THE THOM ISOMORPHISM THEOREM

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Abstract. These notes provide a detailed proof of the Thom isomorphism theorem, which is involved in the construction of the Stiefel-Whitney classes associated to a vector bundle, following the argument given in Chapter 10 of Milnor-Stasheff. The proof will proceed in a way reminiscent of that of de Rham's theorem: we will first establish the result in the case of trivial bundles, then move from there to formally derive the general case.

1. Introduction

The aim of these notes is to provide a careful proof of the Thom isomorphism theorem, which may be stated as follows (underlying $\mathbb{Z}/2$ coefficients are assumed throughout):

Theorem 1.1. *Let $\xi : E \xrightarrow{\pi} B$ be a vector bundle of rank n. Then there exists a unique Thom class $u \in H^n(E, E_0)$, characterized by the property that the image of u under the map $i_b^* : H^n(E, E_0) \to H^n(F_b, (F_b)_0) \cong \mathbb{Z}/2$ induced by the fiberwise inclusion $i_b : (F_b, (F_b)_0) \to (E, E_0)$ is non-zero for all $b \in B$. Furthermore, taking cup products with the Thom class induces an isomorphism, for all $i \geq 0$:*

$$\Phi : H^i(E) \xrightarrow{-\cup u} H^{n+i}(E, E_0).$$

Recall that, in the notation of the above theorem, the Thom isomorphism is used in combination with the i^{th} Steenrod square in defining the i^{th} Stiefel-Whitney class of the vector bundle ξ:

$$w_i(\xi) := \Phi^{-1} Sq^i(u) \in H^i(B),$$

where we canonically identify $H^i(E) \cong H^i(B)$ via the isomorphism induced by the projection map $E \xrightarrow{\pi}$ B. It is thus of great theoretical importance in the development of the machinery of characteristic classes.

This theorem may also be seen to be of geometric significance, in that it may be reinterpreted as providing an isomorphism, for all $i \geq 0$:

$$H^i(B) \xrightarrow{\sim} \tilde{H}^{n+i}(\mathrm{Th}(\xi)),$$

where $\mathrm{Th}(\xi)$ denotes the *Thom space* of the bundle ξ, obtained from the total space E by taking the fiberwise one-point compactification, then identifying all of the resulting points at infinity. In particular, in the case of the trivial bundle $\epsilon^n : B \times \mathbb{R}^n \xrightarrow{\pi} B$, one can see that $\mathrm{Th}(\epsilon^n) \cong \Sigma^n B$, and thus the above isomorphism reduces to the iterated suspension isomorphism:

$$H^i(B) \xrightarrow{\sim} \tilde{H}^{n+i}(\Sigma^n B),$$

which is already guaranteed by the properties of singular cohomology. Thus, the Thom isomorphism theorem can be interpreted as stating that the suspension isomorphism is invariant under twisting, in the sense that the Thom space associated to a vector bundle of rank n is a twisted version of the standard n^{th} suspension of the base space.

In order to prove Theorem (1.1), we will first establish the result in the case of the trivial bundle ϵ^n, and we will then be in a position to extend the result to arbitrary vector bundles. Much of the technical subtleties involved in the proof are contained in the verification of the trivial case, for which we set the ground in sections 2 and 3. The general case will follow relatively quickly in section 4.

2. Refreshers on the Cup and Cross Products

We start by reviewing a few facts about relative cup products and cross products in singular cohomology, which we will be needing in our discussion. Recall that given a space X and cochains $\psi \in C^m(X), \eta \in C^n(X)$, we may define their cup product to the the cochain $\psi \cup \eta \in C^{n+m}(X)$ acting on singular chains $\sigma : \Delta^{n+m} \to X$ via

$$(\psi \cup \eta)(\sigma) := \psi(\sigma|_{[v_0,\dots,v_n]})\eta(\sigma|_{[v_n,\dots,v_{n+m}]}),$$

