Introduction to Varieties and Schemes - or Grothendieck's Paradise

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"Plus encore que ce qu'on appelle les "théorèmes-clef" en mathématique, ce sont les points de vue féconds qui sont, dans notre art, les plus puissants outils de découverte - ou plutôt, ce ne sont pas des outils, mais ce sont les yeux même du chercheur qui, passionnément, veut connaître la nature des choses mathématiques.
Ainsi, le point de vue fécond n'est autre que cet "oeil" qui à la fois nous fait découvrir, et nous fait reconnaître l'unité dans la multiplicité de ce qui est découvert. Et cette unité est véritablement la vie même et le souffle qui relie et anime ces choses multiples."

"Even more so than what we call "key theorems" in mathematics, it is the fertile viewpoints which, in our art, constitute the most powerful tools of discovery - or rather, they are not tools, but they are the very eyes of the researcher who passionately strives to understand the nature of mathematical things.
Thus, the fertile viewpoint provides us with an "eye" which at once leads us towards discovery, and helps up recognize the unity within the multiplicity of what is discovered. And such unity is truly the very life and breath which connects and animates these discoveries."

- Alexander Grothendieck, Récoltes et Semailles.

1 Introduction

To first approximation, algebraic geometry may be described as having developed out of generations of efforts towards the following:

Goal. Study spaces of solutions to systems of polynomial equations with coefficients in a ground ring k via geometric methods.[1]

Endeavors in this direction trace back at least to Descartes, Fermat, and Euler, working over the field of real numbers $k = \mathbb{R}$ and studying vanishing loci of polynomials in two or three variables. The development of complex geometry through work of Abel and Riemann starting in the 19th century initiated a rich interplay between analysis, geometry, and algebra out of which evolved the field of complex algebraic geometry - see for instance [**Wells**]. On the other extreme are Diophantine equations, polynomial equations with coefficients in the ring of integers $k = \mathbb{Z}$ or in the field of rational numbers $k = \mathbb{Q}$, whose study goes back to antiquity, although they weren't fully inserted into a geometric framework until the 20th century. For a remarkably thorough historical overview of the above developments and everything in between, we invite the reader to consult Dieudonné's survey [**Dieudonne**].

These notes grew out of a two-week long summer mini-course whose intention was to provide an "intuitive prelude" to a first course in algebraic geometry[2] - a somewhat conversational overview of the basic language of varieties and schemes, with an emphasis on motivating the concepts at hand and illustrating some key principles through simple examples. Ideally, we hope that these notes will provide an accessible entry-point into this rich subject for young mathematicians who may feel intimidated by the technical background required by standard references, or who may want to whet their appetites with a broad overview before launching into a study of the theory proper.

In 1926, Hilbert declared regarding the set theoretic foundations laid out by Georg Cantor:

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[1] Unless otherwise specified, k will always denote a commutative and unital ground ring.

[2] The specific first course which the authors had in mind was David Ben-Zvi's introduction to scheme theory, a run of which has been documented by Arun Debray, see [**Ben-Zvi**].

"No one shall be able to expel us from the paradise that Cantor created for us."

It is our hope that the coming pages will provide a glimpse of the paradise that Grothendieck, in turn, created for us.

1.1 Initial framework

Taking the above goal as our starting point, let us begin in earnest with a system of polynomial equations with coefficients in some ground ring k:

$$S = \{f_j\}_{j \in J} \subseteq k[(t_i)_{i \in I}].$$

Here, we take our variables $(t_i)_{i \in I}$ to be indexed by some set I, which we will most often be taken to be finite, e.g. $I = \{1, ..., n\}$.

By "space of solutions" to this system, we most naively mean the following set:

$$V(S) := \{\underline{a} \in k^I \mid f_j(\underline{a}) = 0 \ \forall j \in J\},$$

which is customarily called an *affine algebraic set.* Notice that any such vanishing locus comes with a canonical embedding into k^I, which (for finite $|I| = n$) we often denote by $\mathbb{A}^n_k$ and refer to as *affine n-space over k.*

As we let S vary for fixed k and I, we therefore obtain an assignment:

$$V(-) : \big\{\text{Subsets of } k[(t_i)_{i \in I}]\big\} \to \big\{\text{Subsets of } k^I\big\}$$

Exercise 1.1.1. Verify that this assignment is insensitive to passing to ideals, i.e. that $V(S) = V(I)$ whenever $I = (S)$ denotes the ideal of $k[(t_i)_{i \in I}]$ generated by a given subset S.

When dealing with a finite indexing set $I = \{1, ..., n\}$ and a Noetherian ground ring k[3], the above exercise coupled with the Hilbert basis theorem ensures that we lose no generality by restricting ourselves to studying the spaces of solutions associated to *finite* collections of polynomials $f_1, ..., f_r \in k[t_1, ..., t_n]$.

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[3] Both of these assumptions are mild finiteness conditions which will be assumed in most of what follows.

Now, the key - and elementary - insight is that, given a polynomial $f \in k[(t_i)_{i\in I}]$, the sentence

$$f(\underline{x}) = 0$$

"parses", i.e. makes sense, when $\underline{x} = (x_i)_{i\in I}$ is an element of L^I for any field extension L/k, and in fact also when $\underline{x} \in A^I$ for any commutative k-**algebra** $A \in \mathrm{CAlg}_k$: all that is needed is a notion of addition, multiplication, and scaling by elements of k[4]. Hence, a single system of polynomial equations $S \subseteq k[(t_i)_{i\in I}]$ gives rises to an infinite family of "solution spaces" $V_S(A) \subseteq A^I$, one for each k-algebra A - among which the original affine algebraic set is featured as $V(S) = V_S(k)$.

Exercise 1.1.2. Check that any k-algebra homomorphism $A \xrightarrow{\varphi} B$ induces a map between solution sets $V_S(A) \to V_S(B)$ by component-wise "substitution".

Thus, the data one obtains from looking at vanishing loci of a given system of polynomial equations $S \subseteq k[(t_i)_{i\in I}]$ "wherever it makes sense" naturally arranges itself into a functor

$$V_S(-) : \mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$$

$$A \mapsto V_S(A)$$

with functoriality given by the above exercise.

Exercise 1.1.3. Let $R := k[(t_i)_{i\in I}]/(S)$ be the polynomial ring quotient associated to the ideal generated by the system S. Check, using the universal properties of polynomial rings and quotient rings, that we have an isomorphism, naturally in $A \in \mathrm{CAlg}_k$:

$$\mathrm{Hom}_{\mathrm{CAlg}_k}(R, A) \xrightarrow{\cong} V_S(A),$$

Thus, spaces of solutions of the system S valued in a k-algebra A are fully encoded by the relationship between R and A as elements of the category

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[4] In his autobiography "Souvenirs d'Apprentissage" (p.191), André Weil reminisces on a time during WWII when he was assigned to teach young soldiers with little to no knowledge of mathematics as part of a newly instituted "Army Science Training Program". He writes: "One day, one of [the military students] had a question to ask: 'I don't understand what x is'. The question was deeper than he thought, but I did not try to explain to him why that was so."

CAlg_k. We say that the k-algebra R *corepresents* the functor $V_S(-)$.[5]

The system $S = \{f_j\}_{j\in J}$ may therefore be thought of as a “recipe" for producing algebraically defined subsets of infinitely many different ambient spaces[6], all of which naturally fit together into a *(co)representable functor on the category of k-algebras.*

Grothendieck’s notion of scheme gives a geometric framework for thinking about such data. Let us begin somewhat unconventionally by stating the definition of a scheme in French, the language in which the theory was initially developed:

Définition 1.1.0. Un **schéma** est une figure donnant une représentation simplifiée et fonctionelle d’un objet ou d’un processus.[7]

This suggests the following first approximation towards the definition of a scheme in algebraic geometry:

Definition 1.1.1. A **scheme** (in the sense of Grothendieck) is a (geometric) figure offering a simplified and functional representation of an object (of algebraic origin) or (instantiation) process.

