & & \downarrow{\scriptstyle \mathrm{res}^G_{f^{-1}V, f^{-1}U}} \\ G(U) & \xrightarrow[\eta_U]{} & \varinjlim_{W \supset f(f^{-1}(U))} G(W) \end{array}$$

Now, start with an element of $\mathrm{Hom}_{\mathbf{Ab}(Y)}(G, f_*F)$, consisting of a collection of maps $(\varphi_V : G(V) \to F(f^{-1}(V)))_{V \in \mathbf{Top}(Y)}$. Given an open $U \in \mathbf{Top}(X)$, we want to induce a natural map $\varinjlim_{W \supset f(U)} G(W) \to F(U)$. To do so, we follow a procedure summarized in the following diagram:

$$\begin{array}{ccccccc} G(W) & \xrightarrow{\psi_W} & F(f^{-1}(W)) & \xrightarrow{\eta_W} & \varinjlim_{W \supset f(U)} F(f^{-1}(W)) & \longleftarrow & F(f^{-1}(W')) \\ \downarrow & & & & & & \downarrow{\scriptstyle \mathrm{res}^F_{f^{-1}(W'),U}} \\ \varinjlim_{W \supset f(U)} G(W) & - & - & - & - & \dashrightarrow & F(U) \end{array}$$

Given an element in $\varinjlim_{W \supset f(U)} G(W)$, pick a representative (s, W), and map it to $\varinjlim_{W \supset f(U)} F(f^{-1}(W))$ using ψ_W followed by the canonical inclusion

$$\eta_W : F(f^{-1}(W)) \to \varinjlim_{W \supset f(U)} F(f^{-1}(W)).$$

Then, pick a representative (t, W') of $\eta_W\psi_W(s)$, where W' satisfies $W' \supset f(U)$, hence $f^{-1}(W') \supset U$, so we can take the restriction map $\mathrm{res}^F_{f^{-1}(W),U}(t)$ to obtain an element of $F(U)$. $\square$

In particular, taking $j : \{x\} \hookrightarrow X$ to be the inclusion map for a fixed $x \in X$, we see that the skyscraper sheaf functor $\mathbf{Ab} \to \mathbf{Ab}(X), A \mapsto A^x$ at x is right-adjoint to the stalk functor $\mathbf{Ab}(X) \to \mathbf{Ab}, F \mapsto F_x$, as these functors correspond to the pushforward and pullback of j, respectively.

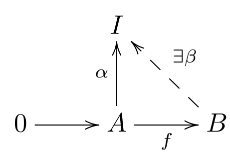
6. Some Homological Algebra

We now present a streamlined version of the theory of derived functors, with an emphasis on the overall structure of the construction. We assume that the reader is acquainted with the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes and chain maps in a fixed abelian category $\mathcal{A}$ and with the notions of chain homotopy and long exact sequence in homology associated to a short exact sequence of chain complexes (the first few sections of [17] are amply sufficient).

The essence of the argument is the following: given a (covariant, additive) left-exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, one may wish to correct for the failure of the functor to be right-exact. Starting with a fixed short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, this can be done by associating to this sequence a *split-exact* short exact sequence of *chain complexes* $0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0$ in $\mathbf{Ch}(\mathcal{A})$ which has the property that the long exact sequence in homology induced by the short exact sequence $0 \to F(I^\bullet) \to F(J^\bullet) \to F(K^\bullet) \to 0$ is bounded below, left-exact and naturally isomorphic in degree zero to the entries $0 \to F(A) \to F(B) \to F(C)$ associated to the short exact sequence we started with, i.e. it is precisely the exactness continuation we sought out at the beginning.

The crucial part of the argument was the construction of $0 \to I^\bullet \to J^\bullet \to K^\bullet \to 0$ from $0 \to A \to B \to C \to 0$, a process which can and is usually done "pointwise", i.e. one object at a time. So fix an object $A \in \mathcal{A}$. A sufficient (but not strictly necessary, as we will see later) condition on the chain complex $I^\bullet$ is for it to be an *injective resolution* of A.
Recall that given any object $B \in \mathcal{A}$, we may define a contravariant, left-exact functor $\mathrm{Hom}_{\mathcal{A}}(\_, B)$, acting on morphisms by pre-composition.

Definition 6.1. An object $I \in \mathcal{A}$ is said to be *injective* if the associated functor $\mathrm{Hom}_{\mathcal{A}}(\_, I)$ is exact; equivalently, I is injective iff given any monomorphism $f : A \to B$, given any map $\alpha : A \to I$, there exists a map $\beta : B \to I$ making the following triangle commute:



Remarks 6.2. The equivalence simply comes from "reading out-loud" what it means for the image of an arbitrary short exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ under the application of the functor $\mathrm{Hom}_{\mathcal{A}}(\_, I)$ to satisfy right-exactness.

A dual construction to the above can be carried out to correct for the potential failure of right-exact functors to be left-exact, by use of projective resolutions. As our main application in these notes will concern the *left-exact* global section functor $\Gamma(X, \_)$, we omit this side of the story, and only refer to it in a non-essential way in (Example 6.15).

Definition 6.3. Given an object $A \in \mathcal{A}$, an *injective resolution* of A is a pair $(I^\bullet, \epsilon : A \to I^0)$, where $I^\bullet$ is a chain complex in $\mathcal{A}$ such that I^k is an injective object for $k \geq 0$, $I^k = 0$ otherwise, and such that the associated chain complex:

$$0 \longrightarrow A \xrightarrow{\epsilon} I^0 \longrightarrow I^1 \longrightarrow ...$$

is exact; equivalently, we require the map $\epsilon : \underline{A} \to I^\bullet$ to be a quasi-isomorphism when taken to be a chain map from $\underline{A} = (... \to 0 \to A \to 0 \to ...)$ viewed as a complex concentrated in degree zero to $I^\bullet$.

