TODAY, I LEARNED...

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Contents

1. November 2020

1.1. 11/30/2020 (homotopical algebra, formal moduli problems).

- Whitehead's theorem can be stated and proven in any model category as the fact that if A, X are both fibrant and cofibrant, then a map $f : A \to X$ is a weak equivalence iff it admits a homotopy inverse (in the model categorical sense of homotopy between maps, using either cylinder or path objects). In this framework, assuming we know what weak/homotopy equivalences should be, the bulk of the work in generalizing Whitehead's theorem to other settings is shifted to exhibiting a model category structure on the category at hand such that the objects one is interested in (e.g. smooth projective varieties) are both fibrant and cofibrant.
- With respect to an appropriate notion of covering, one can think of a simplicial resolution of a commutative ring A as the same thing as a hypercovering of A, meaning roughly the data of a cover of A, together with a covering of each pairwise intersection of elements of the cover, and so on... In this language, a projective resolution is the same thing as a cofinal hypercover, i.e. one such that any other hypercover factors through it.
- I started getting a better feel for what the formal moduli problem X associated to a given elliptic curve E should assign to the ring $k[\epsilon]/(\epsilon^2)$: an element of $X(k[\epsilon]/(\epsilon^2))$ consists of E together with (scheme theoretic) infinitesimal data of order 1 indicating the "germ of a deformation of E". For instance, if we start with a family of elliptic curves over $\mathbb{A}^1$ with fiber E at 0, we can restrict to the tangent space at 0 to get such an element. We

can't expect to get all elements of $X(k[\epsilon]/(\epsilon^2))$ in this way, roughly because some deformations correspond to "power series".

2. December 2020

2.1. 12/01/2020 (complex geometry, sheaf cohomology).

- Identifying $\mathfrak{sl}_2$-representations appearing "in nature" can be a fruitful way to better understand algebraic objects with an underlying vector space structure. For instance, one can recover the hard Lefschetz theorem on the cohomology ring of a compact Kähler manifold by realizing that wedging with the Kähler form can be interpreted as the "e" action of an $\mathfrak{sl}_2$-representation. Another example arises in studying the general structure of semisimple complex Lie algebras: several steps of the structure theorem involve looking at the various $\mathfrak{sl}_2$-module structures obtained by having the subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{h}_\alpha \simeq \mathfrak{sl}_2$ associated to some root $\alpha \in R$ act on $\mathfrak{g}$ via the adjoint action. This tells us for instance that for $k \in \mathbb{Z}$ within a certain range, we get isomorphisms:
$$\mathrm{ad}(e_\alpha) : \mathfrak{g}^{\beta+k\alpha} \xrightarrow{\simeq} \mathfrak{g}^{\beta+(k+1)\alpha}.$$
- Taking cohomology is a construction that makes sense in the category $\mathrm{Shv}_{\mathrm{Ab}}(\mathcal{C})$ of abelian sheaves on any Grothendieck site $\mathcal{C}$: this is still an abelian category and one can run the usual right derived functor on global sections construction. For a scheme X, one usually works with $\mathcal{C}$ the (étale or otherwise) site over X (think: étale maps to X), and the global sections functor $\Gamma : \mathrm{Shv}_{\mathrm{Ab}}(\mathcal{C}) \to \mathrm{Ab}$ is given by evaluation at X. This language carries over to commutative rings: if B is an A-algebra, we can take $\mathcal{C}$ to be A-algebras over B, coverings to be set-theoretically surjective maps, and then, given a B-module M (equivalent to an abelian group object $B \oplus M$ in $\mathcal{C}$), we can take the sheaf cohomology of the representable sheaf $\mathrm{Der}_A(-, M) = \mathrm{Hom}_B(-, B \oplus M)$ in that Grothendieck site:
$$D^q(B, M) := H^q_T(B, \mathrm{Der}_A(-, M)).$$

2.2. 12/02/2020 (homological algebra, formal geometry).

- There are at least three ways to associate a chain complex to a given simplicial abelian group $A_\bullet$, all equivalent up to chain homotopy equivalence. The most "direct" construction is called the Moore complex A_*, for which one takes A_n in degree n and sets the differential to be $\partial := \sum_{i=0}^n (-1)^i d_i$, where $d_i : A_n \to A_{n-1}$, $0 \leq i \leq n$ are the face maps in degree n. Alternatively, one can restrict to the subcomplex $DA_\bullet$ with DA_n generated by degenerate n-simplices (those in the image of a degeneracy map). Perhaps most useful is the normalized chain complex $NA_\bullet$, given in degree n by $NA_n := \bigcap_{i=0}^{n-1} \ker(d_i)$ (note that we omitted d_n), and with differential given by $(-1)^n d_n$. The assignment $A_\bullet \mapsto NA_\bullet$ gives one direction of the Dold-Kan correspondence, which is an equivalence of categories:
$$N : \mathrm{sAb} \xrightarrow{\simeq} \mathrm{Ch}_{\geq 0}(\mathrm{Ab}).$$
- In attempting to generalize the Lie group-Lie algebra correspondence to the settings of group schemes over some field k, one is led to looking for a way to formalize what it means to consider an "infinitesimal neighborhood

of the identity $1 \in G$" - the so called "formal group" associated to G. Under the yoga that algebraic geometry studies functors out of categories of commutative rings, formal geometry focuses on functors of the form

$$X : \mathrm{Art}_k^{\mathrm{aug}} \to \mathrm{Set},$$

where $\mathrm{Art}_k^{\mathrm{aug}}$ denotes the category of local Artinian k-algebras ($k \to A \to k$ composing to the identity on k), or left Kan extensions thereof to functors $\mathrm{CAlg} \to \mathrm{Set}$. The intuition is that if X is a scheme, $x \in X$ is a given closed point, and A is a local Artinian k-algebra, then an A-valued point $f : \mathrm{Spec}(A) \to X$ taking the maximal ideal of A to x is the right way to describe "an arbitrary infinitesimal neighborhood of X in x". As such, the formal group associated to G is defined to be the functor

$$G_1^\wedge : \mathrm{Art}_k^{\mathrm{aug}} \to \mathrm{Grp}$$

given by $G_1^\wedge(R) := G(R) \times_{G(k)} \{1\}$. One may more generally associate to any closed embedding $Y \hookrightarrow X$ a formal scheme $X_Y^\wedge$ called the formal completion of Y inside X. In the language of DAG, this object may be succinctly constructed via the following pullback square:

$$\begin{array}{ccc} X_Y^\wedge & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_{\mathrm{dR}} & \longrightarrow & X_{\mathrm{dR}}. \end{array}$$

- Just came across my new favorite proof of the fundamental theorem of algebra: given a polynomial $p \in \mathbb{C}[z]$ of degree $n \geq 1$, view it as the characteristic polynomial of a matrix A (the so-called companion matrix), which can be assumed to be invertible (o/w done). Existence of a root for p is equivalent to existence of an A-eigenvector, which is equivalent to the $\mathbb{C}P^{n-1}$ endomorphism induced by A having a fixed point. Since $GL_n(C)$ is path connected, A is homotopic to I as $\mathbb{C}P^{n-1}$ endomorphisms, so they have the same Lefschetz number, and $L(I) = \chi(CP^{n-1}) = n > 0$, hence A has a fixed point by the Lefschetz fixed point theorem!

2.3. **12/03/2020 (scheme theory, Chern-Weil theory).**

- Guiding principle: "solutions to Diophantine equations may be thought of as sections of a fiber bundle over the arithmetic curve Spec $\mathbb{Z}$". Suppose we are interested in finding solutions to a given polynomial equation $f \in \mathbb{Z}[x]$ modulo various primes p. We attempt to interpret the problem geometrically as follows. Let $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}\ \mathbb{Z}[x]$ denote the affine line, which may be viewed together with a flat morphism down to Spec $\mathbb{Z}$. The polynomial f defines a closed subscheme $\mathrm{Spec}\ \mathbb{Z}[x]/(f) \subseteq \mathbb{A}^1_{\mathbb{Z}}$, and we may consider its pullback to any fiber corresponding to a prime $p \in \mathbb{Z}$:

$$\begin{array}{ccc} \mathrm{Spec}\ \mathbb{F}_p[x]/(f) & \longrightarrow & \mathrm{Spec}\ \mathbb{Z}[x]/(f) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ \mathbb{F}_p & \longrightarrow & \mathrm{Spec}\ \mathbb{Z} \end{array}$$

Solutions to f mod p correspond to maps $\mathrm{Spec}\ \mathbb{F}_p \to \mathrm{Spec}\ \mathbb{F}_p[x]/(f)$, which necessarily compose to the identity on Spec $\mathbb{F}_p$ in the above square: these look like sections of the RHS vertical map interpreted as a “fiber bundle” over the various closed points of Spec $\mathbb{Z}$. More to follow.

- Chern-Weil theory provides a differential-geometric construction of characteristic classes for vector bundles (and more generally for principal G-bundles) through the machinery of connections. Start with a complex vector bundle with connection (E, ∇) of rank r over a smooth manifold M. View the characteristic polynomial (evaluated at -1) as a conjugation-invariant function on $M_r(\mathbb{C})$, and express it as a sum of homogeneous symmetric polynomials P_k for $0 \leq k \leq r$, which may be alternatively viewed as elements of $\mathrm{Sym}^k(M_r(\mathbb{C}))^{GL_r(\mathbb{C})}$. One may formally enable the P_k's to take in and return differential forms. Letting $F_\nabla \in \mathcal{A}^2(\mathrm{End}(E))$ denote the curvature form of ∇, we may define
$$c_k(E) := [P_k(F_\nabla)] \in H^{2k}_{\mathrm{dR}}(M; \mathbb{C}).$$
This definition turns out to be independent of ∇, and to coincide with the standard definition of $c_k(E) \in H^{2k}(M; \mathbb{Z})$. This procedure produces characteristic classes from other choices of conjugation-invariant homogeneous polynomials, and more generally in the settings of principal G-bundles one may define the so-called Chern-Weil homomorphism to be the following map of graded $\mathbb{C}$-algebras:
$$\mathrm{Sym}^*(\mathfrak{g})^G \to H^*_{\mathrm{dR}}(M; \mathbb{C}).$$
The above map may be interpreted as a “way to produce differential form-valued invariants of principal G-bundles”. In [3], Freed-Hopkins attempt to formalize the mathematical context of these invariants, and show that the Chern-Weil homomorphism is in some precise way “the only natural differential form-valued invariant of principal G-bundles on smooth manifolds”.

2.4. **12/04/2020 (scheme theory, representation theory).**

- Affine schemes are fully determined by their ring of functions: the ring R “knows all there is to know” about Spec R. For instance, if R is a k-algebra over some field, then points of Spec R may be recovered by looking at ring homomorphisms $R \to k'$ for k'/k a field extension. This is not true of non-affine schemes - for instance, $\mathcal{O}(\mathbb{P}^n_k) = k$, yet Spec k only has a single point. To formulate an analogous reconstruction result for schemes, one needs to categorify functions to sheaves. It is then true that a scheme is fully determined by the associated symmetric monoidal category of quasi-coherent sheaves $(\mathrm{QCoh}(X), \otimes)$. The fact that the ring of functions sufficed in the affine case is reflected in the identification $\mathrm{QCoh}(\mathrm{Spec} R) \simeq \mathrm{Mod}_R$. More generally, stacks which may be recovered from their category of quasi-coherent sheaves are called “1-affine”.
- (Reminiscence) Serre's GAGA principle allows us to import methods of complex geometry to bear on projective complex algebraic geometry. It states that for any smooth projective complex algebraic variety, the analytification functor induces an equivalence of categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X^{\mathrm{an}})$, as well as a natural isomorphism on sheaf cohomology groups:
$$H^q(X, \mathcal{F}) \simeq H^q(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}).$$

In particular, the fact that $\mathcal{O}(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{C}$ follows from this isomorphism and the fact that holomorphic functions on compact complex manifolds are constant, together with the identification $\mathcal{O}(\mathbb{P}^n_{\mathbb{C}}) = H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}})$.

- (To be fleshed out at a later date) Given a monoidal category $(\mathcal{C}, \otimes)$, one may consider its Bernstein center, which is the ring $Z(\mathcal{C}) = \mathrm{End}(\mathrm{Id}_{\mathcal{C}})$ of endo-natural transformations of the identity functor. This ring naturally acts on $\mathcal{C}$, and one may expect some form of "spectral decomposition of $\mathcal{C}$" accordingly. In AG terms, form the space Spec $Z(\mathcal{C})$. One may then interpret $\mathcal{C}$ as forming in some sense the data of a "sheaf of categories over Spec $Z(\mathcal{C})$":
$$\begin{array}{c} \mathcal{C} \\ \Big| \\ \mathrm{Spec}\ Z(\mathcal{C}). \end{array}$$
In the context of representation theory, this viewpoint should lead to an intuitive understanding of the decomposition of BGG category $\mathcal{O}$ into blocks:
$$\mathcal{O} \simeq \bigoplus_{\lambda \in \mathfrak{h}^*/(W, \bullet)} \mathcal{O}_\lambda.$$
Compare also to the statement that for a finite group A, we may identify $\mathrm{Rep}(A)$ with the category $\mathrm{Vect}(\hat{A})$ of vector bundles over the character group $\hat{A} = \mathrm{Hom}(A, U(1))$, via the fact that any representation V splits as a direct sum $V \simeq \bigoplus_{\alpha \in \hat{A}} \mathrm{Hom}(V_\alpha, V)$, where V_α denotes the irreducible A-rep associated to $\alpha \in \hat{A}$:
$$\begin{array}{c} \mathrm{Rep}(A) \\ \Big| \\ \hat{A}. \end{array}$$
For much more on categorical centers and their applications to geometric rep theory, see [1].