Somewhat more precisely, we are after the following:

Goal. To a system of polynomial equations $S = \{f_j\}_{j\in J} \subset k[(t_i)_{i\in I}]$ we should be able to associate a scheme X which encodes in a concise fashion the geometric structure contained in the various solution sets associated to the system.

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[5] For readers less familiar with functorial language: the “co” in corepresents is a red herring, and simply has to do with the fact that the functor $V_S(-)$ is covariant, and therefore that a representing object R will witness the above isomorphism through maps out of it rather than into it. Ample categorical background for this and everything to follow may be picked up as needed in [**Aluffi**], sections I.3-I.5 and VIII.1-VIII.2.

[6] In an article titled “The Lives of Alexander Grothendieck”, Edward Frenkel comments that “When we write ‘$x^2 + y^2 = 1$’, we wish into existence a perfect circle. Indeed, each solution of this equation is nothing but a pair of coordinates, x and y, of a point of the unit circle on a plane.” This equation not only produces the familiar unit circle in Euclidian space $\mathbb{R}^2$, but also an infinite family of associated shapes for every other commutative ring. The reader may wish to check as an exercise that the “circle over the complex numbers" $V_{x^2+y^2-1}(\mathbb{C}) = \mathbb{C}^*$ may be identified with a punctured complex line.

[7] Definition issued from the online edition of the French dictionary Le Robert.

In particular, from the data of the above scheme X, we expect to be able to retrieve the solution set $X(\bar{k}) = V_S(\bar{k})$ of solutions in an algebraic closure of k (when k is a field), which is the main object of study in classical algebraic geometry, as well as the solution sets $X(A) = V_S(A)$ associated to various other k-algebras A. The scheme X itself should be a workable geometric object, whose properties and relationships to other schemes elucidates the structure of the solution spaces we started with. The objective of this mini-course is to offer a leisurely overview of some aspects of this immensely successful program.

Let us return once more to the settings of Diophantine geometry, in which we are interested in schemes associated to systems of polynomial equations with integral coefficients. By the above principle, such a scheme X gives rise to a set of so-called "A-points" $X(A)$ for any commutative ring A. In the 1940's, André Weil suggested that, since $X(\mathbb{F}_p)$ for a given prime p and $X(\mathbb{C})$ are issued from the same "recipe", there should be a way to connect the geometry of $X(\mathbb{C})$ as an object of complex geometry with the arithmetic information contained in $X(\mathbb{F}_p)$ (and $X(\mathbb{F}_q)$ for some $q = p^n$ more generally)[8]:

$$\begin{tikzcd}
 & \left(X : \begin{array}{c}\text{scheme assoc. to a system}\\ \text{of integral poly. equations}\end{array}\right) \arrow[dl] \arrow[dr] & \\
\left(X(\mathbb{F}_q) : \begin{array}{c}\text{solutions}\\ \text{mod } q = p^n\end{array}\right) \arrow[rr, dashed, leftrightarrow, "\textit{Weil conjectures}"] & & \left(X(\mathbb{C}) : \begin{array}{c}\text{complex}\\ \text{solutions}\end{array}\right)
\end{tikzcd}$$

The specific conjectures that arose from this observation are known as the *Weil conjectures*[9], and they provided much of the impetus for the development of modern algebraic geometry in the years that followed, leading to their proof by Grothendieck and Deligne in the 1960s.

To conclude this discussion of the basic framework of modern algebraic geometry, let us mention that we have so far only described the *affine* objects

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[8] At the time when Weil formulated his conjectures, the framework of scheme theory hadn't yet been developed, although Weil himself had been working on his own program to expand the foundations of algebraic geometry - see [**Weil**]. As such, Weil's original suggestion was formulated in a different language from the one used here.

[9] In order to actually state the conjectures, we would need to add in smoothness and projectivity hypotheses which were omitted in this broad overview.

of study, i.e. those geometric spaces $V(S)$ or $X = \operatorname{Spec} R$, where $R = k[(x_i)_{i\in I}]/(S)$ is the k-algebra associated to a single system of polynomial equations S. These should be thought of as the building blocks for more general varieties or schemes which locally look like one of these affine pieces - in fact, a lot of interesting geometric phenomena only start emerging in the non-affine settings[10], and much of algebraic geometry is concerned with the study of (smooth) projective varieties, which are varieties that come equipped with a canonical closed embedding into projective space, a non-affine scheme. The way in which affine schemes serve as building blocks for more general schemes is very much analogous to the way smooth manifolds are built out of open subsets of Euclidian spaces via smooth transition maps, or how complex manifolds are built out of open subsets of some $\mathbb{C}^n$ via holomorphic transition maps, a parallel which we shall explore in more depth in section 5.1.

1.2 Some problems of interest in algebraic geometry

Let us come back to earth and discuss some of the typical problems considered in algebraic geometry over a ground field or commutative ring k. Naturally, the problems of interest depend heavily on k. This overview is of course by no means comprehensive, and tries only to give a sense of some of the vantage points from which one may approach the broad landscape of algebraic geometry.

- **(Counting problems)** Given an affine algebraic set $V(S) \subseteq k^I$ associated to a given system $S \subseteq k[(t_i)_{i\in I}]$, we can always ask whether V_S is empty or non-empty, i.e. whether the defining system of equations has any solution with values in k.

  When $k = \mathbb{F}_q$ is a finite field, the ambient space $\mathbb{F}_q^I$ is finite (for finite I), and so one can in practice directly determine whether or not V is empty. The counting problem of determining the number of solutions $|V(\mathbb{F}_{q^r})|$ over $\mathbb{F}_{q^r}$ for varying r is of particular interest, and it is featured for instance in the Weil conjectures.

  When $k = \mathbb{Z}$ or $\mathbb{Q}$, it is an open problem in general to algorithmically determine whether a solution to a given Diophantine equation

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[10] For instance, once the apparatus of (sheaf) cohomology groups has been developed, one may prove that affine schemes have no interesting cohomology, i.e. $H^i(\operatorname{Spec} R, \mathcal{F}) = 0$ for any $i > 0$ and any choice of “coherent coefficients” $\mathcal{F}$.

exists. Determining whether or not the set of integral points associated to a Diophantine equation is empty can quickly lead to some deep mathematics, e.g. Fermat's last theorem answers this question in the negative for the Diophantine equation $x^n + y^n + z^n$, $n \geq 3$. In a sense, finding an integral solution is the "hardest of all": suppose we have an integral solution to the system S, whose associated commutative ring is $R = k[(t_i)_{i \in I}]/(S)$. Then, the corresponding ring homomorphism $R \to \mathbb{Z}$ automatically produces an A-valued solution for every other commutative ring by post-composition with the initial ring map:

$$R \to \mathbb{Z} \to A.$$

TODO: Say something about counting problems in intersection theory.

- **(Relationship to other geometries)** When working with a topological field k such as $\mathbb{R}, \mathbb{C}$, or $\mathbb{Q}_p$ for some prime p, we may investigate what additional structure $X(k)$ can be equipped with, and how the resulting object interacts with X viewed as an algebro-geometric object.

  For $k = \mathbb{C}$, this is the subject of Serre's GAGA (Géométrie Algébrique, Géométrie Analytique) principle, as developed in [**Serre**]. The starting point is to notice that if our system $S \subseteq \mathbb{C}[t_0, ..., t_n]$ consists of reasonable (i.e. smooth) *homogeneous* polynomial equations, then we may view its solution set $X(\mathbb{C})$ inside complex projective space $\mathbb{C}P^n$ and topologize it with respect to the standard topology on $\mathbb{C}^n$. The resulting space, sometimes denoted X^{an}, is a compact Kähler manifold, and Serre's paper is concerned with establishing an equivalence between a certain kind geometric data on X^{an} (coherent analytic sheaves) and the corresponding algebro-geometric data on X (coherent algebraic sheaves)[11]. More directions may be pursued along those lines, such as comparing the classical fundamental group $\pi_1(X^{\text{an}})$ with Grothendieck's étale fundamental group $\pi_1^{\text{ét}}(X)$ in algebraic geometry[12], or exploring which singular cohomology classes $\alpha \in H^*_{\text{sing}}(X^{\text{an}}, \mathbb{Q})$

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[11] Once again, we are only describing the affine version of these structures - Serre's paper really tackles the case of general smooth projective schemes and their analytification.