where $\sigma_{[v_0,\dots,v_n]}$, $\sigma_{[v_n,\dots,v_{n+m}]}$ are the restrictions of σ to the corresponding faces of $\Delta^{n+m} = [v_0, ..., v_{n+m}]$, and where the product is taken in $\mathbb{Z}/2$. This operation factors through the cohomology groups of X and give $H^*(X) = \bigoplus_{i\geq 0} H^i(X)$ the structure of an associative, graded commutative ring with multiplicative unit the element $1 \in H^0(X)$ evaluating to 1 on every 0-chain, with product given componentwise by maps

$$H^n(X) \otimes_{\mathbb{Z}/2} H^m(X) \xrightarrow{-\cup-} H^{n+m}(X) \ .$$

Next, let $\psi \in C^m(X, A), \eta \in C^n(X, B)$ be two *relative* cochains with respect to opens subsets $A, B \subseteq X$ (meaning for instance that ψ vanishes on all m-chains whose image is contained entirely in A). Viewing $C^m(X, A) \subseteq C^m(X)$, resp. $C^n(X, B) \subseteq C^n(X)$ as subsets in the natural way, we observe that the cup product induces a map at the level of cochains:

$$C^m(X, A) \otimes_{\mathbb{Z}/2} C^n(X, B) \xrightarrow{-\cup-} C^{m+n}(X, A) \cap C^{m+n}(X, B).$$

Denoting the intersection in the target as $\hat{C}^{n+m}(X; A, B) := C^{m+n}(X, A) \cap C^{m+n}(X, B)$, one easily verifies that the canonical inclusion of cochain complexes

$$C^*(X, A \cup B) \xrightarrow{l} \hat{C}^*(X; A, B)$$

fits into a short exact sequences of chain complexes:

$$0 \longrightarrow C^*(X, A \cup B) \xrightarrow{l} \hat{C}^*(X; A, B) \longrightarrow \hat{C}^*(A \cup B; A, B) \longrightarrow 0. \tag{2.1}$$

Furthermore, the third term of this sequence is acyclic (has vanishing cohomology in every degree), whence it follows from the LES associated to (2.1) that we get an isomorphism in cohomology

$$H^*(X, A \cup B) \cong H^*(\hat{C}^*(A \cup B; A, B)). \tag{2.2}$$

We may thus consider the following well-defined relative cup product operation on cohomology, for all $n, m \geq 0$:

$$H^n(X,A) \otimes_{\mathbb{Z}/2} H^m(X,B) \xrightarrow{-\cup-} H^{n+m}(X, A\cup B)\ .$$

Equipped with this operation, we are in a position to relate the cohomology rings of two distinct pairs of spaces $(X,A),(Y,B)$. In the following definition, $p_1 : X\times Y \to X$, resp. $p_2 : X \times Y \to Y$ denote the standard projection maps.

Definition 2.3. Let $(X,A),(Y,B)$ be pairs of spaces such that $A \subset X$, $B \subset Y$ are open subsets. Given cohomology classes $a \in H^n(X,A)$, $b \in H^m(Y,B)$, we define their *cross product* to be the element $a\times b \in H^{n+m}(X\times Y, A\times Y \cup X\times B)$ obtained under the following composite:

$$\begin{array}{ccc} H^n(X,A)\otimes_R H^m(Y,B) & \xrightarrow{p_1^*\otimes p_2^*} & H^n(X\times Y, A\times Y)\otimes_R H^m(X\times Y, X\times B) \\ & \searrow^{-\times-} & \downarrow{-\cup-} \\ & & H^{n+m}(X\times Y, A\times Y\cup X\times B). \end{array}$$