Say an abelian category $\mathcal{A}$ has *enough injectives* if every object admits a monomorphism into an injective object, i.e. if for every $A \in \mathcal{A}$ there is an injective object $I \in \mathcal{A}$ and an "exact sequence" $0 \to A \to I$. If $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ admits an injective resolution, which can be obtained as indicated in the following diagram:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \xrightarrow{\epsilon} & I^0 & \overset{d^0:=\tilde{d^0}\pi}{\dashrightarrow} & I^1 & \overset{d^1:=\tilde{d^1}\pi}{\dashrightarrow} & I^2 & \longrightarrow & \ldots \\
 & & & & \pi \searrow & & \nearrow \tilde{d^0} \quad \pi \searrow & & \nearrow \tilde{d^1} & & \\
 & & & & & \mathrm{coker}\epsilon & & \mathrm{coker} d^0 & & &
\end{array}
\tag{6.4}
$$

where $(\mathrm{coker}\epsilon, \pi : I^0 \to \mathrm{coker}\epsilon)$ is the categorical cokernel associated with the morphism ϵ, $\tilde{d^0}$ is a monomorphism from $\mathrm{coker}\epsilon$ into an injective object I^1, and so on.

So let $\mathcal{A}$ be an abelian category with enough injectives. Then the following key lemma ensures that injective resolutions for a given object are unique up to chain homotopy equivalence.

Lemma 6.5. *Let $\varphi : A \to B$ be a morphism in $\mathcal{A}$, and let $I^\bullet$, $J^\bullet$ be injective resolutions for A, B respectively. Then there exists a chain map $\overline{\varphi} : I^\bullet \to J^\bullet$, unique up to chain homotopy, making the following diagram commute:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\epsilon} & I^0 & \longrightarrow & I^1 & \longrightarrow & \ldots \\
 & & \downarrow \varphi & & \downarrow \overline{\varphi} & & \downarrow \overline{\varphi} & & \\
0 & \longrightarrow & B & \xrightarrow{\epsilon'} & J^0 & \longrightarrow & J^1 & \longrightarrow & \ldots
\end{array}
$$

Corollary 6.6. *Given an object A in an abelian category, any two injective resolutions of A are chain homotopy equivalent.*

Proof. (of corollary) Given $(I^\bullet, \epsilon : A \to I^0)$, $(J^\bullet, \epsilon' : A \to J^0)$, simply apply the above lemma twice with $\mathrm{id}_A : A \to A$, once from $I^\bullet$ to $J^\bullet$ and once backwards; take the composites and use uniqueness of the induced chain maps $I^\bullet \to I^\bullet, J^\bullet \to J^\bullet$ up to chain homotopy together with the fact that $\mathrm{id}_{I^\bullet}, \mathrm{id}_{J^\bullet}$ respectively are also satisfactory choices to get the required chain homotopy equivalence $I^\bullet \simeq J^\bullet$. $\square$

We are now ready to present the main construction of this section.

Construction 6.7. Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant, additive, left-exact functor between abelian categories, and assume $\mathcal{A}$ has enough injectives. Define the i^{th} *right-derived functor* of F to be the functor $R^iF : \mathcal{A} \to \mathcal{B}$ which sends an object $A \in \mathcal{A}$ to $H^i(F(I^\bullet))$, the cohomology in degree i of the chain complex $F(I^\bullet)$, where $I^\bullet$ is an injective resolution of A, and sends a morphism $\varphi : A \to B$ to the induced morphism in cohomology $(F(\overline{\varphi}))^* : R^iF(A) \to R^IF(B)$, where $\overline{\varphi} : I^\bullet \to J^\bullet$ is induced by φ as in (Lemma 6.5) above.

This functor is well-defined and additive by chain homotopy invariance and additivity of cohomology and uniqueness of both injective resolutions and induced chain maps between injective resolutions up to chain homotopy equivalence (we also use the fact that $F : \mathcal{A} \to \mathcal{B}$ preserves chain homotopy equivalences).

The construction can be summarized by the following diagram:

$$
\mathcal{A} \longrightarrow \mathbf{Ch}(\mathcal{A}) \xrightarrow{F} \mathbf{Ch}(\mathcal{B}) \xrightarrow{\quad H^i \quad} \mathcal{B}
\tag{6.8}
$$

$$
A \longmapsto I^\bullet \longmapsto F(I^\bullet) \longmapsto H^i(F(I^\bullet)) =: R^iF(A)
$$

It is now possible for us to formalize the "exactness continuation" process we described at the beginning of this section. We first state a technical but routine lemma and refer the reader to [17] for a proof:

Lemma 6.9. *(Horseshoe lemma) Let $\mathcal{A}$ be an abelian category, and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Suppose we are given injective resolutions $I^\bullet, K^\bullet$ for A, C respectively. Then there exists an injective resolution $J^\bullet$ for B which fits into a commutative diagram:*

(6.10)
$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\epsilon} & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots \\
 & & f\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{\epsilon''} & J^0 & \longrightarrow & J^1 & \longrightarrow & \dots \\
 & & g\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \xrightarrow{\epsilon'} & K^0 & \longrightarrow & K^1 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Theorem 6.11. *Let $\mathcal{A}$ be an abelian category with enough injectives, let $F : \mathcal{A} \to \mathcal{B}$ be a covariant, additive left-exact functor between abelian categories, and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Then the right-derived functors of F fit into a long exact sequence:*

(6.12)
$$
\begin{array}{l}
0 \longrightarrow R^0F(A) \longrightarrow R^0F(B) \longrightarrow R^0F(C) \\
\xrightarrow{\ \delta\ } R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \\
\xrightarrow{\ \delta\ } R^2F(A) \longrightarrow \dots
\end{array}
$$

Furthermore, we have an isomorphism $R^0F(A) \cong F(A)$ for all $A \in \mathcal{A}$.