[12] This is another nice manifestation of the blueprint "algebraic geometry over k = geometry over $\bar{k}$ + arithmetic input related to $\text{Gal}(\bar{k}/k)$" for perfect fields k. For instance,

may be obtained algebraically, as "Poincaré duals" of algebraic cycles in X (this is roughly the subject of the Hodge conjecture).

TODO: Say something about analytic geometry over $\mathbb{Q}_p$?

- **(Classification problems)** As with many other branches of mathematics, algebraic geometry is home to several classification problems. For instance, one may want to develop algebraic invariants so as to be able to tell non-isomorphic varieties apart. One may begin by defining numerical invariants such as dimension, genus, etc... But it is also possible to import more sophisticated invariants from the realm of algebraic topology, adapting them to the algebro-geometric settings. Thus, one can make sense of various kinds of cohomology groups, as well as K-theory groups and fundamental groups. Notably, much of the theoretical underpinning for the Weil conjectures consisted in Grothendieck's development of an analogue of singular cohomology for varieties defined over a field of characteristic p, called étale cohomology.

  TODO: Say something about birational geometry/minimal model program?

  Interestingly, algebraic geometry provides a flexible enough framework that some classification problems known as moduli problems may be considered as *algebro-geometric objects in their own right*. Once again, the functor of points approach to algebraic geometry provides a natural starting point. Suppose we set out to classify some type of geometric objects living over a fixed variety X (e.g. algebraic vector bundles of a given rank), and let us call these $\mathcal{B}$-objects. Then, we may define a functor

$$\mathcal{M}_{\mathcal{B},X} : \mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$$

  where to each k-algebra A, we associate *families of $\mathcal{B}$-objects over X parametrized by* $\operatorname{Spec} A$, up to isomorphism. In particular, $\mathcal{M}_{\mathcal{B},X}(k)$ singles out individual $\mathcal{B}$-objects over X. We may alternatively consider

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one may show that we have a short exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & \widehat{\pi_1(X^{\mathrm{an}})} \arrow[r] & \pi_1^{\text{ét}}(X) \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1, \end{tikzcd}$$

where $\widehat{\pi_1(X^{\mathrm{an}})}$ denotes the profinite completion of the classical fundamental group of X^{an}. Unfortunately, we will not be able to say much more about the étale fundamental group in these notes.

an “absolute moduli functor" by taking $X = \operatorname{Spec} k$ - e.g. one may be interested in studying how elliptic curves (the algebro-geometric analogue of Riemann surfaces of genus one) vary in families.

The question then becomes whether this functor is *representable* by some scheme Z, in the sense that there exist isomorphisms:

$$\mathcal{M}_{\mathcal{B},X}(A) \cong \mathrm{Hom}_{\mathrm{Sch}_k}(\operatorname{Spec} A, Z)$$

naturally in A. In some cases, such a scheme does indeed exist, and we then think of Z as a *classifying space* for $\mathcal{B}$-objects over X. Perhaps the most fundamental example is projective n-space over a field k, denoted by $\mathbb{P}^n_k$, which is the scheme classifying lines through the origin in k^{n+1}. In other cases, one can prove that a representing scheme cannot exist - examples include the moduli $\mathcal{M}_g$ of curves of a fixed genus g, or the moduli $\mathrm{Bun}_G(X)$ of principal G-bundles over a variety X. The failure of representability in the category of schemes is often due to the fact that the $\mathcal{B}$-objects we are attempting to classify admit non-trivial automorphisms, which makes them ill-suited to set-valued functors; the remedy is to instead consider $\mathcal{M}_{\mathcal{B},X}$ as a functor valued in *groupoids*, but we shall not be saying much more about this direction (known as the theory of stacks) in these notes.

1.3 Varieties vs schemes

In classical algebraic geometry, we usually fix an algebraically closed ground field $k = \bar{k}$ and take as our geometric object of study the vanishing set

$$V(S) = \{\underline{a} \in k^I \mid f(\underline{a}) = 0 \ \forall f \in S\}$$

associated to some system of polynomial equations $S \subseteq k[(x_i)_{i \in I}]$. In the modern approach to algebraic geometry, pioneered by Weil, Kähler, Serre, and Grothendieck (among others), we take k to be an arbitrary commutative ring, and attempt to study the family of solution sets $\{V_S(A)\}_{A \in \mathrm{CAlg}_k}$ in a uniform, geometric way, by means of associating to the system S a geometric space (called an *affine scheme*) $\operatorname{Spec} R$, where $R = k[(x_i)_{i \in I}]/(S)$. In what follows, we list some of the limitations of the classical framework which motivated the development of the modern language of scheme theory.

- **(Hidden solutions)** Sometimes, when working over a fixed field k, we are unable to distinguish two “fundamentally different” systems of polynomial equations S and S'. Consider for instance the following pair of systems in the case $k = \mathbb{R}$, taken from [**Ducros**]:

$$S : x^2 + y^2 = 0 \text{ , and } S' : \begin{cases} x + y = 0 \\ 2x + 3y = 0. \end{cases}$$

Then, as subsets of $\mathbb{R}^2$, we have that $V(S) = \{(0,0)\} = V(S')$. On the other hand, when looking for solutions over the $\mathbb{R}$-algebra $\mathbb{C}$, we find that $V_S(\mathbb{C}) = \{x = iy\} \cup \{x = -iy\}$ consists of the union of two lines in $\mathbb{C}^2$, while $V_{S'}(\mathbb{C}) = \{(0,0)\}$ still only consists of a single point.

In classical algebraic geometry, this problem is ruled out by restricting one's attention to solution sets over algebraically closed fields. But this takes away the possibility, among other things, to make geometric sense of phenomena of an arithmetic nature, which are often happening over $\mathbb{Z}$, $\mathbb{Q}$, etc...

- **(Hidden symmetries)** Suppose we are looking at a complex algebraic variety $V(S) \subseteq \mathbb{A}^n_{\mathbb{C}}$ whose defining polynomial equations all have real coefficients, so that we could interpret $V(S) = V_S(\mathbb{C})$ as the set of complex points of a more general algebro-geometric object which also admits a set of real points $V_S(\mathbb{R})$. Since $f(\overline{z}) = \overline{f(z)}$ for any polynomial f with real coefficients, we see that the operation of complex conjugation leads to an involution:

$$l : V_S(\mathbb{C}) \xrightarrow{\cong} V_S(\mathbb{C})$$

which fixes the subset of real points $V_S(\mathbb{R}) \subseteq V_S(\mathbb{C})$; in fact, $V_S(\mathbb{R})$ may be recovered from the pair $(V_S(\mathbb{C}), l)$ by passing to fixed points:

$$V_S(\mathbb{R}) = V_S(\mathbb{C})^{l=\mathrm{id}}$$

Thus, by looking at solution sets over more general fields, we uncover symmetries of the geometric object at hand.

The above observation has close connections to the classification theory of real algebraic groups, which are the algebraic analogues of real

Lie groups[13]. Indeed, given a real algebraic group $G_{\mathbb{R}}$, one may equivalently study the corresponding complex algebraic group $G_{\mathbb{C}}$, together with the involution $G_{\mathbb{C}} \xrightarrow{l} G_{\mathbb{C}}$ given by complex conjugation. Due to work of Cartan and Chevalley, the pair $(G_{\mathbb{C}}, l)$ fully determines $G_{\mathbb{R}}$: the classification problem of real algebraic group is therefore equivalent to the classification of complex algebraic groups equipped with an involution, and the latter problem turns out to be more tractable.