3. Technical Preliminaries

Before moving on to the proof of the Thom isomorphism theorem, we will need to establish a technical result. Note that in the case of trivial bundles, the Thom isomorphism reduces to an isomorphism $H^i(B\times\mathbb{R}^n) \xrightarrow{-\cup u} H^{n+i}(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0)$ for an appropriate choice of $u \in H^n(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0)$, where we write $\mathbb{R}^n_0 := \mathbb{R}^n - \{0\}$. The cross product gives us a map:

$$H^0(B)\otimes_{\mathbb{Z}/2} H^n(\mathbb{R}^n,\mathbb{R}^n_0) \xrightarrow{-\times-} H^n(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0),$$

and so our strategy will be to express u as a cross product $u = 1\times e^n$, where $1 \in H^0(B)$ is the unit and e^n is a generator of $H^n(\mathbb{R}^n,\mathbb{R}^n_0)$, with the latter group readily seen from the cohomology LES associated to the pair $(\mathbb{R}^n,\mathbb{R}^n_0)$ $H^n(\mathbb{R}^n,\mathbb{R}^n_0)\cong \mathbb{Z}/2$ to be free on one generator for each $n \geq 1$.

Contemplate the following segment of the cohomology LES associated to the triple $(\mathbb{R},\mathbb{R}_0,\mathbb{R}_-)$, where $\mathbb{R}_-$ denotes the negative reals:

$$H^0(\mathbb{R},\mathbb{R}_-) \longrightarrow H^0(\mathbb{R}_0,\mathbb{R}_-) \xrightarrow{\delta} H^1(\mathbb{R},\mathbb{R}_0) \longrightarrow H^1(\mathbb{R},\mathbb{R}_-).$$

Since $\mathbb{R},\mathbb{R}_-$ are both contractible, we get by exactness that the connecting homomorphism $H^0(\mathbb{R}_0,\mathbb{R}_-) \xrightarrow{\delta} H^1(\mathbb{R},\mathbb{R}_0)$ is an isomorphism. Next, by excision applied to the pair $(\mathbb{R}_0,\mathbb{R}_-)$, we also have an isomorphism

$$H^0(\mathbb{R}_0,\mathbb{R}_-) \xrightarrow{\eta^*} H^0(\mathbb{R}_+)), \tag{3.1}$$

where $\mathbb{R}_+$ denotes the positive reals. Thus we have the following sequence of isomorphisms:

$$H^0(\mathbb{R}_+) \underset{\sim}{\xleftarrow{\eta^*}} H^0(\mathbb{R}_0,\mathbb{R}_-) \underset{\sim}{\xrightarrow{\delta}} H^1(\mathbb{R},\mathbb{R}_0),$$

and we may express the generator of $H^1(\mathbb{R},\mathbb{R}_0)$ as the image of the unit $1 \in H^0(\mathbb{R}_+)$ under the horizontal composite, $e^1 := \delta(\eta^*)^{-1}(1)$. This sleight of hand allows us to establish the following result:

Lemma 3.2. *Let (X,A) be a pair of spaces, with $A \subset X$ open. Then the map*

$$H^m(X,A) \xrightarrow{-\times e^1} H^{m+1}(X \times \mathbb{R}, A \times \mathbb{R} \cup X \times \mathbb{R}_0)$$

given by $a \mapsto a \times e^1$ is an isomorphism for all $m \geq 0$.

Proof. We first establish the result for the pair $(X, \emptyset)$. Consider the following portion of the cohomology LES associated to the triple $(X \times \mathbb{R}, X \times \mathbb{R}_0, X \times \mathbb{R}_-)$:

$$\begin{array}{ccccc} H^m(X \times \mathbb{R}, X \times \mathbb{R}_-) & \longrightarrow & H^m(X \times \mathbb{R}_0, X \times \mathbb{R}_-) & & \\ & & \downarrow{\scriptstyle \delta'} & & \\ & & H^{m+1}(X \times \mathbb{R}, X \times \mathbb{R}_0) & \longrightarrow & H^{m+1}(X \times \mathbb{R}, X \times \mathbb{R}_-) \end{array}$$