2.5. **12/05/2020 (motives).**

- Whatever the category of (pure) motives (over a ground field k) $\mathcal{M}(k)$ may be, it should have the structure of a Tannakian category, which makes it susceptible to a Tannakian reconstruction theorem exhibiting it as the category of representations of an affine group scheme G, called the *motivic Galois group*. The group G should be given by the $\otimes$-automorphism group scheme of a fiber functor, $G := \underline{\mathrm{Aut}}^{\otimes}(\omega)$, where $\omega : \mathcal{M}(k) \to \mathrm{Vect}^{fd}_{\mathbb{Q}}$ should be thought of as a "cohomology theory" valued in $\mathbb{Q}$-vector spaces. Furthermore, to any smooth projective algebraic variety X, there should be associated a sequence of objects $H^i_{\mathrm{mot}}(X)$, which should induce the various classical cohomology groups (de Rham, Betti, crystalline,...) under the application of various realization functors out of $\mathcal{M}(k)$. See [2] for more. Also fascinating: there is an analogy
$$\text{finite } G\text{-sets} : \text{finite étale covs } / \ S \ :: \ G\text{-reps} : \text{Tannakian cat } / \ k$$
i.e. one may think of the Tannakian formalism as a linearization of the theory of étale coverings and étale fundamental groups.

2.6. **12/06/2020 (representation theory).**

- DBZ made an interesting comment today, adding to the "sheaves of categories over the Bernstein center" intrigue (see 12/04/2021): there should be an analogy in the settings of linear operators on (f.d.) vector spaces viewed as "sheaves over $\mathbb{C}$":

  eigenspace : generalized eigenspace (Jordan block) ::
  support on a point : support on the formal nbhd of a point

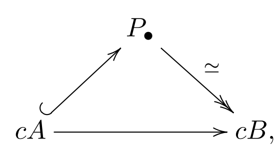
  i.e., the nilpotency of Jordan blocks is geometrically incarnated by sheaf-theoretic infinitesimal behavior. The idea should go as follows: $A : V \to V$ turns V into a $\mathbb{C}[t]$-module, and one may view

$$\begin{array}{c} \mathbb{C}[t] - \text{mod} \\ \Big| \\ \text{Spec } \mathbb{C}[t]. \end{array}$$

  This picture should also apply to category $\mathcal{O} = U\mathfrak{g} - \text{mod}^{f.g.,\mathfrak{n}-locf.,\mathfrak{h}-s.s.}$, wherein generalized central characters should coincide with sheaves scheme theoretically supported on the formal neighborhood of a point in the character group of the torus (so that one should expect the Bernstein center of $\mathcal{O}$ to be $\text{Hom}(T, \mathbb{G}_m)$?). This is now getting closer to the $\text{Rep}(A)$ over $\hat{A}$ picture in the finite group settings from before - and indeed one can show that $\mathbb{C}$-points of Spec $\mathbb{C}[A]$ correspond to characters of A (!).

2.7. **12/07/2020 (homotopical algebra).**

- The construction of the cotangent complex $\mathbb{L}D_{B/A}$ of an A-algebra B consists in viewing the structure map $A \to B$ as a map $cA \to cB$ in sCAlg between the corresponding constant simplicial rings, choosing a projective factorization



  and then defining

$$\mathbb{L}D_{B/A} := \Omega_{P_\bullet/A} \otimes_{P_\bullet} B,$$

  where $\Omega_{-/A} \otimes_- B$ is the abelianization functor on the category $\text{CAlg}_{A/,\backslash B}$, applied component-wise to the simplicial object $P_\bullet$. Viewing this object as a chain complex by Dold-Kan, one may define the *cohomology* of the A-algebra B with coefficients in a B-module M to be:

$$D^i(B/A, M) := H_i(\mathbb{L}D_{B/A} \otimes_B M)$$

  i.e. "take i^{th} homology of the derived functor applied to M". This is the same general procedure that one first encounters when defining the Tor groups for modules over a ring, and somewhat more generally for defining the left derived functors associated to a right exact functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$, provided $\mathcal{A}$ has enough projectives:

$$\mathcal{A} \xrightarrow{\text{fibrant replacement}} \mathbf{Ch}(\mathcal{A}) \xrightarrow{F} \mathbf{Ch}(\mathcal{B}) \xrightarrow{H_i} \mathcal{B}$$

$$A \longmapsto P_\bullet \longmapsto F(P_\bullet) \longmapsto H_i(F(P_\bullet)) =: L_iF(A).$$

One should be able to define a model structure on $\mathbf{Ch}(\mathcal{A})$, resp. $\mathbf{Ch}(\mathcal{B})$ in such a way that this process is an instance of the derived functor construction in the settings of model categories.

2.8. **12/08/2020 (geometric Langlands).**

- ((lots of) missing adjectives) Start with a reductive algebraic group G and a smooth algebraic curve X. The geometric Langlands conjecture roughly states that there should be an equivalence of derived categories

$$\mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}) \xrightarrow{\simeq} \mathcal{D}(\mathrm{Bun}_G)$$

between quasi-coherent sheaves on the moduli stack of $G^\vee$-local systems on X and D-modules on the moduli stack of G-bundles on X. The RHS is called the automorphic side, and categorifies the notion of "automorphic forms", while the LHS is called the spectral side and categorifies the notions of "Galois representations" in the classical settings.

The correspondence should send the skyscraper sheaf associated to an irreducible local system E to a D-module $\mathcal{M}$ which is a *Hecke eigensheaf* for E on Bun_G. This roughly means that $\mathcal{M}$ transforms "like an eigenvector" under the action ("integral transform") of the various Hecke functors

$$H_\lambda : \mathcal{D}(\mathrm{Bun}_G) \to \mathcal{D}(X \times \mathrm{Bun}_G)$$

for $\lambda \in P_+$ a dominant weight of $G^\vee$.

Goal 2.1. Make some sense of what a Hecke eigensheaf should be.

These Hecke functors mentioned above are "integral transforms" induced via a pull-push process by a correspondence at the level of stacks:

$$\begin{array}{ccc} & \mathcal{H} & \\ {\scriptstyle h_1}\swarrow & & \searrow{\scriptstyle h_2} \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{array}$$

where $\mathcal{H}$ is the Hecke stack parametrizing data (P, P', x, β) for P, P' a pair of principal G-bundles P, P' over X trivialized away from $x \in X$ via β. The maps above are respectively given by

$$P \longleftarrow\!\shortmid (P, P', x, \beta) \longmapsto (P', x).$$

Let's run the story over $\mathbb{C}$, where we may work with perverse sheaves on the automorphic side (stacky Riemann-Hilbert?). Here, the Hecke stack admits a stratification $\mathcal{H} = \bigsqcup_{\lambda \in P_+} \mathcal{H}_\lambda$ induced by the Bruhat-type stratification of the affine Grassmannian (which appears when studying the fiber of h_2 over various (x, P')'s). Via this stratification, one may define analogues of IC sheaves IC_λ on $\mathcal{H}$ for each $\lambda \in P_+$. The functor H_λ is then given by

$$H_\lambda : \mathcal{P}_{G(\mathcal{O})}(\mathrm{Bun}_G) \to \mathcal{P}_{G(\mathcal{O})}(X \times \mathrm{Bun}_G)$$

$$H_\lambda(\mathcal{M}) := h_{2,*}(h_1^*(\mathcal{M} \otimes \mathrm{IC}_\lambda)).$$

Now, start with a $G^\vee$-local system E on X and a $G^\vee$-irrepn V_λ associated to $\lambda \in P_+$. Form the balanced product $V_\lambda^E := E \times_{G^\vee} V_\lambda \in \mathrm{LocSys}_{G^\vee}$. We have an exterior product operation:

$$\mathrm{LocSys}_{G^\vee} \times \mathcal{P}_{G(\mathcal{O})}(\mathrm{Bun}_G) \xrightarrow{\boxtimes} \mathcal{P}_{G(\mathcal{O})}(X \times \mathrm{Bun}_G).$$

Definition 2.2. Call $\mathcal{M} \in \mathcal{P}_{G(\mathcal{O})}(\mathrm{Bun}_G)$ a *Hecke eigensheaf* for the $G^\vee$ local system E if it comes with the data of isomorphisms for each $\lambda \in P_+$:

$$\eta_\lambda : H_\lambda(\mathcal{M}) \xrightarrow{\simeq} V_\lambda^E \boxtimes \mathcal{M},$$

which are compatible with the monoidal structure on $(\mathrm{Rep}(G^\vee), \otimes)$.

For more on this story, see section 6.1 of [4].

2.9. **12/09/2020 (geometric Langlands).**

- Abelian class field theory studies the absolute abelian Galois group $\mathrm{Gal}(F^{ab}/F)$ associated to some number field $F/\mathbb{Q}$, and posits the existence of an isomorphism:

$$\mathrm{Gal}(F^{ab}/F) \simeq \left(F^\times \backslash \mathbb{A}_F^\times\right)_{c.c.}$$

where the subscript c.c. denotes connected components, and $\mathbb{A}_F$ is the ring of adèles of F, obtained by putting together the various completions of F.

On the one hand, we have the identification of $\mathrm{Gal}(F^{ab}/F)$ with the abelianization $\mathrm{Gal}(\bar{F}/F)_{ab}$ of the absolute Galois group, whose structure is completely determined by the set $\mathrm{Hom}(\mathrm{Gal}(\bar{F}/F), \mathrm{GL}_1)$ of its 1-dimensional linear representations. Likewise knowledge of $\left(F^\times \backslash \mathbb{A}_F^\times\right)_{c.c.}$ may be reduced to understanding certain of its 1-dimensional representations, or equivalently certain reps of $\mathbb{A}_F^\times$ occurring in $\mathrm{Fun}(F^\times \backslash \mathbb{A}_F^\times)$. We are thus led to the equivalent isomorphism:

$$\left\{\text{1-dim reps of } \mathrm{Gal}(\bar{F}/F)\right\} \simeq \left\{\text{Reps of } \mathrm{GL}_1(\mathbb{A}_F) \text{ in functions on } \mathrm{GL}_1(F^\times)\backslash \mathrm{GL}_1(\mathbb{A}_F^\times)\right\}$$

modulo an appropriate adjective on which reps appear on the RHS.

By Tannakian reconstruction, understanding $\mathrm{Gal}(\bar{F}/F)$ is roughly equivalent to understanding $\mathrm{Rep}^{fd}(\mathrm{Gal}(\bar{F}/F))$ as a Tannakian category. The *classical Langlands correspondence for GL_n* gives us some traction on this category of representations by positing the existence of a correspondence (generalizing the above):

$$\left\{n\text{-dim reps of } \mathrm{Gal}(\bar{F}/F)\right\} \simeq \left\{\text{Reps of } \mathrm{GL}_n(\mathbb{A}_F) \text{ in functions on } \mathrm{GL}_n(F^\times)\backslash \mathrm{GL}_n(\mathbb{A}_F^\times)\right\}$$

which furthermore carries meaningful structure from either side to one another (namely, Frobenius conjugacy classes are matched with Hecke eigenvalues). This is all part of section 1 of [4].

2.10. **12/10/2020 (homotopical algebra).**

- Let X be a complex projective manifold, and notice that any codimension k complex submanifold $Y \subset X$ induces via Poincaré duality an integral form of type (k,k), called an algebraic class. The Hodge conjecture asks whether every cohomology class in $H^{k,k}(X) \cap H^{2k}(X;\mathbb{Q})$ can be obtained as a $\mathbb{Q}$-linear combination of algebraic classes. The only known case of this conjecture for general Kähler manifolds is the Lefschetz theorem on $(1,1)$ classes, which states that taking the Chern class of a line bundle gives a surjection $c_1 : \mathrm{Pic}(X) \twoheadrightarrow H^{1,1}(X) \cap H^2(X;\mathbb{C})_{\mathbb{Z}}$, where $H^2(X;\mathbb{C})_{\mathbb{Z}} := \mathrm{im}(H^2(X;\mathbb{Z}) \to H^2(X,\mathbb{C}))$.
- With respect to the model category structure on the category of non-negatively graded cdga's over $\mathbb{C}$ where weak equivalences are quasi-isomorphisms and fibrations are degree-wise surjective chain maps, a minimal model for $\mathcal{A}^*$ is a valid cofibrant replacement of $\mathcal{A}^*$, i.e. a cofibrant cdga $\mathcal{M}^*$ together with an acyclic fibration $\mathcal{M}^* \twoheadrightarrow \mathcal{A}^*$ factorizing the map $0 \to \mathcal{A}^*$. One defines a minimal cdga $\mathcal{M}^*$ to be one such that there exist elements $(x_i \in \mathcal{M}^i)_{i \in I}$, $1 \leq d_1 \leq d_2 \leq \ldots$, generating $\mathcal{M}^*$ as a cdga, i.e. $\mathcal{M}^* = \bigwedge^* \left(\bigoplus_{i \in I} \mathbb{C} x_i \right)$, and such that $dx_i \in < x_1, \ldots, x_{i-1} >_{>0}$ for each i (where the subscript "> 0" indicates that we are excluding constants). Work of Sullivan shows that minimal models are valid cofibrant replacements with respect to the model category structure described above, and that any simply connected cdga over $\mathbb{C}$ admits a minimal model (which is then necessarily unique up to quasi-isomorphism).

2.11. **12/14/2020 (algebraic groups).**

- The Lie algebra $\mathfrak{g}$ associated to an algebraic group G defined over k is defined to be the Zariski tangent space at the identity $T_1 G$, with Lie bracket given by the adjoint representation of $\mathfrak{g}$ induced by the action of G on itself by conjugation:
$$[x,y] := \mathrm{ad}(x)(y).$$
From the functor of points perspective, for a given k-algebra R, we can define $\mathfrak{g}(R)$ explicitly as follows: write $R[\epsilon] := R[x]/(x^2)$. Then $R[\epsilon]$ has the structure of an augmented R-algebra via
$$R \xrightarrow{r \mapsto r + 0\epsilon} R[\epsilon] \xrightarrow{a + \epsilon b \mapsto a} R.$$
This produces a corresponding sequence of group homomorphisms:
$$G(R) \xrightarrow{i} G(R[\epsilon]) \xrightarrow{\pi} G(R),$$
so that we may set
$$\mathfrak{g}(R) := \ker \left(G(R[\epsilon]) \xrightarrow{\pi} G(R) \right)$$
to consist of "tangent vectors of $G(R)$ starting at $1 \in G(R)$." For more on this, see [10].