The hidden symmetries story outlined above makes sense in greater generality, starting with the observation that $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, and that $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$ is generated by complex conjugation. Thus, for any (perfect)[14] field k and any scheme X defined over k[15], the set $X(\bar{k})$ of $\bar{k}$-points (where $\bar{k}$ denotes an algebraic closure of k) comes equipped with a canonical action of $\mathrm{Gal}(\bar{k}/k)$, in such a way that the set of k-points $X(k) \subseteq X(\bar{k})$ may be recovered as the set of Galois fixed points:

$$X(k) = X(\bar{k})^{\mathrm{Gal}(\bar{k}/k)=\mathrm{id}}. \tag{1}$$

In practice, the object $X(\bar{k})$ is the most “geometrically behaved" solution set associated to X (for instance, $X(\mathbb{C})$ can often be equipped with the structure of a complex manifold). Thus, equation 1 hints at the following general pattern in algebraic geometry:

Principle.

$$(\text{Algebraic geometry over } k) = (\text{Geometry over } \bar{k}) + (\text{Arithmetic input related to } \mathrm{Gal}(\bar{k}/k))$$

- **(Hidden multiplicities)** By passing to the set of solutions of an equation over a field, we forget for instance that the equation $t^2 = 0$ admits the solution $a = 0$ *twice* - thus, if all we look at is $V_{t^2}(k)$, we

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[13] Many algebraic groups come to us as *matrix groups*, i.e. as subsets of $GL_n(k)$ (here, $k = \mathbb{R}$ or $\mathbb{C}$) cut out by certain polynomial equations - e.g. the equation $\det(A) = 1$ cuts out $SL_n(k)$, and the family of equations $A^t A = I$ (one for each matrix entry) cuts out $O(n)$ for $k = \mathbb{R}$.

[14] This is a technical condition for the story to work out nicely, but which is rather permissive: for instance, all finite fields and all fields of characteristic zero are perfect.

[15] At this stage, it is safe to think of X as being a geometric space which locally keeps track of the solution sets $\{V_S(A)\}_{A \in \mathrm{CAlg}_k}$ associated to some system of polynomial equations S with coefficients in k. Namely, there is a notion of A points $X(A)$, which locally looks like $V_S(A)$ for the appropriate system S.

are losing track of "multiplicities" or nilpotent information, which may be interpreted geometrically as "infinitesimal" information of the system (in classical algebraic geometry, this is encoded in the fact that $V(I) = V(\sqrt{I})$).

We will see that the scheme X_{t^2} associated to this system does remember this information, and in particular that it is distinct from the scheme X_t as geometric objects (namely, as "locally ringed spaces"). Here, it is crucial that X_{t^2} is not merely a topological space (we shall see in section 4.2 that the natural topological space associated to a k-algebra does not detect nilpotent information), but that it also comes with the extra structure of a *sheaf of rings*, which we will describe in section 5.3.

Notice that the functorial perspective from earlier is also "discerning enough", as long as we are careful to include solution spaces corresponding to all k-algebras, and not just field extensions of k. Indeed, while $V_{t^2}(L) = V_t(L)$ for any field extension L/k, and in fact for any integral domain, the non-reduced k-algebra $k[t]/(t^2)$ witnesses that these two systems are actually distinct:

$$V_{t^2}(k[t]/(t^2)) = \{at \mid a \in k\} \neq \{0\} = V_t(k[t]/(t^2)).$$

- **(Hidden geometry)**[16] Let us engage in a thought experiment which we hope will shed some light on the ways in which scheme theory provides a framework in which to interpret arithmetic situations in geometric terms. Some of the assertions we make will have to be taken on faith, but this picture should become clearer as the mini-course progresses.

  Suppose we start life looking for an integer solution to a given Diophantine equation $f \in \mathbb{Z}[t_1, .., t_n]$. As mentioned earlier, by the universal property of quotients, this is equivalent to looking for a ring homomorphism:

  $$R := \mathbb{Z}[t_1, ..., t_n]/(f) \to \mathbb{Z}.$$

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[16] This particular discussion was originally inspired by a MathOverflow answer by Minhyong Kim, see `https://mathoverflow.net/questions/38639/thinking-and-explaining/38694#38694`

We set out to re-frame this situation in geometric terms, in accordance with the following principle, which we will spend much of the following sections developing[17]:

Blueprint 1.3.1. ((algebra)^{op} = (geometry))
Fix a commutative ring k. Then the category CAlg_k^{op} may be canonically identified with a geometric category AffSch_k whose objects are called *affine schemes over* k. That is, to every k-algebra R, there corresponds a "geometric space" $\operatorname{Spec} R$ whose "ring of functions" $\operatorname{Hom}(\operatorname{Spec} R, \mathbb{A}^1_k)$ may be canonically identified with R. For any two k-algebras R and A, continuous maps $\operatorname{Spec} A \to \operatorname{Spec} R$ are in 1-1 correspondence with k-algebra homomorphisms $R \to A$ via "pullback of functions":

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[dr, dashed] & \operatorname{Spec} R. \arrow[d] \\
& \mathbb{A}^1_k
\end{tikzcd}$$

As a first "application" of this principle, the ring $R := k[(t_i)_{i\in I}]/(S)$ which we were led to consider when setting out to solve a system of polynomial equations $S = \{f_j\}_{j\in J} \subseteq k[(t_i)_{i\in I}]$ corresponds to an affine scheme $X = \operatorname{Spec} R$, which may be thought of as the "universal solution space" of the system. The act of looking for solutions to the system S valued in some k-algebra A, i.e. looking for k-algebra homomorphisms

$$R \to A,$$

may be interpreted geometrically via the pullback principle as looking for morphisms

$$\operatorname{Spec} A \to \operatorname{Spec} R$$

"picking out" a $\operatorname{Spec} A$-shaped piece out of the solution space $\operatorname{Spec} R$ - this motivates the terminology of "A-points" for such morphisms.

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[17] For now, the reader concerned with rigor may safely think of the "geometric space" $\operatorname{Spec} R$ as labeling the object of CAlg_k^{op} corresponding to $R \in \mathrm{CAlg}_k$, of $\operatorname{Spec} A \to \operatorname{Spec} R$ as labeling the morphism in CAlg_k^{op} corresponding to the k-algebra homomorphism $R \to A$, and of $\mathbb{A}^1_k$ as shorthand for $\operatorname{Spec} k[t]$ (the space on which functions are given by polynomials in one variable with coefficients in k, i.e. the "number line" over k). Later, we shall explain how to actually identify $\mathrm{CAlg}_k^{op} =: \mathrm{AffSch}_k$ with a category of geometric objects, namely with a full subcategory of the category of "locally ringed spaces".

Finally, the k-algebra structure map $k \to R$ may be interpreted as providing a canonical morphism:

$$\begin{tikzcd}\operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} k.\end{tikzcd}$$

We choose to draw the structure map vertically to suggest that one may think of $\operatorname{Spec} R$ as some sort of *bundle* over $\operatorname{Spec} k$. Note that since morphisms $\operatorname{Spec} A \to \operatorname{Spec} R$ were taken to correspond to k-algebra homomorphisms $R \to A$, they must respect the respective morphisms down to $\operatorname{Spec} k$ - i.e. such morphisms "behave like bundle morphisms"[18]. Thus, we arrive at the following:

Principle. Studying systems of polynomial equations with coefficients in k corresponds to doing algebraic geometry over $\operatorname{Spec} k$.

Let us now return to our Diophantine situation "with fresh eyes". Our ring of interest was given by $R := \mathbb{Z}[t_1, ..., t_n]/(f)$, which corresponds to a "geometric space" $X = \operatorname{Spec} R$ living over $\operatorname{Spec} \mathbb{Z}$:

$$\begin{tikzcd}X \arrow[d] \\ \operatorname{Spec} \mathbb{Z}.\end{tikzcd}$$

To find an integral solution for f corresponds to finding a morphism $\operatorname{Spec} \mathbb{Z} \to X$, which, since $\operatorname{Spec} \mathbb{Z}$ is terminal in the category of affine schemes (because $\mathbb{Z}$ is initial in the category of commutative rings), may be interpreted as a *section* of the above structure map.

When trying to find such a solution, one's first instinct is often to try looking for solutions mod p for various primes p: there are only finitely values to check after reducing coefficients mod a given prime, and failing to find a solution mod p guarantees that there can be no integral solution.