Since $X \times \mathbb{R}$ and $X \times \mathbb{R}_-$ both deformation retract to $X \times \{-1\}$, the terms on the extremities vanish, hence δ' must be an isomorphism. Similarly to (3.1) above, we also have an excision isomorphism:

$$H^m(X \times \mathbb{R}_0, X \times \mathbb{R}_-) \xrightarrow{i^*} H^m(X \times \mathbb{R}_+).$$

Hence we may contemplate the following diagram for a given $a \in H^m(X)$, which is graded commutative by naturality of the cross product:

$$\begin{array}{ccccc} H^0(\mathbb{R}_+) & \xleftarrow[\sim]{\eta^*} & H^0(\mathbb{R}_0, \mathbb{R}_-) & \xrightarrow[\sim]{\delta} & H^1(\mathbb{R}, \mathbb{R}_0) \\ \downarrow{\scriptstyle a\times -} & & \downarrow{\scriptstyle a\times -} & & \downarrow{\scriptstyle a\times -} \\ H^m(X) \cong H^m(X \times \mathbb{R}_+) & \xleftarrow[\sim]{i^*} & H^m(X \times \mathbb{R}_0, X \times \mathbb{R}_-) & \xrightarrow[\sim]{\delta'} & H^{m+1}(X \times \mathbb{R}, X \times \mathbb{R}_0), \end{array}$$

Since by construction $e^1 = \delta\eta^{-1}(1)$, it follows from the above that:

$$a \times e^1 = a \times (\delta\eta^{-1}(1)) = \pm\delta'(i^*)^{-1}(a \times 1) = \delta'(i^*)^{-1}(- \times 1)(a),$$

i.e. that the map $- \times e^1$ can be expressed as a composite of isomorphisms, and therefore must be an isomorphism.

We now move on to the case of an arbitrary pair (X, A). Taking cross products at the level of cochains with respect to a representative $z \in C^1(\mathbb{R}, \mathbb{R}_0)$ of the cohomology class $e^1 \in H^1(\mathbb{R}, \mathbb{R}_0)$ gives us a graded commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^m(X,A) & \longrightarrow & C^m(X) & \longrightarrow & C^m(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle -\times z} & & \downarrow{\scriptstyle -\times z} & & \downarrow{\scriptstyle -\times z} & & \\ 0 & \longrightarrow & \hat{C}^{m+1}(X \times \mathbb{R}; X \times \mathbb{R}_0, A \times \mathbb{R}) & \longrightarrow & C^{m+1}(X \times \mathbb{R}, X \times \mathbb{R}_0) & \longrightarrow & C^{m+1}(A \times \mathbb{R}, A \times \mathbb{R}_0) & \longrightarrow & 0, \end{array}$$

where the top row induces the cohomology LES associated to the pair (X, A) and the bottom row corresponds to diagram (2.1) from section 2 above. Further observe that this diagram is compatible with taking coboundaries since z is a cocycle, hence for each $m \geq 0$ we obtain an induced commutative diagram at the level of cohomology, using the isomorphism in (2.2):

$$\begin{array}{ccccccccc}
H^m(X) & \longrightarrow & H^m(A) & \xrightarrow{\delta} & H^{m+1}(X,A) & \longrightarrow & H^{m+1}(X) & \longrightarrow & H^{m+1}(A) \\
\downarrow{\scriptstyle -\times e^1} & & \downarrow{\scriptstyle -\times e^1} & & \downarrow{\scriptstyle -\times e^1} & & \downarrow{\scriptstyle -\times e^1} & & \downarrow{\scriptstyle -\times e^1} \\
H^{m+1}(X\times\mathbb{R}, X\times\mathbb{R}_0) & \longrightarrow & H^{m+1}(A\times\mathbb{R}, A\times\mathbb{R}_0) & \longrightarrow & H^{m+2}(X\times\mathbb{R}, A\times\mathbb{R}\cup X\times\mathbb{R}_0) & & H^{m+1}(X\times\mathbb{R}, X\times\mathbb{R}_0) & \longrightarrow & H^{m+1}(A\times\mathbb{R}, A\times\mathbb{R}_0).
\end{array}$$