2.12. **12/15/2020 (formal groups and Lie algebras).**

- (Variation on a theme) In a variety of contexts, one can fruitfully discuss a correspondence between "spaces" and "groups" according to the following blueprint:

$$\text{Spaces} \underset{\text{exponentiation}}{\overset{\text{differentiation}}{\rightleftarrows}} \text{Groups.}$$

The prototypical example comes from the classical Lie group-Lie algebra correspondence, which is an actual equivalence of categories as long as we restrict our attention to simply connected Lie groups, in which case the above blueprint becomes:

$$\text{LieGrp} \underset{\exp}{\overset{\text{Lie}}{\rightleftarrows}} \text{LieAlg.}$$

Notice that the Baker-Campbell-Hausdorff formula gives us an explicit way to generate a group law from a Lie algebra, in that if $\exp(x)\exp(y) = \exp(z)$, then we may write z explicitly as:

$$z = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \big(\text{higher order brackets in } x, y\big).$$

In AG over a field k of characteristic zero, one may play a similar game, only replacing Lie groups by formal groups, according to the philosophy that small neighborhoods have to be replaced by infinitesimal neighborhoods. We get an analogous equivalence of categories, carrying over the BCH formula "formally":

$$\text{FormalGrp} \underset{\exp}{\overset{\text{Lie}}{\rightleftarrows}} \text{LieAlg}_k.$$

In the land of rational homotopy theory, Quillen tells us that there is an equivalence of categories:

$$\mathcal{S}_{\mathbb{Q}}^{\geq 2} \xrightarrow{\simeq} \text{dgLieAlg}$$

between simply connected spaces up to rational homotopy equivalence and dg Lie algebras (essentially Lie algebra objects in chain complexes). In the land of classical homotopy theory, the loop space construction produces an equivalence of categories on simply connected pointed spaces:

$$\mathcal{S}_*^{\geq 1} \underset{\mathrm{B}}{\overset{\Omega_x}{\rightleftarrows}} \text{Grp}_{\mathbb{E}_1}(\mathcal{S})$$

where the inverse equivalence is given by passing to the classifying space.

(To be further elucidated) This admits an analogue in formal DAG, which provides a similar group-like characterization of formal moduli problems (thought of as infinitesimal moduli spaces in derived algebraic geometry):

$$\text{FMP}_k \xrightarrow{\Omega} \text{Grp}_{\mathbb{E}_1}(\text{FMP}_k)$$

Lurie-Pridham take this a step further and identify the RHS with derived Lie algebras by passing through the world of formal groups:

$$\text{FMP}_k \xrightarrow[\Omega]{\simeq} \text{Grp}_{\mathbb{E}_1}(\text{FMP}_k) \simeq \text{FormalGrp}_k \xrightarrow[\text{Lie}=T_{-,1}]{\simeq} \text{LieAlg}_k$$

2.13. **12/16/2020 (scheme theory).**

- The forgetful functor
$$\mathrm{CAlg}_R \xrightarrow{obliv} \mathrm{Mod}_R$$
admits a left-adjoint given by the symmetric algebra $V \mapsto \mathrm{Sym}^*_R(V)$. One may then think of the assignment
$$M \mapsto \mathrm{Spec}(\mathrm{Sym}^*_R(M))$$
as a "canonical" way of turning an R-module M into a space living over Spec R. By Serre-Swan, for reasonable R, this procedure should produce algebraic vector bundles out of finitely generated projective modules. Over a more general base scheme S, one can make sense of a *relative Spec* construction associating to any sheaf of algebras $\mathcal{A} \in \mathrm{CAlg}(\mathrm{QCoh}(S))$ a scheme
$$\begin{array}{c} \mathrm{Spec}_S(\mathrm{Sym}^*_{\mathcal{O}_S}(\mathcal{A})) \\ \downarrow \\ S. \end{array}$$
Under this assignment, locally free sheaves of finite rank $\mathcal{E}$ correspond to algebraic vector bundles over S. Affine locally on some $U = \mathrm{Spec}\ R \subseteq S$, one retrieves the data of an R-module A such that $\Gamma(U, \mathcal{A}) = A$, which we require to be isomorphic to a polynomial algebra over R in the case where $\mathcal{A}|_U$ is free (corresponding to the usual local trivializability statement after taking relative Spec).

2.14. **12/17/2020 (scheme theory).**

- Oftentimes, adjectives appended to scheme morphisms (e.g. smooth, projective, affine) are ways of carrying over the property we understand for a given variety (defined over a field) to the relative context over a general base scheme, i.e. of making sense of the concept "in families". For instance, a smooth morphism may be defined as a (flat, locally of finite type) scheme morphism such that each geometric fiber is smooth as a scheme over the algebraic closure of the residue field at that point. Exhibit 2, following up on the relative Spec construction above: an affine morphism $X \to S$ may be characterized by the fact that one may find a sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ such that $X \simeq \mathrm{Spec}_S(\mathcal{A})$ is given by a relative Spec (hence affine locally by a classical Spec).

  This philosophy fruitfully applies to arithmetic geometry, for instance in the following context: start with a scheme X locally of finite type over Spec $\mathbb{Z}$ (read: a family of polynomial equations in finitely many variables defined over $\mathbb{Z}$), and suppose that its base change to $\mathbb{C}$ (hence also to the generic fiber $\mathbb{Q}$) is smooth (in the usual Jacobian sense). Say that X has *bad reduction* at a prime $p \in \mathbb{Z}$ if X_p is not smooth over $\mathbb{F}_p$. Then one may prove that X can only have bad reduction at finitely many primes using a theorem of Grothendieck stating that the set of points of Spec $\mathbb{Z}$ at which $X \to \mathrm{Spec}\ \mathbb{Z}$ is smooth is Zariski open, non-empty in our case since it contains the generic point, and thus its complement is Zariski closed in Spec $\mathbb{Z}$ hence finite.

2.15. **12/18/2020 (representation theory).**

- (Following chapter VIII of [14]) Given a complex connected Lie group G, there is a contravariant relationship between the size of $\pi_1(G)$ and the size of $Z(G)$. This relates to the fact that a covering map between Lie groups

$$\begin{array}{c} \tilde{G} \\ \big\downarrow p \\ G \end{array}$$

exhibits G as a quotient of $\tilde{G}$ by the normal discrete subgroup $\ker p \subseteq Z(\tilde{G})$, while the map p induces an inclusion $p_* : \pi_1(\tilde{G}) \hookrightarrow \pi_1(G)$ (hence G has a smaller center and a larger fundamental group than $\tilde{G}$, and vice versa).

  Now, suppose we start with a complex semisimple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ with coroot, resp. coweight lattice $\Lambda_{R^\vee} \subseteq \Lambda^\vee$. If G is any complex connected Lie group with maximal torus T such that $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(T) = \mathfrak{h}$, then the kernel $\Gamma(T)$ of the exponential map $\exp : \mathfrak{h} \to T$ always satisfies $\Lambda_{R^\vee} \subseteq \Gamma(T) \subseteq \Lambda^\vee$. In fact, there is a 1-1 correspondence between complex connected Lie groups G with $\mathrm{Lie}(G) = \mathfrak{g}$ (up to isomorphism) and intermediary subgroups

$$\Lambda_{R^\vee} \subseteq \Gamma \subseteq \Lambda^\vee.$$

  The group corresponding to $\Gamma = \Lambda_{R^\vee}$ is simply connected, and the group corresponding to $\Gamma = \Lambda^\vee$ (called the adjoint group) has trivial center.

  Dually (e.g the weight lattice is dual to the coroot lattice), one may also work with intermediary subgroups

$$\Lambda_R \subseteq X^* \subseteq \Lambda,$$

  where $\Lambda_R \subseteq \Lambda$ denote the root resp. weight lattice of $(\mathfrak{g}, \mathfrak{h})$, and various G's produce various character groups $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$. For a given G, the $\mathfrak{g}$-representations which induce group representations of G are then those whose weights lie inside $X^*(T)$. This is always the case for simply connected G, but may fail for certain representations in general.

2.16. **12/30/2020 (homotopical algebra).**

- The notion of chain homotopy between chain maps can be related to the traditional notion of homotopy between continuous maps as follows: define $I_\bullet$ to be the following chain complex:

$$\dots \to 0 \to \mathbb{Z} \xrightarrow{(+-)} \mathbb{Z} \oplus \mathbb{Z} \to 0 \to \dots ,$$

  where the term $\mathbb{Z} \oplus \mathbb{Z}$ lives in degree 0. (This is equivalently the normalized chain complex associated to the simplicial chains on the standard 1-simplex.)

  Given a chain complex $C_\bullet$, one may consider the total chain complex $C_\bullet \underline{\otimes} I_\bullet$. The data of a chain homotopy between two chain maps $f, g : C_\bullet \to D_\bullet$ may then be expressed as the data of a chain map

$$s : C_\bullet \underline{\otimes} I_\bullet \to D_\bullet$$

with appropriate "boundary conditions" regarding f and g. More explicitly" in degree n, we have

$$(C_\bullet \underline{\otimes} I_\bullet)_n = C_n \oplus C_{n+1} \oplus C_n,$$

and the boundary map is given by the matrix $\begin{pmatrix} d & 1 & 0 \\ 0 & -d & 0 \\ 0 & 1 & d \end{pmatrix}$. In particular, we have natural inclusion maps $C_\bullet \to C_\bullet \underline{\otimes} I_\bullet$. For s to be a chain homotopy between f and g, we then simply require that the following diagram commute:

$$\begin{array}{ccc} C_\bullet & & \\ \downarrow & \searrow^{f} & \\ C_\bullet \underline{\otimes} I_\bullet & \longrightarrow & D_\bullet \\ \uparrow & \nearrow_{g} & \\ C_\bullet & & \end{array}$$

The boundary map then gives rise to the traditional boundary condition $sd + ds = f - g$.

This whole discussion should make sense in the context of model categories: $C_\bullet \underline{\otimes} I_\bullet$ should be a cylinder object associated to the chain complex $C_\bullet$, with obvious structure maps

$$C_\bullet \oplus C_\bullet \to C_\bullet \underline{\otimes} I_\bullet \to C_\bullet,$$

so that the commutativity condition becomes

$$\begin{array}{ccc} C_\bullet \oplus C_\bullet & \xrightarrow{f+g} & D_\bullet \\ \downarrow & \nearrow_{s} & \\ C_\bullet \underline{\otimes} I_\bullet & & \end{array} .$$

3. January 2021

3.1. 01/12/2021 (Goodwillie calculus).

- (Rêve éveillé) Goodwillie calculus starts off with the idea that categories can be fruitfully thought of as "manifolds", among which stable categories correspond to linear manifolds, i.e. vector spaces - with the category Ch_R of chain complexes of R-modules over a ring being the canonical example thereof.

  Following this analogy, one should be able to take the "tangent category" to a category $\mathcal{C}$ at a given object X, encoding "infinitesimal directions" in $\mathcal{C}$ away from X. Taking $\mathscr{S}$ to be the category of spaces and pt to be the one point space, it turns out that

  $$T_{\mathrm{pt}}\mathscr{S} = \mathrm{Ch}_{\mathbb{Z}}$$

  is the category of chain complexes of abelian groups. Furthermore, given any space X, the terminal map $X \to \mathrm{pt}$ can be thought of as a "path" in $\mathscr{S}$, whose "derivative at time zero" outputs as "tangent vector" precisely the chain complex $H_*(X)$ encoding the singular homology groups of X!

3.2. **01/19/2021 (classical Langlands, arithmetic topology).**

- Weil's Rosetta stone hints at a similarity between the features of three classes of mathematical objects: number fields (alt. the space Spec $\mathcal{O}_F$ associated to the ring of integers of that number field), function fields (alt. smooth projective curves over a finite field), and Riemann surfaces (alt. their field of meromorphic functions).

  When looking at things from the étale viewpoint, one is led to realize that $\mathbb{F}_p$ feels more like a circle, in that it has étale fundamental group

$$\pi_1^{\text{ét}}(\text{Spec}\mathbb{F}_p) = \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \hat{\mathbb{Z}}$$

  (generated by the Frobenius element). As such, function fields should really correspond to *Riemann surface bundles over* S^1, which are a special class of 3-manifolds - the idea being that the Frobenius endomorphism should provide gluing data with respect to the usual clutching construction for bundles over S^1. Similar thinking can lead to 3-manifolds associated to number fields.

  This marks the start of arithmetic topology, with ties to the classical Langlands program. For instance, DBZ suggests that an instructive way to think of the passage in the Langlands conjecture from a number field F to the associated space V of "automorphic forms" is by considering the 3-manifold M associated to F, then obtaining the vector space V by feeding M into some 4 dimensional TQFT. Many features of automorphic forms (such as conductors and singularities/ramification behavior along them) can thereby be encoded fruitfully in the language of physics - in the sense of TQFTs.

3.3. **01/25/2021 (HRR via DAG).**

- Start life with a smooth projective variety X defined over a field of characteristic zero k. One may look at the K-theory spectrum $K(X)$ associated to X, globalizing the algebraic K-theory of a ring and encompassing the Grothendieck group $K_0(X)$ of algebraic vector bundles on X, or alternatively at $K(\mathcal{C})$ where $\mathcal{C} = \text{QCoh}(X)$ (where the category $\mathcal{C}$ is compactly generated by perfect complexes, i.e. bounded complexes with locally free cohomology sheaves) on the one hand, and the Hochschild homology complex $HH(X)$ on the other hand, which globalizes the construction $A \otimes^{\mathbb{L}}_{A \otimes A} A$ (a sort of derived self intersection). There is then a trace map relating the two:

$$\text{tr} : K(\mathcal{C}^c) \to HH_*(\mathcal{C}),$$

  where we interpret the k-vector space on the RHS as a spectrum, and where $\mathcal{C}^c$ denotes the subcategory of compact objects in $\mathcal{C}$, so that $K(\mathcal{C}^c) = K(X)$ by the above discussion.