Proof. Let $(I^\bullet, \epsilon : A \to I^0)$, $(K^\bullet, \epsilon' : C \to K^0)$ be injective resolutions for A, C respectively. By the Horseshoe lemma, we may find an injective resolution $(J^\bullet, \epsilon'' : B \to I^0)$ for B producing the commutative diagram (6.10) above. Now, it is immediate from the definition of an injective object that each short exact sequence $0 \to I^n \to J^n \to K^n$ is split-exact, as we can use the monomorphism $0 \to I^n \to J^n$ and the map $\mathrm{id}_{I^n} : I^n \to I^n$ to get a retraction $\beta : J^n \to I^n$ by injectivity of I^n. Since exactness of split short exact sequences is preserved under the application of functors, we obtain a short exact sequence of chain complexes:

$$0 \to F(I^\bullet) \to F(J^\bullet) \to F(K^\bullet) \to 0$$

whose associated long exact sequence in homology is precisely the long exact sequence (6.12). Finally, to obtain an isomorphism $R^0F(A) \cong F(A)$, simply note that by left-exactness of F, we have an exact sequence $0 \to F(A) \to F(I^0) \to F(I^1)$, so that $R^0F(A) := H^0(F(I^\bullet)) = \ker(F(I^0) \to F(I^1)) = \operatorname{im}(F(A) \to F(I^0)) \cong F(A)$. □

In fact, there are a few more properties associated to the family of functors $RF = (RF^i : \mathcal{A} \to \mathcal{B})_i$ making it into what we call a *universal δ-functor*. See (section III.1) of [7] for more details on δ-functors, and chapter IX of [1] for a highly readable treatment of derived functors centered around the Tor and Ext derived functors, which we mention briefly below.

We now point to an alternative way of computing right-derived functors, which we will be putting to use at a later time. Indeed, while uniqueness up to chain homotopy equivalence of injective resolutions make injective objects a great theoretical tool in the construction of right-derived functors, injective objects are not always easy to come by. Noting that $R^iF(I) = 0$ for all $i > 0$ (since the complex $\underline{\mathrm{I}}$ concentrated in degree zero is a valid injective resolution), one may proceed to show that this condition is sufficient for objects to be used as components of a resolution.

Definition 6.13. Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant, additive, left-exact functor between abelian categories, and suppose $\mathcal{A}$ has enough injectives. An object $K \in \mathcal{A}$ is said to be *F-acyclic* if $R^iF(K) = 0$ for all $i > 0$.

We state the following without proof (see for instance [1] [Thm VIII-8-3] for a proof):

Proposition 6.14. *Let $F : \mathcal{A} \to \mathcal{B}$ as above, let A be an object in $\mathcal{A}$, and let*

$$0 \to A \xrightarrow{\epsilon} K^0 \to K^1 \to ...$$

be an exact sequence in $\mathcal{A}$ such that K^n is F-acyclic for all n. Then we have natural isomorphisms $H^i(F(K^\bullet)) \cong R^iF(A)$ for each $i \geq 0$.

Example 6.15. Let **R-Mod** denote the category of R-modules over a commutative ring R. Then **R-Mod** and $\mathbf{Ab} = \mathbb{Z}$**-mod** are abelian categories with enough injectives (see [12] for a short proof of that fact).

Fix an R-module P. We have an adjunction between the additive functors $\_ \otimes_R P : \textbf{R-Mod} \to \mathbf{Ab}$ and $\operatorname{Hom}_R(P, \_) : \mathbf{Ab} \to \textbf{R-Mod}$, via natural bijections:

$$\operatorname{Hom}_{\mathbf{Ab}}(M \otimes_R P, N) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbf{Ab}}(P, N)))$$

Hence $\_ \otimes_R P$ appears to be a left-adjoint, hence right-exact functor, and we may compute its left-derived functors $L_i(\_ \otimes_R P) =: \operatorname{Tor}_i^R(\_, P) : \textbf{R-Mod} \to \mathbf{Ab}$. referred to as the *Tor functors*. Dually, $\operatorname{Hom}_{\mathbf{Ab}}(P, \_)$ is right-adjoint, hence left-exact, and we may use injective resolutions in **Ab** to compute its right-derived functors $R^i(\operatorname{Hom}_{\mathbf{Ab}}(P, \_)) =: \operatorname{Ext}^i_R(P, \_) : \mathbf{Ab} \to \textbf{R-Mod}$, which we call the *Ext functors*.

Both of these constructions play a central role in elementary algebraic topology, providing answers to natural questions one may ask about (singular) homology and cohomology. Namely, the first Tor functor fits into a split short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \to 0$$

known as the *Künneth formula*, which allows one to compute the singular homology groups of a product of two spaces assuming the homology groups of the component spaces are known. In turn, the first Ext functor appears in a split short exact sequence known as the Universal Coefficient Theorem for cohomology and allows one to deduce the singular cohomology groups of a space X with coefficients in a given abelian group G once one has determined the integral homology groups of X:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), G) \to H^n(X; G) \to \mathrm{Hom}_{\mathbf{Ab}}(H_n(X), G) \to 0$$

The above results also hold when $\mathbb{Z}$ is replaced by any commutative ring R, and G is taken to be an R-module. For more on these objects and their interactions, see for instance [8].