We may then apply the 5-lemma to infer the result from the above verification of the case $(X, \emptyset)$, completing the proof. □

Proposition 3.3. *Let (X,A) be a pair of spaces, with $A \subset X$ open. Set $e^n := e^1 \times ... \times e^1 \in H^n(\mathbb{R}^n, \mathbb{R}^n_0)$. Then the map*

$$H^m(X,A) \xrightarrow{-\times e^n} H^{m+n}(X\times\mathbb{R}^n, A\times\mathbb{R}^n \cup X\times\mathbb{R}^n_0)$$

given by $a \mapsto a \times e^n$ is an isomorphism for all $m \geq 0$.

Proof. Simply observe that, by associativity of the cross product, the above map is obtained as a composite of isomorphisms:

$$\begin{array}{ccc}
H^m(X,A) & \xrightarrow{-\times e^1} & H^{m+1}(X\times\mathbb{R}, A\times\mathbb{R}\cup X\times\mathbb{R}_0) \\
 & & \downarrow{\scriptstyle -\times e^1} \\
 & & H^{m+2}(X\times\mathbb{R}^2, A\times\mathbb{R}^2\cup X\times\mathbb{R}^2_0) \\
 & & \downarrow{\scriptstyle -\times e^1} \\
 & {\scriptstyle -\times e^n} & \vdots \\
 & & \downarrow{\scriptstyle -\times e^1} \\
 & & H^{m+n}(X\times\mathbb{R}^n, A\times\mathbb{R}^n\cup X\times\mathbb{R}^n_0).
\end{array}$$

□

4. Proof of the Thom Isomorphism Theorem

We are now in a position to prove the Thom isomorphism theorem. The flavor of the proof is reminiscent of that of the proof of De Rham's theorem in the style of chapter 18 of Lee [1], in which the isomorphism $H^i_{dR}(M) \cong H^i(M;\mathbb{R})$ is first verified for open balls in $\mathbb{R}^n$ via the Poincaré lemma, after which one shows that the De Rham theorem is true for any manifold admitting a basis of opens on which the De Rham theorem holds, and then proceeds to construct such a basis using an exhaustion by compact subsets.

We restate the Thom isomorphism theorem here for convenience:

Theorem 4.1. *Let $\xi : E \xrightarrow{\pi} B$ be a vector bundle of rank n. Then there exists a unique Thom class $u \in H^n(E, E_0)$, characterized by the property that the image of u under the map $i_b^* : H^n(E,E_0) \to H^n(F_b, (F_b)_0) \cong \mathbb{Z}/2$ induced by the fiberwise inclusion $i_b : (F_b, (F_b)_0) \to (E, E_0)$ is non-zero for all $b \in B$. Furthermore, taking cup products with the Thom class induces an isomorphism, for all $i \geq 0$:*

$$\Phi : H^i(E) \xrightarrow{-\cup u} H^{n+i}(E, E_0). \tag{4.2}$$

We first establish the theorem for the trivial bundle $\xi : B \times \mathbb{R}^n \xrightarrow{\pi} B$. By proposition (3.3), we have an isomorphism

$$H^0(B) \xrightarrow{-\times e^n} H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0),$$

so that every element $x \in H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0) \equiv H^n(E, E_0)$ can be expressed uniquely in the form $x = s \times e^n$, where $s \in H^0(B)$. Now, observe that x is a valid Thom class for the bundle ξ provided that the restriction of s to each point in B is non-zero. But this is verified uniquely by the unit element $1 \in H^0(B)$, yielding the existence of a unique Thom class $u := 1 \times e^n \in H^n(E, E_0)$.