  Next, because our variety is smooth, the HKR theorem states that

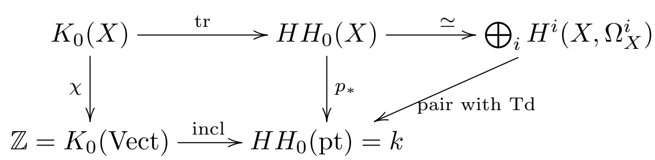
$$HH_*(X) \simeq \bigoplus_i \underline{\Omega^i_X}[i],$$

  which in degree zero specializes to the statement

$$HH_0(X) \simeq \bigoplus_i H^i(X, \Omega^i_X).$$

It turns out that the composite of the trace map followed by the HKR isomorphism recovers precisely the Chern character map on vector bundles (interpreting Chern classes as Dolbeault cohomology classes of equal bidegree).

Now, the trace map is functorial with respect to functors between compactly generated categories which preserve compact objects. The particular instance of such a functor which we care about is the pushforward functor associated to the structure map $X \xrightarrow{p} \mathrm{pt} = \mathrm{Spec}\, k$, which preserves compact objects because X is proper; we therefore end up with a commutative square, which we can piece up with the HKR isomorphism to get the following diagram (focusing on degree 0):



Where the diagonal map is the top degree trace map on cohomology applied to the pairing of a given element of $\bigoplus_i H^i(X, \Omega^i_X)$ with the Todd class of X, viewed as an element of $\bigoplus_i H^{n-i}(X, \Omega^{n-i}_X)$. Commutativity of this diagram together with an explicit interpretation of the inverse of the HKR isomorphism therefore recovers the algebro-geometric Hirzebruch-Riemann-Roch theorem: given an algebraic vector bundle $\mathcal{E}$ over X, one has the relationship:

$$\chi(\mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E})\mathrm{td}(X).$$

3.4. **01/29/2021 (infty categories and chromatic homotopy theory).**

- To say that an object obtained via a UP in classical category theory is unique up to unique isomorphism is to say that the category of possible choices forms a contractible groupoid, meaning a category such that there exists a unique isomorphism between any two objects. This is the discrete analogue of a contractible ∞-groupoid, which is the contractible choice of spaces one obtains when formulating UPs in the ∞-categorical settings.
- (Based on lecture 1 of [7]) To a complex orientable cohomology theory E with a choice of generator $t \in E^2(\mathrm{pt})$, so that $E^*(\mathbb{C}P^\infty) = E^*(\mathrm{pt})[[t]]$, one may associate a formal group law $f^E \in E^*(\mathrm{pt})[[u, v]] = E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ encoding the product rule satisfied by the corresponding theory of Chern classes on line bundles:

$$c_1^E(\mathcal{L} \otimes \mathcal{L}') = f^E(c_1(\mathcal{L}), c_1^E(\mathcal{L}')).$$

This power series in two variables is subject to conditions reflecting the monoidal structure of Pic (X). This structure in turn defines a group operation on the formal affine line $\widehat{\mathbb{A}^1_{E^*(\mathrm{pt})}}$, which turns out to be independent of the choice of t up to "coordinate change". It turns out that the assignment $R \mapsto \mathrm{FGL}(R) \subseteq R[[x, y]]$ is corepresentable by a ring L called the Lazard ring: formal group laws on R are in 1-1 correspondence with ring maps $L \to R$, or scheme-theoretically with affine scheme morphisms Spec $R \to$ Spec L.

As such, we obtain an assignment:

$$\{\text{cplx oriented cohlgy thies}\} \to \{\text{formal groups}\}$$

going from some kind of "spaces" (spectra) to some kind of "algebra". Chromatic homotopy teaches us that there is much to be learned from this assignment, which may be more precisely formulated as mapping into (algebro-geometric objects over) the moduli stack of formal groups $\mathcal{M}_{FG} = (\text{Spec } L)/G$, where G takes into account the aforementioned coordinate changes. For instance, the Landweber exact functor theorem states that any flat morphism Spec $R \to$ Spec L (i.e. any affine scheme flat over Spec L) is guaranteed to correspond to a formal group law coming from some complex oriented cohomology theory - thereby providing us with a rich source of possibly new cohomology theories. For instance, complex K-theory can be obtained as the cohomology theory associated to the formal group law $f(u, v) = u + v + uv \in \mathbb{Z}[[u, v]]$. It turns out that not-necessarily-complex orientable can also be made to fit into the picture as different types of "algebraic geometry over (Spec $L)/G$, namely as quasi-coherent sheaves over it. Whence the following:

Slogan: The structure of the stable homotopy category is controlled by the geometry of the stack (Spec $L)/G$.

3.5. 01/31/2021 (formal group laws).

- Lazard's theorem states that the Lazard ring is a polynomial ring in countably many variables, $L \cong \mathbb{Z}[b_1, b_2, ...]$, so that is geometrically looks like an "infinite dimensional affine space" Spec $L \cong \mathbb{A}^\infty_\mathbb{Z}$. By the UP of L, this isomorphism can be obtained by exhibiting an appropriate formal group law F on the ring $\mathbb{Z}[b_1, b_2, ...]$, corresponding to a ring map $L \to \mathbb{Z}[b_1, b_2, ...]$. The actual proof involves working one degree at a time with the module of indecomposables $QL := I/I^2$, where $I = L_{>0}$, and showing that $QL_{2n} \cong \mathbb{Z}$ for every n. This leads to a nice reinterpretation of the problem in terms of computing certain Ext groups of comodules - more on this later.

4. February 2021

4.1. 02/01/2021 (étale cohomology).

- Here is one way in which étale cohomology is a "better" algebraic invariant for algebraic varieties than sheaf cohomology with respect to the Zariski topology: for X an irreducible algebraic variety, one can directly show using that Zariski opens in X are connected that $H^k(X, \underline{\Lambda}) = 0$ for any $k \geq 1$ and any constant sheaf $\underline{\Lambda}$ (essentially because Λ itself is then flasque). On the other hand, for a smooth variety X over $\mathbb{C}$, one can show that we have isomorphisms for every $k \geq 1$ and every finite abelian group Λ:

$$H^k_{\text{ét}}(X, \underline{\Lambda}) \cong H^k_{\text{sing}}(X, \Lambda).$$

This follows from the fact that étale covers can be refined by complex analytic coverings, in the sense that for any étale cover $(U, u) \xrightarrow{\pi} (X, x)$

one can find a complex analytic neighborhood $x \in V \subset X$ together with a factorization of the corresponding inclusion map:

$$\begin{array}{ccc} & (U,u) & \\ \nearrow & & \searrow^{\pi} \\ (V,x) & \xrightarrow[\text{incl}]{} & (X,x). \end{array}$$

This essentially corresponds to the idea that covering spaces are trivializable in the analytic topology (so that we may map V into a "single sheet" of the covering U over X).

4.2. **02/03/2021 ((co)bar resolution).**

- Start life with a commutative k-algebra R and an R-module M. There is a canonical way to construct a projective resolution of M, called the bar construction, and defined as follows:

$$\mathrm{Bar}_\bullet(M) = (... \to R^{\otimes 3} \otimes M \to R^{\otimes 2} \otimes M \to R \otimes M)$$

(all tensor products are implicitly taken over k). The differential is given in eg degree 2 by

$$d[r_1|r_2|r_3|m] := [r_1r_2|r_3|m] - [r_1|r_2r_3|m] + [r_1|r_2|r_3m].$$

There is an dual construction, called the cobar construction, exhibiting a "universal" resolution of a comodule over a coalgebra by projective comodules. The latter can be used to resolve a crucial lemma (the symmetric 2-cocycle lemma) along the way to proving Lazard's theorem (see note 01/31/2021).

Notice the analogy with the bar-type construction one encounters when trying to form the groupoid associated to a non-free group action of G on X: this involves taking a directed colimit over the following sequence of spaces (with simplicial structure maps mimicking the differential described above):

$$(... \substack{\to\\ \to\\ \to\\ \to} G \times G \times G \times X \substack{\to\\ \to\\ \to} G \times G \times X \rightrightarrows G \times X).$$

For instance, the rightmost top map is given by $[g_1|g_2|x] \mapsto [g_1g_2|x]$, while the rightmost bottom map is given by $[g_1|g_2|x] \mapsto [g_1|g_2x]$. Note that for free actions, the last two terms suffice to define the quotient space X/G.

4.3. **02/08/2021 (underlying interests).**

- (flow of consciousness) Parametrized geometry, deformation theory, homotopy theory: studying mathematical objects in families; studying the structure underlying the dynamics of a deformation problem; studying the essential shape underlying the various realizations/manifestations of an object - the homotopy type/quasi-isomorphism class/motivic homotopy type. Studying the theories that govern such phenomena, in a way that maintains the original motivation visible.

4.4. **02/10/2021 (homotopical algebra).**

- Recall that one may define the cotangent complex associated to a commutative k-algebra A by first choosing a simplicial resolution by smooth algebras $A_\bullet \xrightarrow{\approx} A$ and then defining $\mathbb{L}_A$ to be the left derived functor of the algebraic differential forms functor $(A \mapsto \Omega^1_{A/k})$ applied to $A_\bullet$ - explicitly, letting Tot denote the "Dold-Kan complex" associated to a simplicial ring, we set

$$\mathbb{L}_A := \mathrm{Tot}([n] \mapsto \Omega^1_{A_n}).$$

(This is an application of the blueprint of abstract homotopy theory, which for a general model category dictates that we apply the functor at hand to a cofibrant replacement.)

Let us begin to make precise the slogan that "the cotangent complex of A encodes the deformation theory of A". Let's say that $\tilde{A}$ is an infinitesimal deformation of A if we have a cartesian square of the form:

$$\begin{array}{ccc} k[\varepsilon]/(\varepsilon^2) & \xrightarrow{\varepsilon \mapsto 0} & k \\ \downarrow & & \downarrow \\ \tilde{A} & \longrightarrow & A \end{array}$$

(essentially because the cotangent complex classifies square-zero extensions by its defining UP), corresponding to the algebro-geometric situation

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ k & \longrightarrow & \mathrm{Spec}\ k[\varepsilon]/(\varepsilon^2). \end{array}$$

It turns out that we have a 1-1 correspondence:

$$\big(\text{infinitesimal deformations of } A\big) \cong \mathrm{Ext}^1_A(\mathbb{L}_A, A),$$

with the higher Ext groups encoding whether it is possible to extend a given deformation to a larger Artinian ring of the form $k[\varepsilon]/(\varepsilon^n)$.

This story admits a global analogue following Grothendieck and Illusie's formulation of a global cotangent complex (essentially by resolving the structure sheaf by a sheaf of simplicial algebras $\mathcal{A}_* \to \mathcal{O}_X$ which locally resolves $\mathcal{O}_X$ by polynomial algebras), with a corresponding statement relating the Ext groups $\mathrm{Ext}^*_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X)$ to deformations of X, in a way which recovers the usual Kodaira-Spencer theorem as a special case - stating that for X smooth complex projective, deformations of X are encoded by $H^1(X, T_X)$. In this case, $\mathbb{L}_X$ is simply Ω^1_X concentrated in degree 0, with Ω^1_X a vector bundle (locally free sheaf) of rank equal to the dimension of X, and the reduction to the Kodaira-Spencer cohomology group is essentially given as follows:

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{L}_X, \mathcal{O}_X) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, (\Omega^1_X)^\vee) \cong H^1(X, T_X),$$

where we have used the general fact that for F, G vector bundles over X, we have an identification:

$$\mathrm{Ext}^i_{\mathcal{O}_X}(F,G) \simeq Ext^i_{\mathcal{O}_X}(\mathcal{O}_X, F^* \otimes G).$$

(Induced from the usual hom-tensor adjunction and $F^\vee \otimes G \cong \hom(F,G)$ for $i = 0$.)

For more on this and a host of other exciting ideas, see Toën's DAG survey [18].

4.5. **02/16/2021 (Grothendieck fibration and algebraic de Rham complex).**

- Start with a category $\mathcal{C}$. Grothendieck fibrations are a certain class of functors $\mathcal{E} \to \mathcal{C}$ which are classified by functors $\mathcal{C} \xrightarrow{F} \mathrm{Cat}$, in the sense that each fiber $\mathrm{Fib}(c), c \in \mathcal{C}$ coincides with the designated category $Fc \in \mathrm{Cat}$. Pictorially:

$$\begin{array}{ccc} \mathcal{E} & & \\ \downarrow & & \\ \mathcal{C} & \xrightarrow{F} & \mathrm{Cat} \end{array}$$

(Compare with the universality of maps $X \to BG$ in classifying principal G-bundles over X.)
In ∞-category land, Lurie's formalism allows us to establish an equivalence of categories

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \simeq \mathrm{coCart}/\mathcal{C}$$

between functors $\mathcal{C} \to \mathrm{Cat}_\infty$ and so-called co-cartesian fibrations over $\mathcal{C}$.
- Algebraic differential forms are actually quite friendly. Starting from the observation that polynomials can be formally differentiated over any field, one constructs (for a smooth affine variety $V = \mathrm{Spec}\ k[x_1, ..., x_n]/(f_1, ..., f_m)$) the $\mathcal{O}(V)$-module of algebraic 1-forms as follows:

$$\Omega^1_{V/k} := \mathcal{O}(V) < dx_1, ..., dx_n > /(df_1, ..., df_m),$$

where we follow the classical settings of differential topology and set:

$$df := \sum_{j=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

It turns out that the rank of $\Omega^1_{V/k}$ as an $\mathcal{O}(V)$ module will always equal the dimension of V in the smooth settings. Higher degree algebraic differential forms can be constructed in the naive way, by setting $\Omega^k_V := \Lambda^k \Omega^1_V$. Proceeding naively further, we obtain an algebraic de Rham complex $(\Omega^\bullet_V, d)$ and corresponding de Rham cohomology groups:

$$H^i_{\mathrm{dR}}(V) := H^i(\Omega^\bullet_V, d).$$

In the non-affine case, one constructs a sheaf which locally looks like the above - and one may most concisely obtain Ω^1_X as the pullback $\Delta^*\mathcal{I}/\mathcal{I}^2$ under the diagonal morphism $X \xrightarrow{\Delta} X \times X$. Everything still goes through, with the exception that we need to take the hypercohomology of the resulting de Rham complex of sheaves (really, just pass to an injective resolution in the appropriate derived category):

$$H^i_{\mathrm{dR}}(X) := \mathbb{H}^i(\Omega^\bullet_X).$$

One of the amazing results one can establish is that we have a period isomorphism recovering singular cohomology groups for smooth projective varieties over $\mathbb{C}$:

$$H^i_{\mathrm{dR}}(X) \cong H^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{C}).$$

(And this even though the LHS involved purely algebraic data, while the RHS involves the analytic structure of X: a GAGA-type surprise.)
Another lovely result is that one may recover de Rham cohomology as defined above for not-necessarily affine smooth varieties as the abutement of the so-called Hodge-to-de-Rham spectral sequence whose E_1 page consists of Dolbeault cohomology groups:

$$E_1^{pq} := H^p(X, \Omega^q_X) \implies H^{p+q}_{\mathrm{dR}}(X).$$

An excellent treatment of this story is given in [5].