7. Defining Sheaf Cohomology

We now begin to apply the theory of the previous section to the realm of sheaves of abelian groups. In order to do so, we must first verify that $\mathbf{Ab}$(X) has enough injectives. Recall the statement of (Lemma 4.16) , which reduces exactness of a short exact sequence of sheaves to exactness at the level of stalks, and vice-versa.

Theorem 7.1. *Let X be space. Then the abelian category $\mathbf{Ab}(X)$ has enough injectives.*

Proof. We use the fact that $\mathbf{Ab}$ has enough injectives (see [12] for a proof). Let F be a given element of $\mathbf{Ab}$(X). For each $x \in X$, pick a monomorphism $l_x : F_x \to I^x$ in $\mathbf{Ab}$ from the stalk of F at x to some injective abelian group I^x. Viewing each I^x as a sheaf over $\{x\}$ and letting $j : \{x\} \to X$ be the usual inclusion map, we may consider the direct product sheaf $I := \prod_{x \in X} j_* I^x$ on X as introduced in (Example 2.5), with $I(U) = \prod_{x \in U} I^x$ by construction.

Now, define a morphism $\epsilon : F \to I$ by $\epsilon_U : s \in F(U) \mapsto \prod_{x \in U} l_x(s_x)$. This is a valid sheaf morphism; indeed, by definition, stalks identify elements with their restrictions, so that given an inclusion $U \hookrightarrow V$ and an element $s \in F(V)$, we have $s_x = (\mathrm{res}^F_{V,U} s)_x$ for each $x \in V$, and compatibility of ϵ with restrictions follows. Further, ϵ is a monomorphism: if $\epsilon_U(s) = 0$, then $l_x(s_x) = 0$, so by injectivity of the l_x, $s_x = 0$ for all $x \in U$, hence $s = 0$ by uniqueness of gluings.

So it is enough to check that I is an injective object in $\mathbf{Ab}$(X). Let

$$0 \to F' \to F \to F'' \to 0 \tag{7.2}$$

be an arbitrary short exact sequence in $\mathbf{Ab}$(X). We need to show that the resulting sequence

$$0 \to \mathrm{Hom}_{\mathbf{Ab}(X)}(F'', I) \to \mathrm{Hom}_{\mathbf{Ab}(X)}(F, I) \to \mathrm{Hom}_{\mathbf{Ab}(X)}(F', I) \to 0 \tag{7.3}$$

is in fact exact. Now, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Ab}(X)}(F, I) &= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}(X)}(F, j_* I^x), \text{ by compatibility of Hom with products} \\
&= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}(\{x\})}(j^{-1} F, I^x), \text{ by adjunction of } j_*, j^{-1} \\
&= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}}(F_x, I^x), \text{ by definition of } j^{-1}F \text{ w.r.t. } j : \{x\} \to X
\end{aligned}$$

Further, for each $x \in X$, the sequence $0 \to F'_x \to F_x \to F''_x \to 0$ is exact in $\mathbf{Ab}$ by exactness of (7.2) above and (Lemma 4.16), hence by injectivity of each I_x, each associated sequence

$$0 \to \mathrm{Hom}_{\mathbf{Ab}}(F''_x, I^x) \to \mathrm{Hom}_{\mathbf{Ab}}(F_x, I^x) \to \mathrm{Hom}_{\mathbf{Ab}}(F'_x, I^x) \to 0$$

is exact, which by the above chain of isomorphisms yields exactness of the sequence in (7.3), as needed. □

Our aim is to define the sheaf cohomology groups of a given space X with coefficients in an arbitrary sheaf F on X. Recall from (Section 5) that the global section functor $\Gamma(X, \_) : \mathbf{Ab}(X) \to \mathbf{Ab}$ is right-adjoint to the constant sheaf functor, hence left-exact.

Definition 7.4. Let X be a space, and let F be a sheaf of abelian groups on X. Define the i^{th} *sheaf cohomology group* of X with coefficients in F to be the group obtained as the i^{th} right-derived functor of $\Gamma(X, \_)$ applied to F:

$$H^i(X, F) := R^i(\Gamma(X, \_))(F). \tag{7.5}$$

As mentioned in the introduction, the general notion of sheaf cohomology may not satisfy homotopy invariance. However, homotopy invariance does hold for a certain class of sheaves. Say a sheaf $F \in \mathbf{Ab}(X)$ is *locally constant* if every point $x \in X$ admits a neighborhood $U \ni x$ such that $F|_U$ is isomorphic to a constant sheaf.

Theorem 7.6. *Let $f \simeq g : X \to Y$ be homotopic maps between spaces, and let G be a locally-constant sheaf on Y. Then for each $n \geq 0$, there exists an isomorphism $\theta : H^n(X, f^{-1}G) \to H^n(X, g^{-1}G)$ making the following triangle commute:*

$$\begin{array}{ccc} & H^n(Y,G) & \\ {}^{g^*}\swarrow & & \searrow^{f^*} \\ H^n(X, f^{-1}G) & \xrightarrow{\quad\theta\quad} & H^n(X, g^{-1}G) \end{array} \tag{7.7}$$

The argument is technical and we choose to omit it to keep with the expository spirit of this paper; we refer the curious reader to [14] for a proof. As a corollary, we get the following:

Corollary 7.8. *If $f : X \simeq Y$ is a homotopy equivalence and G is a locally constant sheaf on Y, then we have isomorphisms, for all $n \geq 0$: $H^n(Y, G) \cong H^n(X, f^{-1}G)$. In particular, if $G = A_Y$ is a constant sheaf, then we have isomorphisms, for all $n \geq 0$: $H^n(Y, A_Y) \cong H^n(X, A_X)$.*

Proof. Suppose now that $G = A_Y$ is a constant sheaf on Y. Recall from (Theorem 5.7) that the inverse image functor $f^{-1} : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$ is left-adjoint, hence it commutes with taking colimits. Since the stalk operator is a special kind of colimit, it follows that at each point $x \in X$, we have an isomorphism $(f^{-1}G)_x = G_{f(x)} = A$, and the result follows. □

8. Flasque Sheaves

As discussed in (Construction 6.14), $H^i(X, F)$ is computed by finding an injective (or $\Gamma(X, \_)$-acyclic) resolution $(I^\bullet, \epsilon : F \to I^0)$ for the abelian sheaf F and taking cohomology of the resulting sequence $\Gamma(X, I^\bullet)$ in degree i. The naive way one would go about doing this is to iteratively apply the construction described in the proof that $\mathbf{Ab}$(X) has enough injectives by following diagram (6.4): first obtain a monomorphism $\epsilon : F \to I^0$, check if the resulting cokernel is injective, look for a new monomorphism $\text{coker}\epsilon \to I^1$ if not, and so on. The problem in practice is that the cokernel of a sheaf morphism $\varphi : F \to G$ in $\mathbf{Ab}$(X) is defined as the *sheafification* of the natural assignment ($\text{coker}\varphi : U \mapsto \text{coker}(\varphi_U)$), which can rapidly become an unwieldy object to identify as the subobject of some new injective object. The objective of this section is to present some conditions for acyclicity that can sometimes improve one's prospects of computing the cohomology of a given sheaf.

Definition 8.1. Say a sheaf $F \in \mathbf{Ab}$(X) is *flasque* if for every inclusion $U \hookrightarrow V$, the associated group homomorphism $F(V) \to F(U)$ is surjective.

Equivalently, a sheaf F is flasque iff $F(X) \to F(U)$ is surjective for every open inclusion $U \subset X$, i.e. iff every local section can be extended to a global one. Note in particular that if F is flasque, then for any open $U \subset X$, the restricted sheaf $F|_U$, $V \mapsto F(V)$ is also flasque. We first show that injectivity is a stronger condition relative to being flasque:

Lemma 8.2. *Every injective abelian sheaf is flasque.*

Proof. Let I be an abelian sheaf, and define the direct product sheaf $F := \prod_{x\in X} j_* I_x$ as in (Theorem 7.1), so that $F(U)$ corresponds to the abelian group of maps $\{f : U \to \prod_{x\in U} F_x\}$. Note that F is trivially flasque, as local sections may be extended by zero. Further, we have a natural sheaf map $\epsilon : I \to F$ via the assignment $s \in I(U) \mapsto (s_x)_{x\in U}$. Hence we obtain a short exact sequence:

$$0 \to I \xrightarrow{\epsilon} F \to F/I \to 0$$

Observe that this sequence automatically splits because I is injective, and note that whenever a flasque sheaf can be expressed as a direct sum of sheaves, the direct summands are automatically flasque. Hence it follows that I is flasque, as claimed. ☐

From the following theorem and (Proposition 6.14), it follows that we can use resolutions of flasque sheaves to compute the derived functors of $\Gamma(X, \_)$. This is rather convenient, as one can expect the question of whether the local sections of a given sheaf systematically extend to global ones to be easier to answer than that of whether the sheaf is injective.

Proposition 8.3. *Let F be a flasque abelian sheaf. Then F is $\Gamma(X, \_)$-acyclic.*

Proof. We follow the argument given in [13] [Proposition 6.75].

(i) We first show that the global section functor is exact when applied to any short exact sequence $0 \to F' \xrightarrow{l} F \xrightarrow{\varphi} F'' \to 0$ in which F' is assumed to be flasque. We need only check that the map $\varphi_X : F(X) \to F''(X)$ is surjective, and we do so by a standard application of Zorn's Lemma. So pick an arbitrary element $s'' \in F''(X)$, and consider the collection

$$\mathfrak{X} := \{(s, U) \mid U \in \mathbf{Top}(X), s \in F(U) \text{ satisfies } \varphi_U(s) = \text{res}^{F''}_{X,U} s''\},$$

partially ordered via $(U', s') \subseteq (U, s)$ whenever $U' \subset U$ and $\text{res}^{F'}_{U,U'} s = s'$.

$\mathfrak{X}$ is non-empty as it automatically contains $(0, \varnothing)$. Given a chain $\{(s_i, U_i)\}$ in $\mathfrak{X}$, one may look at the family of local sections $\{s_i \in F(U_i)\} \cup \{s_{ij} := \mathrm{res}_{U_i, U_{ij}} s_i \in F(U_{ij})\}$ in $U := \bigcup_i U_i$ (where $U_{ij} = U_i \cap U_j$). This family is compatible under restrictions, so that there exists a unique gluing (U, s) satisfying $\mathrm{res}_{U,U_i} s = s_i$ for all i and $\varphi_U(s) = s''$, hence (U, s) is a valid upper bound for $\{(s_i, U_i)\}$.