It remains to check that the induced map

$$H^i(B \times \mathbb{R}^n) \xrightarrow{-\cup u} H^{n+i}(E, E_0) \tag{4.3}$$

is an isomorphism. By the Künneth theorem, every element $y \in H^i(B \times \mathbb{R}^n)$ can be expressed uniquely in the form $y = z \times 1$, where $z \in H^j(B)$. Hence, we see that

$$y \cup u = (z \times 1) \cup (1 \times e^n) = (z \cup 1) \times (1 \cup e^n) = z \times e^n = (- \times e^n)(z),$$

so since the map $- \times e^n$ is an isomorphism it follows that the map in (4.3) is an isomorphism, as needed.

The idea is now to use the existence of local trivializations for vector bundles to patch together the Thom classes of local trivializations into global Thom classes, ensuring in the process that the desired isomorphism (4.2) is preserved. In what follows, say a vector bundle $\xi : E \xrightarrow{\pi} B$ is a *Thom bundle* is the Thom isomorphism theorem holds for ξ. Then we have the following intermediary result:

Proposition 4.4. *If a vector bundle $\xi : E \xrightarrow{\pi} B$ admits a finite open cover $B_1, ..., B_k$ of the base space B such that the restriction to each B_i and each intersection $B_{ij} \equiv B_i \cap B_j$ is a Thom bundle, then ξ is a Thom bundle.*

Proof. We establish the case $k = 2$; the result then follows by induction on k. Indeed, given an open cover $B_1, ..., B_k$ satisfying the above hypothesis, we may consider the open cover with two components $B := B_k, B' := \cup_{i=1}^{k-1} B_i$, so that $\xi|_B, \xi|_{B'}$ are both Thom bundles by induction, resp. $\xi|_{B \cap B'} = \xi|_{\cup_{i=1}^{k-1}(B_k \cap B_i)}$ is a Thom bundle, whence it follows that ξ is a Thom bundle once the result holds for $k = 2$.

So let $B = B^1 \cup B^2$ be an open cover of B such that $\xi_{B^1} : E^1 \to B^1$, $\xi_{B^2} : E^2 \to B^2$, $\xi_{B^{12}} : E^{12} \to B^{12}$ are Thom bundles. We may then consider the following relative Mayer-Vietoris LES in cohomology:

$$... \longrightarrow H^{n-1}(E^{12}, E^{12}_0) \xrightarrow{\delta} H^n(E, E_0) \longrightarrow H^n(E^1, E^1_0) \oplus H^n(E^2, E^2_0) \longrightarrow H^n(E^{12}, E^{12}_0) \longrightarrow ... \tag{4.5}$$

By hypothesis, there exist unique Thom classes $u_k \in H^n(E^k, E_0^k)$, $k = 1, 2$. From the following commutative diagram, for any $b \in B$:

$$\begin{array}{ccccc} H^n(E^1, E_0^1) & & & & \\ & \searrow^{l*} & & \searrow^{l^*} & \\ & & H^n(E^{12}, E_0^{12}) & \longrightarrow & H^n(F_b, (F_b)_0), \\ & \nearrow_{l*} & & \nearrow_{l^*} & \\ H^n(E^2, E_0^2) & & & & \end{array}$$

we see that both restriction l^*u_1, l^*u_2 satisfy the condition for being a Thom class on $\xi|_{B^{12}}$. By uniqueness, it follows that we must have $l^*u_1 = l^*u_2 = u_{12}$, and hence the pair (u_1, u_2) is sent to 0 under the rightmost map of (4.5). Hence, by exactness, there exists an element $u \in H^n(E, E_0)$ which restricts to u_1, resp. u_2 in $H^n(E^1, E_0^1)$, resp. $H^n(E^2, E_0^2)$, and which by a similar argument as above constitutes a valid Thom class for ξ. Uniqueness of such an element u follows by exactness of (4.5) together with the fact that $H^{n-1}(E^{12}, E_0^{12}) = 0$. □

Since any vector bundle $\xi : E \xrightarrow{\pi} B$ over a compact base space admits a finite open cover of B by local trivializations, our work so far completes the proof of the Thom isomorphism theorem for compact spaces. In order to establish the general case, we resort to a lemma relating the cohomology groups of a space to those of its compact subsets.