4.6. **02/18/2021 (formal moduli problems, HKR theorems).**

- (Variation on 12/15/20) The philosophy of formal moduli problems dictates that we should always be able to find some dg Lie algebra encoding the infinitesimal behavior of a given FMP. Begin with a scheme X, a point $x \in X$, and consider the formal completion $X^\wedge_x$ as a formal moduli problem. Then the corresponding dg Lie algebra turns out to be the shifted tangent complex $T_x[-1]$ associated to the group (in an appropriate homotopical sense) $\Omega_x X$. The Lie algebra structure on $T_x[-1]$ may be obtained by thinking about Atiyah classes, a formalization of the attempt to import an analogue of Chern-Weil theory to the land of AG.

  The underlying philosophy is the following: every space in AG can be fruitfully thought of as $X = BG$ for some "group" G, in such a way that every sheaf on X can be interpreted as carrying an action of G. From this perspective, Grothendieck-Riemann-Roch really provides a character formula for G-representations in this setting, in a way which interpolates between classical characters associated to representations of finite groups (think: $V \to BG$) and Chern characters associated to vector bundles (think: $E \to X$). More to come...

4.7. **02/22/2021 (Grothendieck duality).**

- Start life with a reasonable morphism of schemes $X \xrightarrow{f} Y$ (i.e. of finite type between Noetherian schemes). Then at the level of the respective derived categories of complexes of quasi-coherent cohomology sheaves, aside from the usual adjunction $\mathbb{L}f^* \dashv \mathbb{R}f_*$, Grothendieck teaches us that we also obtain an extraordinary pullback functor fitting in an adjunction $\mathbb{R}f_* \dashv f^\times$. That is, we have the data of a natural isomorphism:

$$\hom(\mathbb{R}f_* A, B) \cong \hom(A, f^\times B)$$

  for any $A \in \mathcal{D}_{q.c.}(X), B \in \mathcal{D}_{q.c.}(Y)$.

Let's specialize to the case where $X \xrightarrow{f} \mathrm{Spec}\, k$ is a smooth and proper scheme over a field k. Then one has that $f^{\times}\mathcal{O}_{\mathrm{Spec}\, k} = \omega_X = \Omega^n_X[n]$ is given by volume forms on X, concentrated in degree $n = \dim X$. The above adjunction applied to a quasi-coherent sheaf $\mathcal{F}$ over X and to $\mathcal{O}_{\mathrm{Spec}\, k}$ translates to:

$$\mathrm{hom}(\mathbb{R}f_*\mathcal{F}, \mathcal{O}_{\mathrm{Spec}\, k}) \cong \mathrm{hom}(\mathcal{F}, \Omega^n_X[n]), \text{ a.k.a.}$$

$$(\mathbb{R}f_*\mathcal{F})^\vee \cong \mathrm{hom}(\mathcal{F}, \Omega^n_X[n]).$$

Taking $(i-n)^{th}$ cohomology, we recover the familiar statement of Serre duality (remembering that shifting complexes results in shifts in cohomology degree):

$$H^{n-i}(X, \mathcal{F})^\vee \cong \mathrm{Ext}^i(\mathcal{F}, \Omega^n_X).$$

In the more general case where X is no longer assumed to be smooth, but only Cohen-Macaulay, we still recover the dualizing sheaf ω_X concentrated in degree n as $f^{\times}\mathcal{O}_{\mathrm{Spec}\, k}$, and therefore obtain the more general form of Serre duality:

$$H^{n-i}(X, \mathcal{F})^\vee \cong \mathrm{Ext}^i(\mathcal{F}, \omega_X).$$

As such, Grothendieck duality can be seen as a "maximal generalization" of Serre duality in relative form and in not necessarily smooth or proper context - in the general absolute case, $f^{\times}\mathcal{O}_{\mathrm{Spec}\, k}$ outputs the dualizing complex ω_X, which may no longer be concentrated in a single degree. For an excellent treatment of this story, see [11].

4.8. **02/24/2021 (∞-category of spectra).**

- (Ressenti) Elements of the ∞-category of spectra, analogously to derived R-modules, admit a "minimal" characterization in terms of their homotopy groups and k-invariants, where the latter encode the data of the maps of spectra necessary to "reconstruct" a given spectrum "one homotopy group at a time", by means of iterated pushout diagrams of the form:

$$\begin{array}{ccc} X_{\leq(n+1)} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X_{\leq n} & \xrightarrow{k_n} & \Sigma^{n+2}\pi_{n+1}(X). \end{array}$$

  The trick is, while these k_n maps are relatively straightforward to construct in the case of chain complexes, maps of spectra can be much more subtle (e.g. something as elementary as $\pi_*\mathrm{hom}_{\mathrm{Sp}}(\mathbb{F}_2, \mathbb{F}_2)$ already outputs the Steenrod algebra $\mathcal{A}$), and so while this construction is formally analogous to the situation in $\mathcal{D}_R$, it may be of lesser use in actually pinpointing the weak equivalence class of a given spectrum.

4.9. **02/24/2021 (Serre duality, proper morphisms).**

- In its most elementary form, Serre duality posits the existence of an isomorphism, for any coherent sheaf $\mathcal{F}$ over a (smooth?) projective scheme X over a field k,

$$\mathrm{hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee,$$

where ω_X denotes the dualizing sheaf of X. This isomorphism is obtained from the data of a non-degenerate pairing given by the composite:

$$\hom(\mathcal{F}, \omega_X) \otimes H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \xrightarrow{\text{tr}} k$$

where the first map is the "Yoneda pairing" applying the map given by the first term to the given cohomology class of the second term, while the second map is a trace map, to be thought of as the AG analogue of integrating a top degree form over a compact manifold (with properness here playing the role of compactness).

For dualizable $\mathcal{F}$ (e.g. $\mathcal{F}$ an algebraic vector bundle over X), we may shuffle things around to formulate the above pairing in the form:

$$H^0(X, \mathcal{F}^\vee \otimes \omega_X) \otimes H^n(X, \mathcal{F}) \to k,$$

whose sibling pairings for different cohomological degrees recover the "usual" form of Serre duality.

- A brisk walk through useful facts on proper morphisms: in terms of "categorical invariants", proper morphisms are useful because proper pushforward is guaranteed to preserve coherent sheaves. Compare this to the non-proper morphism $\mathbb{A}^1_k \to \text{Spec } k$, under whose pushforward the structure sheaf is sent to the infinite dimensional vector space $k[t]$ over k. In terms of scheme adjectives, one can show that a map is proper and affine iff it is a finite map, meaning that affine locally it consists of finite ring maps - thus proper morphisms between affine schemes are rather uninteresting. On the other hand, we get lots of interesting examples of proper morphisms by first realizing that projective space is proper over its field of definition, and that as a result projective schemes are guaranteed to be proper.

  One quickly recovers a fundamental theorem in valuation theory in this framework: given a morphism $\mathbb{P}^n \xrightarrow{f} \mathbb{P}^m$ between projective spaces, cut out by homogeneous polynomials $f_1, ..., f_m$ in terms of a choice of coordinates on the domain, we are guaranteed that the map is proper because the domain is proper and the target is separated, and as such it has closed image. Thus the image of f may be described as the vanishing locus of homogeneous polynomials $g_1, ..., g_s$ with coordinates in the target - as such, we can systematically "eliminate the parameters" involved in defining f.

4.10. 02/26/2021 (∞-category of chain complexes).

- Here's a "geodesic" way to form the ∞-category $\mathcal{D}_R$ of derived R-modules, starting from the category Ch_R of chain complexes of R-modules: view Ch_R as enriched over chain complexes of abelian groups, by equipping each $\underline{\hom}_{\text{Ch}_R}(M, N)$ with a chain complex structure, essentially with R-linear maps $M \to \Sigma^n N$ living in degree n. Now, apply the truncation functor $\tau_{\geq 0}$ hom-wise, and follow this up with a hom-wise Dold-Kan construction, resulting in an enrichment of Ch_R over simplicial abelian groups, hence simplicial sets after applying a forgetful functor. This category may now be ∞-localized at weak equivalences to obtain the derived ∞-category of R-modules.

  Notice that we seem to have "killed off" half of the information we started with by applying a truncation functor. It turns out that this process is not really destructive, a realization that crucially relies on the fact that $\mathcal{D}_R$ is a stable ∞-category, and that as such the suspension/shift functor is an autoequivalence, allowing us to "lift back up" negative information from the dead - compare with the notion that spectra are "infinite loop-spaces", in that a given spectrum X may be presented as $X = \Omega^n \Sigma^n X$ for any $n \geq 0$. (Story to be made more precise at a later time.)

More generally, for any pointed ∞-category $\mathcal{C}$, one may form a category $\mathrm{Sp}(\mathcal{C})$ of "spectrum objects" of $\mathcal{C}$ consisting of infinite loop objects in an analogous sense to the above (with the classical case corresponding to $\mathcal{C} = \mathcal{S}$ and $\mathrm{Sp}(\mathcal{S}) = \mathrm{Sp}$). The resulting category $\mathrm{Sp}(\mathcal{C})$ is a stable ∞-category (the "stabilization of $\mathcal{C}$"), and generalized Brown representability theorem characterizes cohomology theories on $\mathcal{C}$ as being precisely classified by elements of $\mathrm{Sp}(\mathcal{C})$. As such, the stabilization procedure feels like a way to pass from a "category of spaces" to a category of "coefficients" with respect to which to measure these spaces. This is part of section 1.4 in [6].

4.11. **02/27/2021 (derived Satake).**

- Let's try to reinterpret the bounded derived category of $G(\mathcal{O})$-equivariant sheaves on the affine Grassmanian $\mathcal{G}\mathrm{r}_G$ in the special case where $G = \mathbb{G}_m$, so that $\mathcal{G}\mathrm{r}_{\mathbb{G}_m} = \mathbb{C}((t))/\mathbb{C}[[t]] \cong \mathbb{Z}$. As sheaves on an ind-scheme, they can only be supported on a finite subset of points, hence

$$\mathrm{Sh}_{\mathbb{G}_m}(\mathcal{G}\mathrm{r}_{\mathbb{G}_m}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Sh}(\mathrm{pt}/\mathbb{C}^*).$$

Now, a theorem of Bernstein-Lunts gives us an equivalence component-wise:

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Sh}(\mathrm{pt}/\mathbb{C}^*) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Mod}^{\mathrm{fg}}_{C_*(\mathbb{C}^*)},$$

where $C_*(\mathbb{C}^*) = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ denotes the cdga of cochains on $\mathbb{C}^*$. In DAG, the latter category at each i may be reinterpreted as the category of coherent sheaves on the affine derived scheme $\mathrm{Spec}\ \mathrm{Sym}(\mathbb{C}[1])$. Remembering that $\mathbb{G}_m^\vee = \mathbb{G}_m$ so that $\mathfrak{g} = \mathfrak{g}^\vee = \mathbb{C}$, we may further identify this category with $\mathrm{Coh}(\mathfrak{g}^\vee[-1])$ under the AG viewpoint on a vector space V as $\mathrm{Spec}\ \mathrm{Sym}(V^*)$. Thus

$$\mathrm{Sh}_{\mathbb{G}_m}(\mathcal{G}\mathrm{r}_{\mathbb{G}_m}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Coh}(\mathfrak{g}^\vee[-1]) \simeq \mathrm{Coh}_{\mathbb{G}_m}(\mathfrak{g}^\vee[-1]),$$

under the usual correspondence between $\mathbb{Z}$-grading and $\mathbb{G}_m$-representation structure. Finally, the HKR theorem enables us to identify, for a general algebraic group G:

$$\mathrm{pt} \times_G \mathrm{pt} = \Omega_{\mathrm{pt}} G = \mathcal{L}G \times_G \mathrm{pt} \simeq \mathbb{T}_G[-1] \times_G \mathrm{pt} = \mathfrak{g}[-1].$$

Therefore, we conclude with an identification:

$$\mathrm{Sh}_{\mathbb{G}_m}(\mathcal{G}\mathrm{r}_{\mathbb{G}_m}) \simeq \mathrm{Coh}_{\mathbb{G}_m}(\mathrm{pt} \times_{\mathbb{G}_m} \mathrm{pt}).$$

It turns out that this toy example leads to a sensible derived enhancement of the classical geometric Satake equivalence, in the sense that by work of Bezrukavnikov-Finkelberg, we have for any reductive group G an equivalence of derived categories compatible with the geometric Satake equivalence in the following sense:

$$\begin{array}{ccc} \mathrm{Sh}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G) & \xrightarrow{\simeq} & \mathrm{Coh}_{G^\vee}(\mathrm{pt}\times_{G^\vee}\mathrm{pt}) \\ \uparrow & & \uparrow \pi_* \\ \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G) & \xrightarrow[\simeq]{} & \mathrm{Rep}(G^\vee)\simeq \mathrm{Coh}_{G^\vee}(\mathrm{pt}). \end{array}$$

Taking a stacky perspective on the RHS as

$$(\mathrm{pt}\times_{G^\vee}\mathrm{pt})/G^\vee \simeq \mathrm{Loc}_{G^\vee}(\mathbb{D}\sqcup_{\mathbb{D}^*}\mathbb{D}),$$

i.e. as the moduli stack of local systems on the "ravioli" obtained from gluing two formal disks $\mathbb{D} = \mathrm{Spec}\ k[[t]]$ away from their closed point, and likewise reinterpreting the space on the LHS as the moduli stack

$$G(\mathcal{O})\backslash G(K)/G(\mathcal{O}) = \mathrm{Bun}_G(\mathbb{D}\sqcup_{\mathbb{D}^*}\mathbb{D})$$

of principal G-bundles on the ravioli, we see that this equivalence outputs a sort of local geometric Langlands correspondence, in the form of an equivalence of derived categories:

$$\mathrm{Sh}(\mathrm{Bun}_G(\mathbb{D}\sqcup_{\mathbb{D}^*}\mathbb{D})\simeq \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\mathbb{D}\sqcup_{\mathbb{D}^*}\mathbb{D})).$$

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4.12. **02/28/2021 (étale fundamental group).**

- Start life with a smooth projective variety X over $\mathbb{C}$. Then the Riemann existence theorem leads to the data of an isomorphism:

$$\pi_1^{\text{ét}}(X)\cong \pi_1^{\widehat{\text{top}}(X(\mathbb{C}))}$$

between the étale fundamental group of X and the profinite completion of the topological fundamental group of its analytification. Together with a base change theorem, this tells us that any smooth projective curve of genus g over an algebraically closed field of characteristic zero admits $2g$ topological generators (and something along similar lines can be said in characteristic p can be said using a specialization argument).