So let (U, s) denote the maximal element in $\mathfrak{X}$. We claim that $U = X$, in which case we are done. Otherwise, we work towards a contradiction. So pick $x \in X - U$. By (Lemma 4.16), surjectivity of $\varphi : F \to F''$ implies surjectivity of $\varphi_x : F_x \to F''_x$ at the level of stalks. Hence $s''_x = \varphi_x(t_x)$ for some $V \ni x$ open, $t \in F(V)$, such that we then have $\varphi(t) = \mathrm{res}_{X,V} s''$.
In particular, on $U \cap V$, we have $\mathrm{res}_{U,U\cap V} s - \mathrm{res}_{V,U\cap V} t \in \ker \varphi_{U\cap V} = (\mathrm{Im} l)_{U\cap V}$. Hence viewing the difference as an element in $F'(U \cap V)$ with F': flasque, we can find an extension to a section $r \in F'(V)$ satisfying

$$\mathrm{res}^F_{U,U\cap V} s = \mathrm{res}^F_{V,U\cap V} t + \mathrm{res}^F_{V,U\cap V} l(r)$$

on $F(U \cap V)$ by construction.
Thus we get a compatible pair of local sections $(s, U), (t + l(r), V)$ on $U \cup V \supsetneq U$, which extends to a global section $\overline{r} \in F(U \cup V)$ satisfying $\varphi(\overline{r}) = s''$ by uniqueness of gluings, thereby contradicting maximality of (s, U).

(ii) We claim that from the existence of an exact sequence $0 \to F' \to F \to F'' \to 0$ with F, F' flasque, we can deduce that F'' is flasque. Let $U \subset X$ be an arbitrary open subset. First observe that the fact that F', and hence that $F'|_U$ is flasque, ensures us that $\varphi_U : F(U) \to F''(U)$ is surjective by (i), from the short exact sequence $0 \to F'(U) \to F(U) \to F''(U) \to 0$ resulting from the application of the $\Gamma(U, \_)$ functor to the provided exact sequence. Now, we have a commutative square:

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi_X} & F''(X) \\ {\scriptstyle \mathrm{res}^F_{X,U}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^{F''}_{X,U}} \\ F(U) & \xrightarrow[\varphi_U]{} & F''(U) \end{array}$$

Where $\mathrm{res}^F_{X,U}$ is surjective because F is flasque, hence by the above comment, $\mathrm{res}^{F''}_{X,U}$ must be surjective, as needed.

We are now ready to prove acyclicity. Let F be an arbitrary flasque sheaf, with $l : F \to G$: monomorphism into G: injective, hence flasque sheaf by (Lemma 8.2). Taking $H := G/F$, we get an exact sequence

$$0 \to F \to G \xrightarrow{\varphi} H \to 0 \tag{8.4}$$

where H is flasque by (ii) above. Now, G is injective, so $H^n(X, G) = 0$ for all $n \geq 1$. At $n = 1$, the long exact sequence in cohomology associated to the sequence resulting from applying the global section functor to (8.4) yields that $H^1(X, F) = \mathrm{coker}(\varphi_X : G(X) \to H(X)) = 0$, using (i) together with the fact that $H^1(X, G) = 0$. In particular, this tells us that $H^1(X, F) = 0$ for any flasque sheaf. We then proceed by induction on n, observing that for $n \geq 1$, the same long exact sequence in cohomology gives us an isomorphism:

$$... \to 0 \to H^n(X, H) \xrightarrow{\sim} H^{n+1}(X, F) \to 0 \to ...$$

which, together with the fact that H is flasque, completes the induction and yields $H^n(X, F) = 0$ for all $n \geq 1$. □

9. Čech Cohomology

Construction 9.1. Let F be an abelian sheaf on X, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Write $U_{\alpha_0 \dots \alpha_k}$ for the intersection $U_{\alpha_0} \cap \dots \cap U_{\alpha_k}$, and define the group of *k-cochains* of $\mathcal{U}$ with coefficients in F, denoted $C^k(\mathcal{U}, F)$, to consist of the functions

$$c^k \in \prod_{\alpha_0 < \dots < \alpha_k} F(U_{\alpha_0 \dots \alpha_k})$$

under pointwise addition, where a total ordering has been defined on A. That is, cochains in $C^k(\mathcal{U}, F)$ assign to each ordered tuple $(\alpha_0, ..., \alpha_k)$ from A an element of $F(U_{\alpha_0 \dots \alpha_k})$. Next, define the *coboundary maps* $\delta : C^k(\mathcal{U}, F) \to C^{k+1}(\mathcal{U}, F)$ to be given by the assignment:

$$\delta : c^k \mapsto \big(\ \delta(c^k) : (\alpha_0, ..., \alpha_{k+1}) \mapsto \sum_{i=0}^{k+1} (-1)^i c^k(\alpha_0, ..., \hat{\alpha_i}, ..., \alpha_{k+1})|_{U_{\alpha_0 \cdots \alpha_{k+1}}}\ \big)$$

where each $c^k(\alpha_0, ..., \hat{\alpha_i}, ..., \alpha_{k+1})|_{U_{\alpha_0 \cdots \alpha_{k+1}}}$ denotes the image of $c^k(\alpha_0, ..., \hat{\alpha_i}, ..., \alpha_{k+1})$ under the restriction map $F(U_{\alpha_0 \dots \hat{\alpha_i} \dots \alpha_{k+1}}) \to F(U_{\alpha_0 \dots \alpha_{k+1}})$ provided by the inclusion $U_{\alpha_0 \dots \alpha_{k+1}} \hookrightarrow U_{\alpha_0 \dots \hat{\alpha_i} \dots \alpha_{k+1}}$. A direct calculation yields $\delta^2 = 0$.
Finally, define the k^{th} *Čech cohomology group* of $\mathcal{U}$ with coefficients in F to be:

$$\check{H}^k(\mathcal{U}, F) := H^k(C^\bullet(\mathcal{U}, F), \delta).$$

Remark 9.2. Two other variants of Čech cohomology are sometimes defined, differing from the above in the choice of domains for the cochains. Instead of using a total ordering on A, one could take the domain to be *all* tuples $(\alpha_0, ..., \alpha_k)$ in A^{k+1}, including repeated entries. One could also consider an *alternating* formulation of $C^k(\mathcal{U}, F)$, where we identify $c^k(\alpha_{\sigma(0)}, ..., \alpha_{\sigma(k)}) = \mathrm{sgn}(\sigma) c^k(\alpha_0, ..., \alpha_k)$ for all permutations $\sigma \in S_{k+1}$, and set $c^k(\alpha_0, ..., .\alpha_k) = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$. It turns out that the above three ways of constructing $C^k(\mathcal{U}, F)$ all produce isomorphic Čech cohomology groups - see [5] for a proof of this fact. We choose the total ordering version for its geometric appeal, which is hinted at below.

The combinatorial nature of this construction can be further emphasized by viewing k-cochains in $C^k(\mathcal{U}, F)$ as functions whose domain is the *nerve* $N(\mathcal{U})$ of the open cover $\mathcal{U}$, namely the abstract simplicial complex obtained by declaring $\{\alpha_0, ..., \alpha_k\}$ to be a simplex in $N(\mathcal{U})$ whenever the intersection $U_{\alpha_0 \dots \alpha_k}$ is nonempty. Then, $C^k(\mathcal{U}, F)$ can be interpreted as the direct product of abelian groups

$$C^k(\mathcal{U}, F) = \prod_{\sigma \in N(\mathcal{U}), \dim \sigma = k} F(U_\sigma)$$

ranging over all the k-simplices in the nerve of $\mathcal{U}$, where we write U_σ for the intersection $U_{\alpha_0 \dots \alpha_k}$, with $\sigma = \{\alpha_0, ..., \alpha_k\}$. Further, the coboundary map can be understood as sending a k-cochain to a (k+1)-cochain acting on (k+1)-simplices as the usual coboundary homomorphism, with the caveat that the functional first takes its values on the groups associated to the faces, before the alternating sum is taken in the group associated to the (k+1)-simplex via the provided restriction maps.

We wish to express the Čech cohomology of X with coefficients in F as the categorical direct limit in **Ab** taken over all open covers of X. First observe that the collection of open covers of X can naturally be partially ordered by refinements. Recall that given two open covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}, \mathcal{V} = \{V_\beta\}_{\beta \in B}$ of X, we say that $\mathcal{U}$ is a *refinement* of $\mathcal{V}$ if, for every $\alpha \in A$, we have $U_\alpha \subseteq V_\beta$ for some $\beta \in B$. We declare $\mathcal{V} \leq \mathcal{U}$ whenever $\mathcal{U}$ is a refinement of $\mathcal{V}$.

Now, whenever $\mathcal{V} \leq \mathcal{U}$, with respective indexing sets B, A as above, we have a *refinement map* $\lambda : A \to B$ satisfying $U_\alpha \subseteq V_{\lambda(\alpha)}$ for all $\alpha \in A$. This map can be used to define a map at the level of cochain complexes:

$$\lambda^{\#} : C^k(\mathcal{V}, F) \to C^k(\mathcal{U}, F)$$

$$c^k \mapsto \big((\lambda^{\#} c^k) : (\alpha_0, ..., \alpha_k) \mapsto c^k(\lambda(\alpha_0), ..., \lambda(\alpha_k))|_{U_{\alpha_0 \cdots \alpha_k}} \in F(U_{\alpha_0 \dots \alpha_k})\big),$$

well-defined because of the inclusion $U_{\alpha_0 \dots \alpha_k} \subseteq V_{\lambda(\alpha_0) \dots \lambda(\alpha_k)}$. A direct computation shows that $\lambda^{\#} : C^\bullet(\mathcal{V}, F) \to C^\bullet(\mathcal{U}, F)$ is a valid chain map, hence it descends to a collection of valid maps on cohomology:

$$\lambda^* : \check{H}^k(\mathcal{V}, F) \to \check{H}^k(\mathcal{U}, F)$$

for all $k \geq 0$.

Further, we note that any two refinement maps $\lambda, \mu : A \to B$ yield homotopic chain maps $\lambda^{\#} \simeq \mu^{\#} : C^\bullet(\mathcal{V}, F) \to C^\bullet(\mathcal{U}, F)$, with chain homotopy provided by

$$h : C^\bullet(\mathcal{V}, F) \to C^{\bullet - 1}(\mathcal{U}, F),$$

$$c^k \mapsto \big((hc^k) : (\alpha_0, ..., \alpha_{k-1}) \mapsto \sum_{i=0}^{k-1} c^k(\lambda(\alpha_0), ..., \lambda(\alpha_i), \mu(\alpha_i), ..., \mu(\alpha_{k-1}))|_{U_{\alpha_0 \cdots \alpha_k}}\big)$$

The verification of the equality

$$\mu^{\#} - \lambda^{\#} = \delta h + h \delta$$

as maps $C^k(\mathcal{V}, F) \to C^k(\mathcal{U}, F)$ for all $k \geq 0$ is involved but straightforward.