Lemma 4.6. *Let $\mathcal{C}$ be a direct system under inclusions of compact subsets of a space B, and let $\xi : E \xrightarrow{\pi} E_0$ be a vector bundle. Then the canonical map $H^i(B) \to \varprojlim_{\mathcal{C}} H^i(C)$ is an isomorphism for all $i \geq 0$, and similarly $H^i(E, E_0)$ maps isomorphically to $\varprojlim_{\mathcal{C}} H^i(\pi^{-1}(C), \pi^{-1}(C)_0)$ for all $i \geq 0$.*

Proof. First observe that in homology, since the topological i-simplex Δ^i is compact, every singular i-chain $\sigma : \Delta^i \to B$ lands in a compact subset $\sigma(\Delta^i) \subseteq B$. Hence, the canonical map $\varinjlim_{\mathcal{C}} H_j(C) \to H_j(B)$ is an isomorphism. The result then follows from the duality of homology and cohomology over a field, together with the fact that the hom functor is compatible with limits and colimits in the sense that

$$\mathrm{Hom}_{\mathbb{Z}/2}(\varinjlim_{\mathcal{C}} H_j(C), \mathbb{Z}/2) \cong \varprojlim_{\mathcal{C}} \mathrm{Hom}_{\mathbb{Z}/2}(H_j(C), \mathbb{Z}/2).$$

The result for $H^i(E, E_0) \xrightarrow{\sim} \varprojlim_{\mathcal{C}}(\pi^{-1}(C), \pi^{-1}(C)_0)$ follows similarly. □

Remark 4.7. The validity of the above lemma relies crucially on the fact that the underlying ring of coefficients $\mathbb{Z}/2$ is a field, giving a direct duality between homology and cohomology.

Now, since the union of any two compact sets is again compact, the restriction of $\xi : E \xrightarrow{\pi} B$ to any component $C \subseteq B$ of the system $\mathcal{C}$ is a Thom bundle, hence admits a Thom class $u_C \in H^n(\pi^{-1}(C), \pi^{-1}(C)_0)$. Then, by the above lemma, there exists a unique element $u \in H^n(E, E^0)$ which is mapped to $(u_C)_{C \in \mathcal{C}} \in \varprojlim_{\mathcal{C}} H^n(\pi^{-1}(C), \pi^{-1}(C)_0)$, and hence which constitutes the unique Thom class in $H^n(E, E_0)$.

Furthermore, by construction of u, we get for each compact subset $C \subseteq B$ and for all $i \geq 0$ we have a commutative square:

$$\begin{array}{ccc} H^i(E) & \xrightarrow{-\cup u} & H^{i+n}(E, E_0) \\ \downarrow{\scriptstyle l^*} & & \downarrow{\scriptstyle l^*} \\ H^i(\pi^{-1}(C)) & \xrightarrow{-\cup u_C} & H^{i+n}(\pi^{-1}(C), \pi^{-1}(C)_0). \end{array}$$

Strapping these together for all $C \in \mathcal{C}$, we get an induced commutative square:

$$\begin{array}{ccc} H^i(E) & \xrightarrow{-\cup u} & H^{i+n}(E, E_0) \\ \downarrow{\scriptstyle l^*} & & \downarrow{\scriptstyle l^*} \\ \varprojlim_{\mathcal{C}} H^i(\pi^{-1}(C)) & \xrightarrow{\prod_{\mathcal{C}}(-\cup u_C)} & \varprojlim_{\mathcal{C}} H^{i+n}(\pi^{-1}(C), \pi^{-1}(C)_0), \end{array}$$

where each vertical map is an isomorphism by the above lemma and where the horizontal map is an isomorphism by construction, completing the proof of the general case.

References

[1] Lee, John M. Introduction to Smooth Manifolds. Graduate Texts in Mathematics. Springer. 2013. 35-36

[2] Milnor, J. W. Stasheff, J. D. Characteristic Classes. Princeton University Press and University of Tokyo Press. 1974.