This story, together with the existence of a SES for any variety X defined over a subfield $k\subset\mathbb{C}$:

$$1\longrightarrow \pi_1^{\text{ét}}(X_{\mathbb{C}})\longrightarrow \pi_1^{\text{ét}}(X_k)\longrightarrow \mathrm{Gal}(\bar{k}/k)\longrightarrow 1,$$

produces interesting arithmetic data of topological origin in the case where $X=\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\}$ is the projective line over $\mathbb{Q}$ with three points removed: in this case, the SES together with the base change theorem posit the existence of a SES:

$$1\longrightarrow (<\gamma_0,\gamma_1,\gamma_\infty\mid \gamma_0\gamma_1\gamma_\infty=1>)^\wedge\longrightarrow \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\})\longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\longrightarrow 1.$$

Let H denote the profinite group on the LHS. Then the semi-direct product structure on $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\})$ corresponding to this group extension is determined by the data of a map:

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to\mathrm{Aut}(H),$$

which upon linearization of the RHS produces a linear representation of the absolute Galois group of $\mathbb{Q}$.

5. March 2021

5.1. **03/01/2021 (de Rham stack and $\mathcal{D}$-modules).**

- Given a scheme X, we may associate to it its de Rham stack X_{dR}, given as a functor of points by the formula
$$X_{\mathrm{dR}} := X(R/N(R))$$
for any $R \in \mathrm{CAlg}$, where $N(R)$ denotes the nilpotent radical of R. In spirit, this is constructing X_{dR} out of X by collapsing or canonically identifying all infinitesimal neighborhoods in X (where infinitesimal is to be read in the AG sense of nilpotency data). We attempt to begin making sense of the analogy:
$$X \ : \ \mathcal{O}_X \ :: \ X_{\mathrm{dR}} \ : \ \Omega_X,$$
where Ω_X denotes the dg algebra of differential forms on X. The idea is to view X_{dR} as admitting as "underlying dg manifold" the shifted tangent bundle $T_X[1]$, so that its structure sheaf (for smooth projective X?) should be given by
$$\mathcal{O}_{X_{\mathrm{dR}}} = \mathrm{Sym}(T_X[1]^\vee) = \mathrm{Sym}(\Omega^1_X[-1]) = \Omega_X.$$
This leads to a nice categorification tower linking "linearizations of X" at various categorical levels to "linearizations of X with added flatness conditions" (with flatness here to be understood in the sense of flat connections):

Categorical level	Linearization of X	Flat linearization of X	Linearization of X_{dR}
1	$\mathcal{O}_X$	Ω_X	$\mathcal{O}_{X_{\mathrm{dR}}}$
2	$\mathrm{QCoh}(X)$	$\mathcal{D}_X$-mod	$\mathrm{QCoh}(X_{\mathrm{dR}})$
3	$\mathrm{ShCat}(X)$	$\mathrm{CrysCat}(X)$	$\mathrm{ShCat}(X_{\mathrm{dR}})$

Here, $\mathrm{ShCat}(X)$ denotes the 2-category of quasi-coherent sheaves of categories, which for reasonable X is equivalent to $\mathcal{O}_X$-mod-mod; resp. $\mathrm{CrysCat}(X)$ is the "2-category of crystals of categories" (whatever this might mean independently of the identification with $\mathrm{ShCat}(X_{\mathrm{dR}})$). In particular, one may think of $\mathcal{D}_X$-mod as "the sheaf theory one obtains from $\mathrm{QCoh}(X)$ by insisting that the sheaves be equipped with a flat connection", and such a requirement is precisely encoded by the notion of a quasi-coherent sheaf on the de Rham stack X_{dR}; furthermore, this behavior persists at different categorical levels. This story is told in some more details in Safronov's shifted geometric quantization paper [13]

5.2. **03/11/2021 (Hochschild homology).**

- Let A be an algebra over a field k. According to [16], Hochschild homology should be thought of not as an invariant of A as a k-algebra, but rather as an invariant of Mod_A as a k-linear dg-category - in those settings, one may recover $\mathrm{HH}_*(A)$ by taking the categorical trace of Mod_A viewed as a dualizable object in dgcat_k with dual $\mathrm{Mod}_{A^{\mathrm{op}}}$. In this monoidal category, the unit object is given by Vect_k, endomorphisms of which are determined by the image of the line k concentrated in degree zero. Furthermore, one may show that the evaluation and coevaluation maps associated to Mod_A compose to the unit endomorphism sending k to $A \otimes^{\mathbb{L}}_{A\otimes A^{op}} A$. This is great, because this complex precisely returns the Hochschild homology groups

$\mathrm{HH}_*(A)$ upon passing to homology! As such, in analogy with the notion of trace for dualizable elements of Vect_k (namely finite-dimensional vector spaces), we have realized one precise sense in which Hochschild homology may be thought of as a categorified notion of dimension.

5.3. **03/12/2021 (Chern classes).**

- There are at least five different approaches to defining Chern classes in algebraic geometry, fitting into the paradigm that one is after an appropriate collections of maps (for a given smooth projective variety X over $\mathbb{C}$)

$$c_i : \mathrm{K}_0(X) \to \mathrm{H}^*(X)$$

from the Grothendieck group of algebraic vector bundles on X to some choice of algebro-geometric cohomology theory valued at X, or alternatively to the de Rham or singular cohomology of $X(\mathbb{C})$ viewed as a complex Kähler manifold.

Start by fixing a rank n complex vector bundle E on $X(\mathbb{C})$. In section 3.2 of [15], I describe the algebro-topological construction of the Chern classes of E viewed as integral classes in singular cohomology, which consists of first defining them for the universal bundle on $BU(n)$, then pulling them back via the classifying map corresponding to E in $[X, BU(n)]$.

Next, Chern-Weil theory provides a second approach to defining Chern classes as elements of the de Rham cohomology of X, as described in entry 2.3: one first equips E with some C^∞ connection : $\mathcal{A}^0(E) \to \mathcal{A}^1(E)$

6. April 2021

6.1. **04/02/2021 (factorization homology).**

- (First steps) Roughly speaking, factorization homology should be thought of as a device enabling us to "integrate" any $\mathbb{E}_n$-algebra $A \in \mathrm{Alg}^B_{\mathbb{E}_n}(\mathcal{V})$ over a given n-manifold M, viewed as an element of the sym. mon. ∞-category $\mathcal{M}\mathrm{fld}^B_n$, where B denotes a choice of "structure on the tangent bundle" (for instance, $B = \mathrm{pt}$ gives framed manifolds). The output is an element

$$\int_M A \in \mathcal{V}.$$

One may define factorization homology concisely by means of a left Kan extension. Namely, start with the full subcategory of disks $\mathcal{D}\mathrm{isk}^B_n$, so that A may be interpreted as a functor $A : \mathcal{D}\mathrm{isk}^B_n \to \mathcal{V}$, and simply define $\int_M A$ to be the image of M under the diagonal functor

$$\begin{array}{ccc} \mathcal{D}\mathrm{isk}^B_n & \xrightarrow{A} & \mathcal{V} \\ \downarrow & \nearrow_{\int_{(-)} A} & \\ \mathcal{M}\mathrm{fld}^B_n & & \end{array}$$

which is the left Kan extension of A along the vertical inclusion, defined explicitly pointwise (as for all left Kan extensions) as the colimit of the following composite of functors:

$$\int_M A := \operatorname{colim}\left((\mathcal{D}\mathrm{isk}_n^B)_{/M} \xrightarrow{\text{fgt}} \mathcal{D}\mathrm{isk}_n^B \xrightarrow{A} \mathcal{V} \right).$$

This machine is wonderful. For the simple case $M = S^1$, $n = 1$, and $\mathcal{V} = \mathrm{Mod}_k$, so that $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_k) = \mathrm{Alg}_k$ is a classical k-algebra, one retrieves the Hochschild homology of A as a chain complex of k-modules:

$$\int_{S^1} A = HH_*(A).$$

One may take $\int_{S^1} A$ to be the definition of A for more general A. A beautiful result further down the road states that for any perfect stack X, we have an equivalence

$$\int_M \mathrm{QCoh}(X) \simeq \mathcal{O}(\mathrm{Map}(S^1, X)),$$

where S^1 denotes the simplicial circle viewed as a constant derived stack. This statement therefore retrieves the DAG perspective on Hochschild homology as functions on the derived loop space $\mathcal{L}X = \mathrm{Map}(S^1, X)$ of X.

6.2. **04/02/2021 (complete Segal spaces).**

- A (small) category may be encoded by the data of a simplicial set

  $$X : \Delta^{\mathrm{op}} \to \mathscr{S}\mathrm{et}$$

  satisfying the axiom that for every $n, m \geq 0$, the diagram

  $$\begin{array}{ccc} X_{n+m} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & X_0 \end{array}$$

  induced by the following maps in the simplex category

  $$\begin{array}{ccc} [m+n] & \xleftarrow{k \mapsto m+k} & [n] \\ {\scriptstyle k \mapsto k} \uparrow & & \uparrow {\scriptstyle 0 \mapsto 0} \\ [m] & \xleftarrow[0 \mapsto m]{} & [0] \end{array}$$

  is a pullback square. This is encoding the fact that 1-categories admit unique composition of composable sequences of morphisms. In these settings, the data beyond $n + m = 3$ (associativity) is redundant. Notice that the hom-spaces of X can be extracted by taking the pullback of the following diagram:

  $$\begin{array}{ccc} \hom_X(x, y) & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow {\scriptstyle (x,y)} \\ X_1 & \longrightarrow & X_0 \times X_0 \end{array}$$

  In the context of simplicial **spaces**, i.e. functors of the form

  $$X : \Delta^{\mathrm{op}} \to \mathscr{S}$$

(where we may think of $\mathscr{S}$ as referring to $s\mathscr{S}$et or Top up to localization), one may state the Segal axiom as requiring that the above diagrams be **homotopy pullback squares**. A model for ∞-categories is then obtained by restricting one's attention to complete Segal spaces, namely to simplicial spaces satisfying the Segal axiom together with an additional condition requiring that the (appropriately defined) map

$$\mathrm{path}_X(x,y) \to \hom_X(x,y)^{\mathrm{inv}}$$

is an equivalence for every $x, y \in X_0$ - this effectively formalizes the notion that "invertible morphisms should correspond to paths between points" in an ∞-category.

6.3. **04/06/2021 (Chern classes).**

- Let X be a smooth projective algebraic variety over $\mathbb{C}$, which may alternatively be viewed as a compact Kähler manifold X^{an} by passing to its analytification. Then there are several ways to define a theory of Chern classes on X, beginning with an axiomatic characterization. Here is a diagram illustrating some of the possible approaches and how they relate to one another:

$$\begin{array}{ccccccc}
 & & & \nearrow & H^{2i}_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Z}) & \longrightarrow & H^{2i}_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}) \\
 & & & & \downarrow & & \downarrow \simeq \\
K^0(X) & \xrightarrow{c_i} & A^i(X) & \longrightarrow & H^{i,i}(X)_{\mathbb{Z}} & \longrightarrow & H^{2i}_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C}) \\
 & & & & \updownarrow & & \downarrow \simeq \\
 & & & \searrow & H^{i,i}(X) & \longrightarrow & \bigoplus_{p+q=2i} H^{p,q}(X)
\end{array} .$$

6.4. **04/07/2021 (genus, lifting criteria in AG).**

- Let X be a smooth projective curve. Then Serre duality posits that there is an isomorphism:

$$H^0(X, \Omega^1_X) \simeq H^1(X, \mathcal{O}_X)^\vee.$$

This statement indicates that in these settings, arithmetic genus (which is defined as the rank of $H^1(X, \mathcal{O}_X)$) and geometric genus (defined as the dimension of $H^0(X, \Omega^1_X)$) coincide. When $X = V(f)$ is given by a homogeneous polynomial of degree d, we further have that this genus is given by the formula

$$g(X) = \frac{(d-1)(d-2)}{2}.$$

- Let's formalize the notion of "path lifting" in the context of algebraic geometry. Here, paths should be interpreted as nilpotent thickenings, which in the context of A-algebras for some commutative ring A can be expressed as ring maps $B \to B/I$ where $I \subset B$ is an ideal consisting of nilpotent elements - to be thought of geometrically as the closed inclusion Spec $B/I \hookrightarrow$ Spec B of Spec B/I into an infinitesimal thickening of it.

Now, let X be an A-scheme, thought of as a functor $X : \mathrm{CAlg}_A \to \mathcal{S}\mathrm{et}$. Then, given an infinitesimal thickening $B \to B/I$ in CAlg_A, one may ask for the corresponding map

$$X(B) \to X(B/I)$$

to be:

- surjective, which is equivalent to requiring X to be a **smooth** A-scheme;
- injective, which is equivalent to requiring the map $X \to \mathrm{Spec}\ A$ to be **unramified**;
- bijective, which is equivalent to requiring the map $X \to \mathrm{Spec}\ A$ to be **étale**.