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[18] This is one place among many in these notes wherein we are inviting the reader to try their luck at the arcane art of "squinting", so as to hopefully gather some intuition about a given situation based on their past mathematical experience on other waters.

Let us interpret this geometrically. For a given prime p, the canonical "reduction mod p" ring map $\mathbb{Z} \twoheadrightarrow \mathbb{F}_p$ corresponds to an "inclusion" $\operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}$ (essentially because epimorphisms become monomorphisms in the opposite category). This inclusion should be thought of as "picking out a point" $[p] \in \operatorname{Spec} \mathbb{Z}$ corresponding to the prime p. Now, an $\mathbb{F}_p$-valued solution $a^1 \in \mathbb{F}_p^n$ of f corresponds to a ring map $R \to \mathbb{F}_p$, which (again because $\operatorname{Spec} \mathbb{Z}$ is terminal) may be interpreted as a commutative triangle:

$$\begin{tikzcd}
 & X \arrow[d] \\
\operatorname{Spec} \mathbb{F}_p \arrow[ru, dashed, "a^1"] \arrow[r] & \operatorname{Spec} \mathbb{Z}.
\end{tikzcd} \tag{2}$$

i.e. as a *lift* of the inclusion $\operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}$ along the structure map of X. Thus, failure to find a solution mod p features as a *local obstruction at a point to forming a global section.*

Exercise 1.3.1. Readers with a topology background may expect more to be true for the above triangle, namely for the lift to "respect fibers". For us, this may be made precise as follows (using only the formal correspondence described in the $\mathrm{AffSch}_k \simeq \mathrm{CAlg}_k^{op}$ blueprint): define the *fiber of X over* $\operatorname{Spec} \mathbb{F}_p$ to be the following pullback[19]:

$$X_{\mathbb{F}_p} := \varprojlim \left(\begin{tikzcd}
 & X \arrow[d] \\
\operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd} \right).$$

(1) Using the categorical equivalence $\mathrm{AffSch}_k \simeq \mathrm{CAlg}_k^{op}$, show that

$$X_{\mathbb{F}_p} = \operatorname{Spec}(R \otimes_{\mathbb{Z}} \mathbb{F}_p) = \operatorname{Spec} \mathbb{F}_p[t_1, ..., t_n]/(\bar{f}).$$

where $\bar{f}$ denotes the polynomial obtained by reducing the coefficients of f modulo p.

(2) Arguing by working in the opposite category once again, prove that

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[19] This is a special instance of a ubiquitous procedure in algebraic geometry known as base-change, which in the simplest instance is the geometric analogue of changing the ring of coefficients of our polynomial equations.

any lift as in equation 2 must in fact factor through $X_{\mathbb{F}_p}$ in the pullback square:

$$\begin{tikzcd}
X_{\mathbb{F}_p} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}.
\end{tikzcd}$$

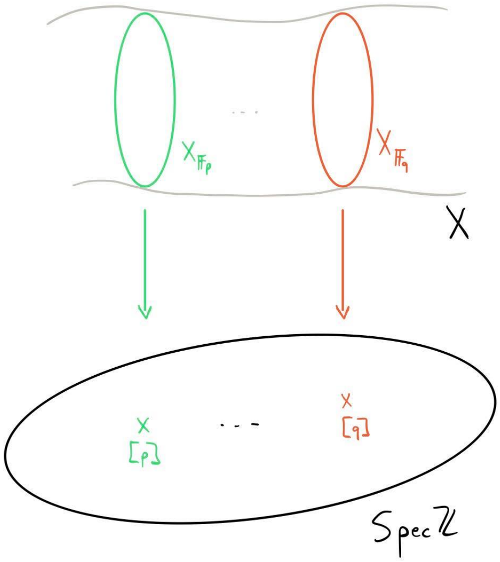

Figure 1: Visualizing X as a “bundle over $\operatorname{Spec} \mathbb{Z}$”

Supposing we have found lifts for every prime p, we may then ask whether these local lifts “glue together to produce a global section”, i.e. whether they all come from a single integral solution. This is in general a hard problem in number theory known as a local-to-global problem[20]. To start with, we may attempt the more tractable “local-to-formal” problem[21], which consists in asking, near each prime p, whether our

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[20] This problem is often formulated in the context of looking for rational solutions to a system of equations defined over the ring of integers of a number field (a generalization of the situation $\mathbb{Z} \subseteq \mathbb{Q}$). For more in this direction, a good first step might be to look at the Hasse-Minkowski theorem.

[21] In the literature, “local” usually means “formal”, i.e. assumes that the step we are

pointwise lift may be extended to a lift over a "formal neighborhood" of $[p] \in \operatorname{Spec} \mathbb{Z}$.

More specifically, we begin by noticing that the reduction mod p ring map factors through "reduction mod p^2":

$$\mathbb{Z} \twoheadrightarrow \mathbb{Z}/(p^2) \twoheadrightarrow \mathbb{F}_p,$$

which may geometrically be viewed as successive inclusion maps

$$\operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}/(p^2) \hookrightarrow \operatorname{Spec} \mathbb{Z}.$$

For reasons that will become clear in later sections, we may fruitfully think of $\operatorname{Spec} \mathbb{Z}/(p^2)$ as a "first order infinitesimal neighborhood" around the point $\operatorname{Spec} \mathbb{F}_p$ inside $\operatorname{Spec} \mathbb{Z}$. Iterating this process produces "N^{th}-order infinitesimal neighborhoods" for every N:

$$\operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}/(p^2) \hookrightarrow \ldots \hookrightarrow \operatorname{Spec} \mathbb{Z}/(p^N) \hookrightarrow \operatorname{Spec} \mathbb{Z}.$$

Exercise 1.3.2. (1) Check that finding a lift:

$$\begin{tikzcd} & X \arrow[d] \\ \operatorname{Spec} \mathbb{Z}/(p^2) \arrow[r] \arrow[ur, dashed, "a^2"] & \operatorname{Spec} \mathbb{Z}. \end{tikzcd} \tag{3}$$

extending the lift $a^1 : \operatorname{Spec} \mathbb{F}_p \to X$ from equation 2 corresponds to finding a $(\mathbb{Z}/(p^2))$-valued solution a^2 to f which reduces to a^1 mod p.

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about to outline has already been cleared. Interestingly, the local-to-formal problem is automatically satisfied provided that the scheme at hand is smooth - this is a consequence of the so-called *infinitesimal lifting criterion* for smoothness.

(2) Check that finding a compatible system of lifts

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{Z}/p \arrow[r, "a^1"] \arrow[d] & X \\
\operatorname{Spec}\mathbb{Z}/p^2 \arrow[ur, "a^2"] \arrow[d] & \\
\cdots \arrow[d] & \\
\operatorname{Spec}\mathbb{Z}/p^N \arrow[uuur, "a^N"'] \arrow[d] & \\
\cdots &
\end{tikzcd}$$

for every $N \geq 1$ corresponds to finding a solution to f valued in the ring of *p-adic integers* $\mathbb{Z}_p$ and that conversely any $\mathbb{Z}_p$-valued solution of f gives rise to such a compatible family of a^N's. Geometrically, we may therefore think of $\mathbb{Z}_p$-valued solutions to f as lifts of the structure map over a *formal disk* around p:

$$\begin{tikzcd}
 & X \arrow[d] \\
\operatorname{Spec}\mathbb{Z}_p \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z}.
\end{tikzcd}$$

Being able to find such extensions over formal disks around each prime p is what we referred to as the "local-to-formal" problem. Going any further, i.e. determining when all of these lifts over formal disks may be "coherently integrated out" into a single integral solution, requires digging into more serious number theory.

2 Affine Varieties

2.1 The Zariski Topology

Our starting point will consist of solutions to multivariate polynomials in n-dimensional space. Let k be any algebraically closed field, and denote by $\mathbb{A}^n_k$ all n-tuples of elements of k. We will denote an n-tuple $(t_1, \dots t_n)$ as $\underline{t}$[22].