Hence any two refinement maps associated to a given refinement induce the same maps on cohomology. It follows that we have a valid direct system of abelian groups, for $k \geq 0$ fixed:

$$\big(\check{H}^k(\mathcal{U}, F), \lambda^*_{\mathcal{V}, \mathcal{U}}\big), \tag{9.3}$$

where $\mathcal{U}$ ranges over all open covers of X, and for each refinement $\mathcal{V} \leq \mathcal{V}$, $\lambda^*_{\mathcal{U}, \mathcal{V}}$ is the map on cohomology

$$\lambda^*_{\mathcal{U}, \mathcal{V}} : \check{H}^k(\mathcal{V}, F) \to \check{H}^k(\mathcal{U}, F)$$

induced by any refinement map between the underlying indexing sets of $\mathcal{U}$ and $\mathcal{V}$. This direct system is well-defined by the above discussion together with the chain homotopy invariance of cohomology. We may then define:

Definition 9.4. For any $n \geq 0$, the n^{th} *Čech cohomology group* of X with coefficients in $F \in \mathbf{Ab}(X)$ is defined to be the direct limit:

$$\check{H}^n(X, F) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U}, F),$$

ranging over the direct system of abelian groups (9.3).

A judicious choice of open cover yields isomorphisms between Čech cohomology as defined above and regular sheaf cohomology which bypass the direct limit step of the above process:

Theorem 9.5. *(Leray) Let F be an abelian sheaf on X, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Suppose that $\check{H}^k(U_{\alpha_0 \dots \alpha_k}, F) = 0$ for all $k \geq 0$ and for all $\{\alpha_0, ..., \alpha_k\} \in N(\mathcal{U})$. Then we have an isomorphism, for all $n \geq 0$:*

$$\check{H}^n(\mathcal{U}, F) \cong H^n(X, F).$$

The above theorem is attributed to Leray, who in the 1940s formulated for the first time the notion of a sheaf in one of his papers [10]. Combined with homotopy invariance of sheaf cohomology with coefficients in a locally constant sheaf as stated in (Theorem 7.8), this result gives us a useful corollary. Say an open cover $\mathcal{U}$ of a space X is *good* if all finite intersections of $\mathcal{U}$ are contractible, hence homotopy equivalent to a one point space.

Corollary 9.6. *Let F be a locally-constant sheaf on X, and let $\mathcal{U}$ be a good cover of X. Then we have an isomorphism, for all $n \geq 0$:*

$$\check{H}^n(\mathcal{U}, F) \cong H^n(X, F).$$

Example 9.7. Consider the constant sheaf $\mathbb{Z}_{S^1}$ on the circle $S^1 \subset \mathbb{C}$. Construct an open cover $\mathcal{U} = \{U_0, U_1, U_2\}$ of S^1 with each U_i taken to be an open arc covering a little more than a third of S^1, so that each U_{ij} is a small, contractible arc and U_{012} is empty. Then $\mathcal{U}$ is a good cover of S^1, hence by the above corollary can compute the sheaf cohomology of S^1 with coefficients in the constant sheaf $\mathbb{Z}_{S^1}$ using the isomorphisms $\check{H}^*(\mathcal{U}, \mathbb{Z}_{S^1}) \cong H^*(S^1, \mathbb{Z}_{S^1})$.

The computation now becomes highly tractable. A 0-chain, i.e. an element $c^0 \in C^0(\mathcal{U}, \mathbb{Z}_{S^1})$, consists of a choice of locally constant $\mathbb{Z}$-valued function on each U_i: connected $(i < j)$, hence we clearly have $C^0(\mathcal{U}, \mathbb{Z}_{S^1}) = \mathbb{Z}^3$. Similarly, an element $c^1 \in C^1(\mathcal{U}, \mathbb{Z}_{S^1})$ is a choice of a value of $\mathbb{Z}$ for each of the three connected opens U_{ij}, so again $C^1(\mathcal{U}, \mathbb{Z}_{S^1}) = \mathbb{Z}^3$. Hence the Čech complex associated to $\mathcal{U}$ and F is given by:

$$... \to 0 \to \mathbb{Z}^3 \xrightarrow{\delta} \mathbb{Z}^3 \to 0 \to ...$$

where δ sends c^0 to the 1-cochain (δc^0) given by its actions

$$(\delta c^0)(U_{01}) = (c^0(U_1) - c^0(U_0))|_{U_{01}},$$
$$(\delta c^0)(U_{02}) = (c^0(U_2) - c^0(U_0))|_{U_{02}},$$
$$(\delta c^0)(U_{12}) = (c^0(U_2) - c^0(U_1))|_{U_{12}}.$$

With restriction maps being just identity mappings as elements of $\mathbb{Z}$, so that as an element of $\mathbb{Z}^3$, $\delta c^0 = (c^0(U_1) - c^0(U_0), c^0(U_2) - c^0(U_0), c^0(U_2) - c^0(U_1))$. It then appears that

$$H^0(S^1, \mathbb{Z}_{S^1}) \cong \check{H}^0(\mathcal{U}, \mathbb{Z}_{S^1}) = \ker(\delta : C^0(\mathcal{U}, \mathbb{Z}_{S^1}) \to C^1(\mathcal{U}, \mathbb{Z}_{S^1})) \cong \mathbb{Z}, \text{ and}$$
$$H^1(S^1, \mathbb{Z}_{S^1}) \cong \check{H}^1(\mathcal{U}, \mathbb{Z}_{S^1}) = \operatorname{coker}(\delta : C^0(\mathcal{U}, \mathbb{Z}_{S^1}) \to C^1(\mathcal{U}, \mathbb{Z}_{S^1})) \cong \mathbb{Z},$$

coinciding with singular cohomology with $\mathbb{Z}$ coefficients, as is the case for constant sheaves with respect to any abelian group.

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