Thus, étale A schemes are precisely those which admit unique path lifting in the sense of algebraic geometry, pictorially corresponding to the existence of a unique diagonal extension:

$$\begin{array}{ccc} \mathrm{Spec}\ B/I & \longrightarrow & X \\ {\scriptstyle \mathrm{cl}}\downarrow & \nearrow & \\ \mathrm{Spec}\ B & & \end{array}$$

This is the so-called nilpotent lifting condition.

Here is an interesting example from arithmetic geometry: suppose we start with a scheme X smooth over $\mathrm{Spec}\ \mathbb{Z}$, thought of as an equation to be solved over the integers. The existence of a solution mod p corresponds to the existence of an $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ point $\mathrm{Spec}\ \mathbb{F}_p \to X$. Since X is smooth, we are guaranteed that the solution mod p can be lifted to a solution mod p^2, using the nilpotent lifting condition on the situation

$$\begin{array}{ccc} \mathrm{Spec}\ \mathbb{Z}/p\mathbb{Z} & \longrightarrow & X \\ {\scriptstyle \mathrm{cl}}\downarrow & \nearrow & \\ \mathrm{Spec}\ \mathbb{Z}/p^2\mathbb{Z} & & \end{array}$$

Iterating for increasing powers of p and passing to the limit, we actually obtain the existence of a $\mathbb{Z}_p$-point $\mathrm{Spec}\ \mathbb{Z}_p \to X$. Thinking of $\mathrm{Spec}\ \mathbb{Z}_p$ as a p-adic formal disk, this is telling us that for smooth schemes, pointwise solutions can be lifted to solutions on "formal neighborhoods". One may then ask whether this "local-to-formal" extension can be "integrated out" into a "local-to-global" extension, i.e. into an integer solution to the original equation.

6.5. 04/11/2021 (categorical group actions).

- Given a group G, let BG denote the "delooping" category of G, i.e. the category with one object (denoted by pt) and with morphism set $\mathrm{end}_{BG}(\mathrm{pt}) = G$. Then one may express the notion of an "object equipped with a G-action" internally to any category $\mathcal{C}$ as the data of a functor

$$BG \xrightarrow{X} \mathcal{C},$$

which is equivalent to the choice of an object $X \in \mathcal{C}$ together with a collection of endomorphisms $g\cdot \in \mathrm{end}_{\mathcal{C}}(X)$ compatible with the group structure on G.

In particular, one should be able to express the notion of "group action on a category" as the data of a functor

$$BG \xrightarrow{\mathcal{C}} \mathrm{Cat}.$$

The adequate notion may require a loosening of the notion of "equality" between the composite of endofunctors $(g\cdot) \circ (h\cdot)$ and $gh\cdot$, for instance as the data of corresponding natural transformations.

6.6. **04/16/2021 (Milnor numbers, vanishing cycles, matrix factorization).**

- Start with a Henselian DVR $S = \mathrm{Spec}\ R$ with perfect residue field k and fraction field K. (The running example will be Spec $\mathbb{Z}_p$, a Henselian DVR of mixed characteristic. Think of an algebro-geometric formal neighborhood of a point: this is one of the "smallest" settings in which to study families of algebraic varieties, and we are particularly interested in "smooth degeneration to a singular fiber" over the closed point). Suppose we are given a flat and proper morphism $X \to S$, with X a regular scheme, such that the generic fiber X_K is smooth over K. Then Bloch's conductor conjecture states that we have an equality:

$$\chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) = [\Delta_X, \Delta_X]_S + \mathrm{Sw}(X_{\bar{K}}),$$

where $\mathrm{Sw}(X_{e\bar{t}a})$ is the Swan conductor, which carries arithmetic data about $X_{\bar{K}}$; the Euler characteristics are taken with respect to ℓ-adic cohomology; and $[\Delta_X, \Delta_X]_S$ denotes a localized self-intersection number.

The above is to be thought of as a deformation theoretic/possibly singular generalization of the "ℓ-adic Lefschetz fixed point formula", which states that for any smooth projective variety X over an algebraically closed field, we have that

$$\chi(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) = [\Delta_X, \Delta_X],$$

where $h^{p,q}(X)$ denotes the rank of the Dolbeault cohomology group $H^q(X, \Omega_X^p)$.

In their paper on the Bloch conductor formula[20], Toën-Vezzosi suggest an approach via non-commutative geometry to a special class of cases (those with "unipotent monodromy") in which the Swan conductor vanishes. The argument proceeds by building a dg category of matrix factorizations associated to a choice of uniformizer π for R, whose "Euler characteristic" recovers the difference $\chi(X_{\bar{k}}) - \chi(X_{\bar{K}})$. A general formalism of traces in the nc-context then leads to the desired formula.

One source of motivation for considering categories of matrix factorization as a reasonable approach to this problem comes from the notion of Milnor number in differential topology. Given a function $f : X \to \mathbb{C}$ with an isolated critical point at 0, the Milnor number is a geometric invariant of the singular fiber $f^{-1}(0)$. Near a point $x \in f^{-1}(0)$, it may be computed as the Euler characteristic of the cohomology ring $H^*(B_\delta(x) \cap f^{-1}(\epsilon))$ for small $\delta, \epsilon > 0$. The latter cohomology groups admit an interpretation in terms of nearby cycles of the constant sheaf $\Psi_f \underline{\mathbb{C}}$, and the Euler characteristic may be alternatively obtained as the dimension of the Jacobian ring of f, given by the quotient of the stalk of X at x by the ideal generated by the partial derivatives of f at that point, or (appealingly) as the ring of functions on the derived critical locus $\mathrm{DCrit}(f) = \Gamma(df) \cap \{0\} \subseteq T^*X$.

Now, start life with a complex manifold X together with a holomorphic function (the "potential") $W : X \to \mathbb{C}$. The Landau-Ginzburg model in quantum field theory corresponds to a 2d TQFT LG_B (via the "B-model") which assigns to a point the dg category of matrix factorizations $\mathrm{MF}(W)$, whose elements are given by complexes $(P^\bullet, d)$ satisfying the relation $d^2 = W\cdot$. To start building some intuition for this category, notice that these chain complexes are acyclic whenever $W' \neq 0$, so that when interpreted as "living over X" they are supported along the critical locus $\mathrm{Crit}(W) \subseteq X$. As an extended TQFT, LG_B takes the following values:

Categorical level	Input	Output
0	T^2	Milnor number of (X, W)
1	S^1	(Jacobian ring of W) $= \Phi_W \underline{\mathbb{C}}$
2	pt	$\mathrm{MF}(W)$

Thus, $MF(W)$ appears as a "twice categorified" avatar of the Milnor number, which is built to capture invariants of singular fibers appearing as degenerations of smooth fibers. Toën and Vezzosi's main achievement was to transport the intuition from the transcendental settings (over $\mathbb{C}$) to the mixed characteristic settings, developing the necessary framework at the level of nc-schemes.

6.7. **04/21/2021 (topos theory).**

- A meta-theorem in topos theory states that "any mathematical structure which can be described by "geometric" axioms in a suitable language admits a classifying topos" - for a more precise statement, see Chapter VIII of [8]. For instance, there exists a classifying topos for ring objects in topoi. Following the usual representability pattern, this consists of a topos $\mathcal{R}$ together with a (universal) ring object $R \in \mathcal{R}$ such that, for any topos $\mathcal{E}$, geometric morphisms $\mathcal{E} \xrightarrow{f} \mathcal{R}$ are in 1-1 correspondence with ring objects in $\mathcal{E}$ via pullback $f^*R \in \mathcal{E}$ of the universal ring object:

$$\hom_{\mathrm{Topoi}}(\mathcal{E}, \mathcal{R}) \cong \{\text{ring objects } \mathcal{O}_\mathcal{E} \in \mathcal{E}\}.$$

This specific example is useful in formulating the analogue of "ringed spaces" in these settings: a **ringed topos** is simply defined to be a topos $\mathcal{X}$ equipped with a ring object $\mathcal{O}_\mathcal{X} \in \mathcal{X}$ (thought of as the structure sheaf of $\mathcal{X}$), or equivalently a topos $\mathcal{X}$ together with the data of a geometric morphism $\mathcal{X} \to \mathcal{R}$. When $\mathcal{X} = \mathrm{Sh}(X)$ is the category of sheaves associated to a Grothendieck site, $\mathcal{O}_\mathcal{X}$ being a ring object translates to the corresponding sheaf being a sheaf of rings on X, as expected. This is part of the general yoga that $\mathcal{A}$-objects in a category of (pre)sheaves of sets correspond to (pre)sheaves factoring through $\mathcal{A}$, where $\mathcal{A}$ is an appropriate category such as abelian groups or rings.

6.8. **04/22/2021 (TQFT, $\mathcal{B}$-models).**

- Suppose we are handed an $n \geq 2$ dimensional TQFT Z, and let us evaluate it on a codimension 2 sphere to obtain a category $Z(S^{n-2})$. Then $Z(S^{n-2})$ can always be equipped with a monoidal structure via the "little disks in larger disks" picture, and because of the choice of co/dimension it turns out that the resulting pair $(Z(S^{n-2}), *)$ is always *symmetric monoidal*. As such, we may construct an algebro-geometric space out of it via the (categorified) Spec construction:

$$\mathcal{M}_Z^1 := \text{Spec } (Z(S^{n-2}), *)$$

This space allows us to formally associate a new TQFT in any dimension m to Z, called the $\mathcal{B}$-model associated to Z and denoted by $\mathcal{B}_Z^m$, according to the blueprint

$$\mathcal{B}_Z^m(M^{m-k}) := \text{kQCoh}(\text{Map}(M, \mathcal{M}_Z^1)).$$

Here, M is viewed as a homotopy type and interpreted as a constant derived stack, placing it in the same world as $\mathcal{M}_Z^1$; Map implicitly refers to locally constant maps; the terminology "kQCoh" refers to the k-category of sheaves of $(k-1)$-categories, a notion currently being formalized by Stefanich [17] and possibly others. In low codimensions, we therefore obtain the following outputs:

Categorical level	Input	Output
0	M^k	$\mathcal{O}(\text{Map}(M, \mathcal{M}_Z^1))$
1	N^{k-1}	$\text{QCoh}(\text{Map}(N, \mathcal{M}_Z^1))$
2	P^{n-2}	$\text{2QCoh}(\text{Map}(P, \mathcal{M}_Z^1))$

Of particular interest is the case $m = n+1$. In this case, it turns out that the original TQFT Z may be viewed as a **boundary condition** for the $(n+1)$d TQFT $\mathcal{B}_Z^{n+1}$ - this implies in particular that for any codimension 2 manifold N^{n-1}, the category $Z(N^{n-2})$ sheafifies/spectrally decomposes over the space $\text{Map}(N, \mathcal{M}_Z^1)$.

6.9. **04/23/2021 (categorified HKR, Koszul duality).**

- Start life with a perfect derived scheme X over a field of characteristic zero. Then the derived HKR theorem states that there is an equivalence of derived stacks:

$$\mathcal{L}X \simeq \mathbb{T}X[-1],$$

where the LHS denotes the derived free loop space of X, and the RHS denotes the total space of the tangent complex of X, shifted by -1 - a space which may alternatively presented as the relative Spec:

$$\text{Spec }_X(\text{Sym}^\bullet_{\mathcal{O}_X}(\mathbb{T}_X^*[1])),$$

where $\mathbb{T}_X^*[1]$ now denotes the cotangent complex shifted by 1, viewed as a sheaf over X.

In particular, for X a smooth scheme, we have that $\mathbb{T}_X^* = \Omega^1_{X/k}$ is the sheaf of Kähler differentials, so that passing to functions yields the classical version of the HKR isomorphism:

$$HH_*(X) = \Omega^*_{X/k}$$

(using on the LHS that functions on the derived free loop space coincide with Hochschild homology).

Now, it turns out that this story admits a categorification (in the sense of a passage from functions to sheaves). Namely, it turns out that there exists an equivalence of ∞-categories:

$$\mathrm{IndCoh}(\mathcal{L}X) \simeq \mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathbb{T}_X[-2])\text{-mod}.$$

This equivalence may be obtained as a special case of a more general result which states that given any finite rank vector bundle E over a smooth Noetherian scheme X, there exists an equivalence of ∞-categories:

$$\mathrm{IndCoh}(E[-1]) \simeq \mathrm{Sym}^\bullet_{\mathcal{O}_X}(E[-2])\text{-mod}.$$

One retrieves categorified HKR at $E = \mathbb{T}_X$.

This result is to be thought of as a parametrized version/an incarnation in families of the classical form of Koszul duality, which states that given a finite dimensional vector space V over k, if we set $S_V := \mathrm{Sym}^\bullet_k(V[-2])$ and $\Lambda_V := \mathrm{Sym}^\bullet_k(V^*[1])$, then there exists an equivalence of ∞-categories:

$$\mathrm{IndCoh}(\Lambda_V) \simeq S_V\text{-mod}.$$

This story, and much more, is beautifully explored in Rustam Antia's thesis [12]

6.10. **04/29/2021 (∞-categorical Yoneda embedding, homology theories).**

- Let $\mathcal{C}$ be an ∞-category, and denote by $\hat{\mathcal{C}}$ the ∞-category of presheaves on $\mathcal{C}$ taking values in $\mathscr{S} = \infty\text{-}\mathcal{G}\mathrm{pd}$. Then there is an ∞-categorical version of the Yoneda embedding, which posits the existence of a fully-faithful functor

$$h : \mathcal{C} \to \hat{\mathcal{C}}$$

exhibiting $\hat{\mathcal{C}}$ as the completion of $\mathcal{C}$ with respect to all (small) colimits. This means that, for any test ∞-category $\mathcal{D}$, if we denote by $\underline{\mathrm{hom}}_!(\hat{\mathcal{C}}, \mathcal{D})$ the full ∞-subcategory of $\underline{\mathrm{hom}}(\hat{\mathcal{C}}, \mathcal{D})$ consisting of those functors which preserve all (small) colimits, then restricting along the Yoneda embedding gives an equivalence

$$h^* : \underline{\mathrm{hom}}_!(\hat{\mathcal{C}}, \mathcal{D}) \xrightarrow{\simeq} \underline{\mathrm{hom}}(\mathcal{C}, \mathcal{D}).$$

This result leads as a special case to the Eilenberg-Steenrod characterization of (ordinary) homology theories as being determined by their value at a point. Namely, we can take $\mathcal{D} = *$ to be the one point ∞-category, with $\hat{*} \simeq \mathscr{S}$, and let $\mathcal{D} = \mathcal{D}_{\mathbb{Z}}$ denote the derived ∞-category of (chain complexes of) abelian groups. Now, one way to describe a homology theory is as (the homotopy categorical truncation of) a colimit preserving functor

$$H : \mathscr{S} \to \mathcal{D}_{\mathbb{Z}}.$$

from the ∞-category of spaces to $\mathcal{D}_{\mathbb{Z}}$. Then, by the UP of the Yoneda embedding, it follows that any such functor is determined up to equivalence by its value at a point, $H(*) \in \mathcal{D}_{\mathbb{Z}}$.