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[22] In general, our coordinates will be given by $(t_1, \dots t_n)$ during this section and the next.

We will equip $\mathbb{A}^n_k$ with the *Zariski topology*, which is defined as follows.

Let S be a subset of $k[t_1, \dots t_n]$, and define $V(S)$ to be the points in $\mathbb{A}^n_k$ where all elements of S vanish:

$$V(S) = \{\underline{t} \in \mathbb{A}^n_k \,|\, \text{For any} f \in S,\ f(\underline{t}) = 0\}$$

We will call sets of the form $V(S)$ *algebraic sets.*

Exercise 2.1.1. Give an example of two ideals I, and J such that $V(I) = V(J)$ but $I \neq J$.

Proposition 2.1.1. *The union of two algebraic sets is algebraic:*

$$V(S_1) \cup V(S_2) = V(S_1 S_2)$$

where $S_1 S_2 := \{fg \,|\, f \in S_1,\ g \in S_2\}$

An intersection of arbitrarily many algebraic sets is algebraic:

$$\cap_\alpha V(S_\alpha) = V(\cup_\alpha S_\alpha)$$

The empty set and $\mathbb{A}^n_k$ *are algebraic sets. Therefore, algebraic sets define a topology on* $\mathbb{A}^n_k$*.*

We call this topology the Zariski topology.

Proof. For the first claim notice that if $\underline{t} \in V(S_1) \cup V(S_2)$ then $\underline{t} \in V(S_1)$ or $V(S_2)$, without loss of generality let $x \in V(S_1)$. Then $f(x) = 0$ for any $f \in S_1$, but then $f(\underline{t})g(\underline{t}) = 0$ for any $f \in S_1$ and any $g \in S_2$. So $V(S_1) \cup V(S_2) \subseteq V(S_1 S_2)$. On the other hand if $\underline{t} \in V(S_1 S_2)$, then if $\underline{t}$ is not an element of $V(S_2)$ there is some element g in $V(S_2)$ such that $g(\underline{t}) \neq 0$. But then $f(\underline{t})g(\underline{t}) = 0$ for any $f \in S_1$, and so $f(\underline{t}) = 0$ for any $f \in S_1$. So $\underline{t} \in V(S_1)$.

For the second equality $\underline{t} \in \cap_\alpha V(S_\alpha)$ means exactly that $f(\underline{t}) = 0$ for any $f \in S_\alpha$ for any α.

The last claim is because $\mathbb{A}^n_k = V(\emptyset)$ and $\emptyset = V(k[t_1, \dots t_n])$.

□

For a given $f \in k[t_1, \dots t_n]$ one defines $D(f)$ to be the complement of $V(f)$. The "D" could be taken to mean "does not vanish". By the proposition $\cap_\alpha V(S_\alpha) = V(\cup_\alpha S_\alpha)$, and so for some collection of polynomials $S = \{f_\alpha\}$ we have that $V(S)^c = \cup_\alpha D(f_\alpha)$. So it follows that sets of the form $D(f)$ are a basis for the Zariski topology.

Example 2.1.2. Consider the 1 dimensional case. Then since $k[t]$ is a PID, any ideal is generated by some $f \in k[t]$. So $V(f)$ is a finite set of points, so the Zariski topology is the cofinite topology on k[23].

Example 2.1.3. In the 2-dimensional case, things are slightly more complicated but still manageable, because any nontrivial prime ideal in $k[x, y]$ is either of the form $(f(t_1, t_2))$ for some irreducible polynomial f, or of the form $(t_1 - \alpha, t_2 - \beta)$ for some scalars $\alpha, \beta \in k$. These will correspond to curves and individual points, respectively, and we will see later that this implies that any nontrivial closed set of $\mathbb{A}_k^2$ is a finite union of points and curves.

Exercise 2.1.2. Prove the slightly unfortunate fact of nature that the Zariski topology on $\mathbb{A}_k^2$ is not the product topology on $\mathbb{A}_k^1 \times \mathbb{A}_k^1$. (Note that this a completely general phenomenon for products of varieties and schemes, in fact schemes are worse, the underlying set isn't even the product set. This provides some motivation for the "functor of points" perspective when dealing with algebraic groups, a topic for a later date.)

To further explain the Zariski topology, we need to introduce some topological concepts which are prevalent in algebraic geometry, though somewhat less common elsewhere, namely Noetherianness and irreducibility.

Definition 2.1.4. A topological space X is **irreducible** if whenever $X = X_1 \cup X_2$, and X_1 and X_2 are both closed, then $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$.

Exercise 2.1.3. Show that if a space is Hausdorff and irreducible then it must consist of a single point.

Exercise 2.1.4. Show that if a space is irreducible then it's connected.

Example 2.1.5. $V(xy)$ is *not irreducible*: both $V(x)$ and $V(y)$ are closed in $V(xy)$ and $V(x) \cup V(y) = V(xy)$. Note that $V(x)$ and $V(y)$ are irreducible. In general for an irreducible polynomial f, $V(f)$ is irreducible.

Definition 2.1.6. A topological space is **Noetherian** if any decreasing sequence of closed subsets stabilizes, i.e. for any sequence:

$$Z_1 \supset Z_2 \supset Z_3 \supset \dots$$

there is some r_0 such that for any $r \geq r_0$ we have $Z_r = Z_{r+1}$.

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[23] Recall that the *cofinite topology* on a set X is defined by declaring the closed sets to be finite sets in X - equivalently, by declaring the open sets to be those subsets whose complements are finite.

Important Exercise 2.1.5. Show that $\mathbb{A}_k^n$ is Noetherian, and that a closed subset of any Noetherian space is Noetherian. Show that $\mathbb{R}^n$ equipped with its standard topology is not Noetherian unless $n = 0$.

Proposition 2.1.7. *Any closed subset of a Noetherian topological space X is uniquely given by a finite set of irreducible components.*

Proof. Consider the collection of nonempty closed subsets of X which cannot be expressed as a union of finitely many irreducible sets. This collection contains a minimal element Y because X is Noetherian. Y is not irreducible so it can be written $Y = Y' \cup Y''$, with Y' and Y'' closed, one not containing the other, and they can be each be written as a union finitely many irreducible sets by minimality - but then, so can Y. Thus, the collection of nonempty closed subsets of X which cannot be expressed as a union of irreducible sets must be empty.

Now suppose a nonempty closed subset Y of X can be written as a union of irreducible closed sets two ways:

$$Y = \bigcup_{i=1}^{r} Y_i \quad \text{and} \quad Y = \bigcup_{j=1}^{s} Y'_j.$$

Then $Y_1 = \cup_{j=1}^{s}(Y_1 \cap Y'_j)$. But Y_1 is irreducible, so $Y_1 = Y'_{j_0}$ for some j_0. We can then apply this same argument to $\bigcup_{i=2}^{r} Y_i = \bigcup_{j=1, j \neq j_0}^{s} Y'_j$ to show that the decompositions are the same (the process terminates by finiteness of the decompositions). $\square$

Irreduciblity is an important notion for one because many questions in algebraic geometry can be dealt with by only considering irreducible varieties, similar to how many topological questions can be solved by passing to connected components. Furthermore, irreduciblity gives us a notion of dimension.

Definition 2.1.8. The **dimension** of an algebraic set is defined to be the maximum length n of a chain of increasing, distinct, and irreducible closed subsets $Z_0 \subset Z_1 \ldots \subsetneq Z_n$.

2.2 Ideals and Coordinate Rings

Given an ideal I in $k[t_1, \ldots t_n]$, we can construct a closed subset $V(I) \subseteq \mathbb{A}_k^n$. Conversely, given a closed subset V of $\mathbb{A}_k^n$ we can define an ideal $I(V)$ by setting

$$I(V) = \{f \in k[t_1, \dots t_n] \mid f(\underline{t}) = 0 \ \forall \underline{t} \in V\}.$$

This ideal is very important for the following reason: first, if one was to speak of "the algebraic functions on $\mathbb{A}^n_k$", the obvious candidate for this is $k[t_1, \dots t_n]$: polynomial functions in n variables. Next, if V is a closed subset of $\mathbb{A}^n_k$, then two algebraic functions f, g define the same function when restricted to V if and only if they are equal in the ring $k[t_1, \dots t_n]/I(V)$! We call the ring $k[t_1, \dots t_n]/I(V)$ the **coordinate ring** of V, and denote it by $\mathcal{O}(V)$. Its elements are called **regular functions** or **global functions** V. This ring plays a central role in the study of affine varieties. Note that the coordinate ring is always a commutative, finitely generated, reduced k-algebra: the fact that it is a commutative k-algebra is clear, it's finitely generated because it's a quotient of a finitely generated ring, and the fact that it is always reduced (i.e. that it contains no nilpotents) follows from the fact that if for $\underline{t} \in V$ $f^N(\underline{t}) = 0$ for some $f \in k[t_1, \dots t_n]$ and $N \in \mathbb{N}$ then $f(\underline{t}) = 0$.

With a little more vocabulary we can state a very important correspondence.

Definition 2.2.1. If I is an ideal in a ring R, then set $\sqrt{I} := \{r \in R \mid r^n \in I, \text{for some } n \in \mathbb{N}\}$. We call this the *radical of* I.

Exercise 2.2.1. Verify that $\sqrt{I}$ is an ideal in R.

Important Exercise 2.2.2. Show that for, if V is a closed set in affine space $\mathbb{A}^n_k$, then V is irreducible if and only if $I(V)$ is a prime ideal of $k[t_1, ..., t_n]$.

The following theorem gives us an order reversing correspondence between ideals that are a radical of some ideal and closed subsets of affine space.

Theorem 2.2.2. ***(Hilbert's Nullstellensatz)*** *For any ideal I in $k[t_1, \dots t_n]$ one has that:*

$$I(V(I)) = \sqrt{I}$$

A particular case of this is the *Weak Nullstellensatz*, which says that the maximal ideals of $k[t_1, \dots t_n]$ are all of the form $(t_1 - \alpha_1, \dots, t_n - \alpha_n)$. The latter is exactly the ideal of functions that vanish at $(\alpha_1, \dots \alpha_n)$, thereby establishing a bijection:

$$\Big(\text{Maximal ideals of } k[t_1, \dots t_n]\Big) \leftrightarrow \Big(\text{Points of } \mathbb{A}^n_k\Big).$$

Furthermore combining the Nullstellensatz with 2.2.2 we see that we have a bijection:

$$\Big(\text{Prime ideals of } k[t_1, \dots t_n]\Big) \leftrightarrow \Big(\text{Irreducible closed subsets of } \mathbb{A}^n_k\Big).$$

Important Exercise 2.2.3. Show that for a general algebraic set X there is a bijection between the maximal ideals of $\mathcal{O}(X)$ and the points of X.

2.3 Morphisms Between Affine Varieties

The last major thing to discuss is what morphisms between algebraic sets look like, and the relation of these morphisms to the coordinate ring. We will define a morphism between two affine varieties to be a map given by polynomials. Precisely speaking, let $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$ be two algebraic sets. A function $\varphi : X \to Y$ is a morphism of algebraic sets if there exists m polynomials $\varphi_1, \dots \varphi_m \in k[t_1, \dots t_n]$ such that for any $\underline{t} = (t_1, \dots t_n) \in X$ we have:

$$\varphi(\underline{t}) = (\varphi_1(\underline{t}), \dots \varphi_m(\underline{t})).$$

Thus we have a category of algebraic sets! We will denote this category by AffVar_k.

Example 2.3.1. The map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $t \mapsto t^n$ is a morphism.

The projection map $\mathbb{A}^2_k \to \mathbb{A}^1_k$, $(t_1, t_2) \mapsto t_1$ is a morphism.

The identity map is a morphism.

The composition of morphisms is a morphism.

The map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto e^z$ is *not* a morphism of algebraic sets.(A proof may not be obvious to you but this should *feel* reasonable.)

An element of $\mathcal{O}(X)$ is the same thing as a morphism $X \to \mathbb{A}^1_k$.

Important Exercise 2.3.1. Show that any morphism is continuous in the Zariski topology.

An extremely important issue is how these maps relate to morphisms of coordinate rings. Let X and Y be as above. Let $\varphi : X \to Y$ be a morphism of algebraic sets, defined by m polynomials $\varphi_1, \dots \varphi_m \in k[t_1, \dots t_n]$ as above. Let f be an element of $\mathcal{O}(Y)$, then $f \circ \varphi$ is a function from $X \to k$. Furthermore $f \circ \varphi$ is defined by polynomials so in fact it's an element of $\mathcal{O}(X)$. Thus we get a map $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Furthermore, notice that precomposition is k-linear, and $f \cdot g \circ \varphi = f \circ \varphi \cdot g \circ \varphi$ so it's multiplicative so $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a k-algebra homomorphism.

Important Exercise 2.3.2. Show that $\mathrm{id}^* : \mathcal{O}(X) \to \mathcal{O}(X)$ is the identity map, and show that $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. In other words, passing from affine algebraic sets to coordinate rings is (contravariantly) functorial.

Theorem 2.3.2. *Let X, Y be two affine algebraic sets. Then passing to rings of regular functions induces a canonical bijection*

$$\mathrm{Hom}_{\mathrm{AffVar}_k}(X, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{CAlg}_k}(\mathcal{O}(Y), \mathcal{O}(X))$$

$$(X \xrightarrow{\varphi} Y) \mapsto (\mathcal{O}(Y) \xrightarrow{\varphi^*} \mathcal{O}(X)).$$

In other words, the functor

$$\mathcal{O}(-) : \mathrm{AffVar}_k \to \mathrm{CAlg}_k^{op}$$

is fully faithful.

Proof. Let us prove surjectivity; injectivity may be checked similarly. Start with an arbitrary k-algebra homomorphism

$$\varphi^* : \mathcal{O}(Y) = k[t_1, ..., t_m]/I(Y) \to k[x_1, ..., x_n]/I(X) = \mathcal{O}(X).$$

We seek to define a morphism $\psi : X \to Y$ such that $\psi^* = \varphi^*$. To this end, set $\psi_i := \varphi^*(\overline{t_i})$: each ψ_i is a regular function on X, and we may bring them together to form a morphism

$$\psi := (\psi_1, ..., \psi_m) : X \to \mathbb{A}_k^m.$$

Let us verify that ψ is in fact a map to $Y \subseteq \mathbb{A}_k^m$. Since $Y = V(I(Y))$, it suffices to show that for any $\underline{x} \in X$ and $f \in I(Y)$, we have that $f(\psi(\underline{x})) = 0$. By definition,

$$\psi(\underline{x}) = (\psi_1(\underline{x}), ..., \psi_m(\underline{x})) := (\varphi^*(\overline{t_1})(\underline{x}), ..., \varphi^*(\overline{t_m})(\underline{x})),$$

so that

$$\begin{aligned} f(\psi(\underline{x})) &= f(\varphi^*(\overline{t_1})(\underline{x}), ..., \varphi^*(\overline{t_m})(\underline{x})) \\ &= \varphi^*(f)(\underline{x}) = 0, \end{aligned}$$

since $\varphi^*(f) \in I(X)$ and $\underline{x} \in X$. Finally, for each $i = 1, ..., m$, we have that

$$\psi^*(\overline{t_i}) = \overline{t_i} \circ \psi = \overline{t_i} \circ (\varphi^*(\bar{t}_1), \dots \varphi^*(\bar{t}_m) = \varphi^*(\bar{t}_i)$$

so that ψ^* and φ^* have the same coordinate functions, and therefore they are equal. $\square$

Exercise 2.3.3. (1) Check the equality $f(\varphi^*(\overline{t_1})(\underline{x}), ..., \varphi^*(\overline{t_m})(\underline{x})) = \varphi^*(f)(\underline{x})$ asserted above for the explicit example $f := t_1^2 + t_2 \in k[t_1, t_2]$.
(2) Complete the proof of the theorem by checking injectivity.