7. May 2021

7.1. **05/02/2021 (Stable ∞-categories, AG).**

- An ∞-category $\mathcal{C}$ is called **stable** if it has a zero object, admits finite co/limits, and if any commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W & \longrightarrow & Z \end{array}$$

is Cartesian iff it is coCartesian (i.e. it is a pullback square iff its is a pushout square).

A key example is the stable infinity category of derived A-modules for some commutative ring A, obtained as the Dwyer-Kan localization $\mathcal{D}(A) := L(\mathrm{Ch}_A, \mathrm{q.iso})$. This notion expresses all of the properties of the 1-truncation $\tau_1\mathcal{D}(A)$ as a triangulated category "internally" to the theory, in that all of the extra structures involved in a triangulated category appears as properties in the ∞-categorical settings. For instance, the distinguished triangles are given precisely by the biCartesian squares, for any given morphism $X \xrightarrow{f} Y$ in $\mathcal{D}(A)$:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y/X, \end{array}$$

with the shift structure map given by further taking the suspension of X on the RHS and invoking the UP of pushouts:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y/X & \longrightarrow & \Sigma X, \end{array}$$

Furthermore, the octahedral axiom is elucidated through the fact that given a composite of morphisms $X \to Y \to Z$ in a stable ∞-category, the central square in the following diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y/X & \longrightarrow & Z/X & \longrightarrow & \Sigma X \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & Z/Y & \longrightarrow & \Sigma Y \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & \longrightarrow & \Sigma Z
\end{array}$$

is biCartesian.

- Start life with a commutative ring R (the story may also be told for R a k-algebra, replacing $\mathbb{Z}$ by k throughout). Let's make precise the idea that "any element $f \in R$ provides a function on Spec R". By the UP of polynomial rings, such an element $f \in R$ is equivalent to the data of a ring homomorphism $\mathbb{Z}[x] \to R$ sending x to f. Passing to affine schemes, this is the same data as a map

$$\text{Spec } R \xrightarrow{f} \mathbb{A}^1.$$

Evaluation of f at a point of Spec R may also be interpreted geometrically: any point $\mathfrak{p} \in \text{Spec } R$ may be identified with the image of the map $\text{Spec } \kappa(\mathfrak{p}) \to \text{Spec } R$, where $\kappa(\mathfrak{p}) = \text{Frac}(R/\mathfrak{p})$ is the residue field of R at $\mathfrak{p}$, and the map is opposite to the composite $R \twoheadrightarrow R/\mathfrak{p} \to \kappa(\mathfrak{p})$. Now, we may consider the diagram

$$\begin{array}{ccc}
\text{Spec } \kappa(\mathfrak{p}) & \longrightarrow & \text{Spec } R \\
 & \searrow & \downarrow{\scriptstyle f} \\
 & & \mathbb{A}^1,
\end{array}$$

which, upon passing to commutative rings, is equivalent data by the same UP as above to an element $f(\mathfrak{p}) \in \kappa(\mathfrak{p})$: this is precisely the evaluation of f at the point $\mathfrak{p}$, taking values in the appropriate residue field.

7.2. **05/11/2021 (Gelfand and Koszul).**

- The slogan that "spaces may be studied via their ring of functions" may be said to trace back to Gelfand's representation theorem from the early 1940's, stating the following: let X be a compact Hausdorff space, and write $C^0(X)$ to denote the ring of continuous $\mathbb{C}$-valued functions on X. If $\text{mSpec}(C^0(X))$ denotes the maximal spectrum of X (viewed as a topological space by giving it the coarsest topology with respect to which the map $\varphi_f : \text{mSpec}(C^0(X)) \to \mathbb{C}$, $\mathfrak{m} \mapsto \bar{f} \in C^0(X)/\mathfrak{m} \cong \mathbb{C}$ is continuous for every $f \in C^0(X)$), then we have a homeomorphism:

$$X \xrightarrow{\cong} \text{mSpec}(C^0(X))$$

$$x \mapsto \mathfrak{m}_x := \ker(\text{ev}_x).$$

This assignment produces a fully faithful embedding

$$\mathrm{CpctHaus} \to \mathrm{BanachAlg}$$

of the category of compact Hausdorff spaces into the category of Banach $\mathbb{C}$-algebras (by equipping each $C^0(X)$ with the sup norm), whose essential image consists of commutative Banach algebras which admit an involution satisfying the axioms of a C^*-algebra. This suggests that it may be reasonable to enlarge our notion of "space" by considering arbitrary, not necessarily commutative Banach algebra, and exploring what kind of structures/theorems may be established in these broader settings. This turns out to be a surprisingly fruitful philosophy referred to as "non-commutative geometry".

Suppose we set out to look for an analogue of Gelfand's representation theorem in the algebro-geometric settings, starting life with some k-scheme X. From the get-go, the fact that e.g. $\mathcal{O}(\mathbb{P}^n_k) \cong k$ indicates that trying to study a variety via its ring of global functions will not do. This, together with cohomological considerations, indicates that the appropriate linearization procedure comes from considering the (bounded) derived category of perfect complexes on X, denoted by $\mathcal{D}(X)$ in [19] - a k-linear dg category which may be built out of derived categories of bounded complexes of f.g. projective modules (denoted $\mathcal{D}(A)$ for some commutative k-algebra A) via a (homotopy) limit operation taken in $\mathrm{Cat}_k^{\mathrm{dg}}$ with respect to an affine open covering of X.

Now for the miracle: it can be proven that, as long as X is quasi-compact and quasi-separated (read: AG-analogue of compact and Hausdorff), the dg category $\mathcal{D}(X)$ admits a single compact generator $E_X \in \mathcal{D}(X)$, i.e. a compact object in $\mathcal{D}(X)$ with the property that the internal hom functor

$$\hom(E, -) : \mathcal{D}(X) \to \mathcal{D}(k)$$

is conservative. This is a situation in which Koszul duality may intervene. Namely, let us consider the dg k-algebra $B_X := \mathrm{end}_{\mathcal{D}(X)}(E_X)$. Then the fact that E_X is a compact generator guarantees that we obtain an equivalence of dg categories:

$$\mathcal{D}(X) \simeq \mathcal{D}(B_X).$$

Thus, we may think of the dg algebra B_X as an analogue of of $C^0(X)$ in the AG context, leading to the following analogy:

$$\text{Cpct Hausdorff } X : \mathbb{C}\text{-algebra } C^0(X) \ :: \ \text{qcqs } X : \text{dg } k\text{-algebra } B_X.$$

One should however keep in mind the caveat that the scheme X may not be actually be recoverable from the data of the dg category $\mathcal{D}(B_X)$ - namely, Morita equivalence is not an "injective" invariant of schemes. Nonetheless, this perspective suggests that we enlarge the realm of algebraic geometry by setting out to study general k-linear dg categories as "non-commutative schemes" in their own right. This is the approach that Toën and Vezzosi take towards a partial proof of the Bloch conductor conjecture, mentioned in subsection 6.6.

7.3. **05/13/2021 (deformation theory over a DVR).**

- The situation considered by the Bloch conductor conjecture, as described in subsection 6.6, appears naturally in various algebro-geometric contexts. For instance, given a lft scheme $X \to \mathrm{Spec}\ \mathbb{Z}$ which is smooth over the generic fiber $\mathrm{Spec}\ \mathbb{Q}$, we discussed in subsection 2.14 that X could only possibly have bad reduction (i.e. fail to be smooth) at finitely many primes p. Choose such a prime. “Zooming in” near p, by focusing on the formal disk $\mathrm{Spec}\ \mathbb{Z}_p$ around $\mathrm{Spec}\ \mathbb{F}_p \hookrightarrow \mathrm{Spec}\ \mathbb{Z}$, we arrive to precisely a situation where we have a Henselian DVR $A = \mathbb{Z}_p$ of mixed characteristic over which lives a family $\mathcal{X} \to \mathrm{Spec}\ \mathbb{Z}_p$ with smooth fiber X_t over the generic point $\mathrm{Spec}\ \mathbb{Q}_p$ degenerating to a singular fiber X_0 over the closed point $\mathrm{Spec}\ \mathbb{F}_p$. Pictorially, by further base changing to the geometric fibers, we arrive to the following diagram:

$$\begin{array}{ccccc} X_t & \longrightarrow & \mathcal{X} & \longleftarrow & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\ \bar{\mathbb{Q}}_p & \longrightarrow & \mathrm{Spec}\ \mathbb{Z}_p & \longleftarrow & \mathrm{Spec}\ \bar{\mathbb{F}}_p \end{array}$$

7.4. **05/17/2021 (Balmer spectrum and Gelfand’s categorification).**

- Let us state a more precise version of the algebro-geometric analogue of the Gelfand representation theorem hinted at in subsection 7.2, issued from Balmer’s survey on tensor triangulated (tt) categories in [9]. The spirit is the following: it is possible to assign a so-called Balmer spectrum $\mathrm{Spc}\ \mathcal{T}$ to any tensor triangulated category $\mathcal{T}$ (i.e. $\mathcal{T}$ is a triangulated category admitting a symmetric monoidal structure which is exact in each variable), and in particular to the derived category $\mathcal{D}^{\mathrm{perf}}(X)$ of perfect complexes associated to any scheme X: prime ideals in the classical settings are replaced by prime thick-tensor (tt) ideals, and the basis for the closed subsets of the Zariski topology is given by the subsets

$$\mathrm{supp}(X) := \{\mathcal{P} \in \mathrm{Spc}\ \mathcal{T} \mid X \notin \mathcal{P}\},$$

for X ranging over $\mathcal{T}$ (for instance, classical points feature as skyscraper sheaves $i_*\mathcal{O}_{\mathrm{Spec}\ \kappa(x)}$).

We then have the following theorem of Thomason, which provides a concrete incarnation of the philosophy that categories of sheaves, as categorification of rings of functions, provide a more sensitive linearization procedure for algebro-geometric objects:

Theorem 7.1. *Let X be a qcqs scheme. Then there is a homeomorphism:*

$$|X| \xrightarrow{\cong} Spc\ (\mathcal{D}^{perf}(X))$$

given by sending $x \in X$ to the prime tt-ideal

$$\mathcal{P}(x) = \{Y \in \mathcal{D}^{perf}(X) \mid Y_x \cong 0\}$$

given as the kernel of the residue functor $\mathcal{D}^{perf}(X) \to \mathcal{D}^b(\kappa(x))$.

(We should note however that this result only recovers X as a topological space, and in particular it doesn’t account for the structure sheaf.)

7.5. **05/21/2021 (Euler and Riemann).**

- Let X be a smooth projective curve defined over an algebraically closed field k, and consider the category $\mathrm{Coh}(X)$ of coherent sheaves on X. This category comes with two distinguished objects, the structure sheaf $\mathcal{O}_X$ and the dualizing sheaf ω_X, and we may more generally consider the tensor powers $\omega_X^{\otimes n}$ for $n \in \mathbb{Z}$.

  A canonical numerical invariant which can be associated to elements $\mathcal{F} \in \mathrm{Coh}(X)$ is their Euler characteristic, defined as:

$$\chi(\mathcal{F}) := \sum (-1)^i h^i(X, \mathcal{F}).$$

  Using the Riemann-Roch theorem, we may compute that

$$\chi(\omega_X^{\otimes n}) = \chi(\mathcal{O}_X) + \deg(\omega_X^{\otimes n}) = (1-g) + n(2g-2).$$

  In particular, the function

$$\chi(\omega_X^{\otimes -}) : \mathbb{Z} \to \mathbb{Z}$$

  is a linear polynomial in n, with coefficients depending only on the genus of X. We may view it as the restriction to $\mathbb{Z} \subseteq \mathbb{C}$ of the function of one complex variable

$$f : \mathbb{C} \to \mathbb{C}$$

$$s \mapsto (2g-2)s + 1 - g.$$

  Now, observe that Serre duality gives rise to the following identity, for each $n \in \mathbb{Z}$:

$$\begin{aligned} \chi(\omega_X^{\otimes n}) &= h^0(\omega_X^{\otimes n}) - h^1(\omega_X^{\otimes n}) \\ &= h^1(\omega_X^{\otimes -n+1}) - h^0(\omega_X^{\otimes -n+1}) \\ &= \chi(\omega_X^{\otimes -n+1}). \end{aligned}$$

  This extends to a functional equation for f:

$$f(s) = -f(-s+1).$$

  Notice also that f has a unique zero at $s = 1/2$, and that the genus of g may be recovered as the special value $f(1) = g - 1$.

  Abstractly, working in the derived context and letting $\pi : X \to \mathrm{Spec}\ k$ denote the structure morphism, the dualizing sheaf of X could have been obtained via exceptional pullback of the structure sheaf of the ground field, $\omega_X = \pi^! k$, and the Euler characteristic could have been defined as the following composite:

$$\chi : \mathrm{Coh}(X) \xrightarrow{\pi_*} \mathrm{Coh}(\mathrm{Spec}\ k) \simeq \mathrm{Vect}_k \to K_0(\mathrm{Vect}_k) \cong \mathbb{Z}.$$

  Working with l-adic cohomology instead of coherent cohomology and following the latter framework, this story can be carried out with respect to a smooth algebraic variety X defined over $\bar{\mathbb{F}}_q$ so as to recover the Riemann zeta function ζ_X associated to X, the central object of interest for the Weil conjectures.

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