The next natural question then is what rings can be realized as coordinate rings of algebraic sets? We saw at the beginning of this section that the coordinate ring is always a commutative, reduced, finitely generated k-algera.

Exercise 2.3.4. Show that, conversely, any finitely generated reduced commutative k-algebra may be expressed as $\mathcal{O}(V)$ for some affine algebraic set V.

Putting everything together, we have arrived at an equivalence of categories:

$$\mathcal{O} : \mathrm{AffVar}_k \xrightarrow{\simeq} \big(\text{f.g. reduced commutative } k\text{-algebras}\big)^{op}$$

This is an extremely important initial theorem. First of all, this illustrates a driving force behind much of algebraic geometry: geometry provides intuition, while algebra provides a strong technical arsenal. Furthermore, this is a manifestation of a key principle which can be summed up by the following:

Blueprint 2.3.3.

$$(\text{Geometry}) \simeq (\text{Algebra})^{op}$$

The language of schemes is precisely what you get when you enlarge our equivalence of categories to include all commutative rings.

3 Varieties in Projective Space

3.1 Projective Space

In this section we will see an important place in which one does algebraic geometry: projective space. In general, one constructs varieties by gluing affine varieties together, though we lack the language at this instant to develop this in full generality, and it's work that might as well be done in the full generality of scheme theory. However, many many explicit examples in algebraic geometry are what are called quasi-projective varieties. In this section we will discuss what this is with a main focus on examples.

First, we define projective space: n-dimensional projective space over a field, denoted $\mathbb{P}^n_k$, is defined to be $n+1$-tuples of elements of k, which are not all 0, and two such $n+1$-tuples will be equivalent if they are same up to a scalar multiple. That is, for $\lambda \neq 0$, we set:

$$(t_0, \dots t_n) \sim (\lambda t_0, \dots \lambda t_n)$$

Equivalently, one could say that points of $\mathbb{P}^n_k$ are lines through the origin in $\mathbb{A}^{n+1}_k$. Oftentimes we will denote points of $\mathbb{P}^k_n$ with homogenous coordinates:

$$[t_0 : \dots : t_n]$$

where we remember that these coordinates are only defined up to scaling.

This is already an example of an object that is made up of affine varieties, but isn't itself affine. We may produce a covering of $\mathbb{P}^k_n$ by copies of $\mathbb{A}^n_k$ as follows: denote by U_i the set of all points of $\mathbb{P}^n_k$ where the ith homogeneous coordinate is nonzero. Then, each point can be written in a unique way in homogeneous coordinates by scaling the coordinates so the ith coordinate is 1:

$$[t_0 : \dots : t_n] = [t_0/t_i : \dots t_{i-1}/t_i : 1 : t_{i+1}/t_i : \dots t_n/t_i]$$

Then the map:

$$[t_0/t_i : \dots t_{i-1}/t_i : 1 : t_{i+1}/t_i : \dots t_n/t_i] \mapsto (t_0/t_i, \dots t_{i-1}/t_i, t_{i+1}/t_i, \dots t_n/t_i)$$

gives a bijection $U_i \to \mathbb{A}^n_k$.

So far, this discussion made no mention of a topology or of algebraic functions, so let us now lay the ground work for this. First, we must figure out how to talk about zero sets of polynomials. Consider the ring $k[t_0, \dots t_n]$, and denote by $k[t_0, \dots t_n]_d$ the collection of all k-linear combinations of degree d monomials, where a monomial $t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ is declared to have degree $\alpha_0 + \alpha_1 + \cdots \alpha_n$. We call $k[t_0, \dots t_n]_d$ is the set of **homogeneous elements of degree d**. Let $f \in k[t_0, \dots t_n]$, and let $[t_0 : \dots : t_n]$ be some homogeneous coordinates for a point $\underline{t}$ in $\mathbb{P}^n_k$. For varying $\lambda \neq 0$, the tuple $[\lambda t_0 : \dots : \lambda t_n]$ defines the same point $\underline{t} \in \mathbb{P}^n_k$. If we just try to plug in the homogeneous coordinates into f, we have that $f(t_0, \dots t_n) \neq f(\lambda t_0, \dots \lambda t_n)$ in general. However, if f is homogeneous of degree d, we have that:

$$f(\lambda t_0, \dots \lambda t_n) = \lambda^d f(t_0, \dots t_n)$$

and since $\lambda \neq 0$, f has a well-defined notion of vanishing on homogeneous coordinates. So for some collection S of homogeneous elements of $k[t_0, \dots t_n]$ (possibly of different degrees), we can define $V(S)$ to be the set of common zeroes of elements of S as before, and we call such sets **projective algebraic sets**. Irreducible algebraic sets will be called **projective varieties** and an open subset of a projective variety will be called a **quasi-projective variety**.

Exercise 3.1.1. Verify that projective algebraic sets form a topology on $\mathbb{P}^n_k$.

Exercise 3.1.2. Describe the topology on $\mathbb{P}^1_k$.

Important Exercise 3.1.3. Verify that the maps described above $U_i \to \mathbb{A}^n_k$ are homeomorphisms with respect to the subspace topology on $U_i \subseteq \mathbb{P}^n_k$. This implies the following:

Corollary 3.1.1. *All affine varieties are quasi-projective, and furthermore all quasi-projective varieties admit an open cover by affine varieties!*

A first instinct then is to seek an example of a quasi-projective which is not affine.

Exercise 3.1.4. Show that $\mathbb{A}^n_k \setminus (0,0)$ is not an affine variety.

3.2 Regular Functions and Morphisms

A first issue with projective varieties is that speaking of their algebraic functions is slightly more complicated. There is no nice set of global functions for a projective variety: one could look at the **homogeneous coordinate ring** $k[t_0, \dots t_n]/I(X)$, where $I(X)$ is defined as in the affine case, but this ring in general isn't even isomorphism invariant. Without knowing what morphism are, it should be believable that the map $\mathbb{P}^1_k \to \mathbb{P}^2_k$ given by:

$$[t_0 : t_1] \mapsto [t_0^2 : t_0 t_1 : t_1^2]$$

is a morphism. In fact, it's an isomorphism onto its image. But the homogeneous coordinate ring of $\mathbb{P}^1_k$ is just $k[t_0, t_1]$, and the homogeneous coordinate ring of the image of the map is $k[t_0, t_1, t_2]/(t_0t_2 - t_1^2)$, and these rings are not isomorphic! Furthermore, speaking of morphisms is slightly trickier, there is no nice (geometry) $\simeq$ algebra^{op} equivalence of categories, and the coordinates are slightly more complicated. We will fix all of this with a suitable notion of "regular functions".

First, we address the case of a regular function on an affine variety in $\mathbb{A}_k^n$. We say a function $f : X \to k$ is **regular** at a point $p \in X$ if there is a neighborhood U of p in X and polynomials $g, h \in k[t_1 \dots t_n]$ such that h does not vanish on U and $f|_U = g/h$, i.e. "f is locally given by a rational function". A function is said to be regular on a subset Z of X if it's regular at every point of Z. Note that for a fixed open set $U \subseteq X$ the regular functions on U form a ring.

Example 3.2.1. The function $t \mapsto 1/t$ is a regular function on $\mathbb{A}_k^1 \backslash \{0\} \subseteq \mathbb{A}_k^1$.

For quasiprojective varieties, we can do something similar. Let X be a quasiprojective variety in $\mathbb{P}_k^n$. Given $f : X \to k$, we say that f is regular at a point $p \in X$ if there exist two homogeneous polynomials of equal degree $g, h \in k[t_0, \dots t_n]$, with h nonvanishing in a neighborhood U of p, such that $f|_U = g/h$. (Make you sure you understand why we demanded that g and h have the same degree!)

Exercise 3.2.1. Show that the two definitions of regular function agree, meaning that if X is quasi projective, and $f : X \to k$, and $p \in X$, and U is some open affine neighborhood of p, then f being regular at p as a function on the affine U is the same as being regular at p as a function on a quasiprojective variety.

Proposition 3.2.2. *Any regular function $f : X \to k$ is continuous in the Zariski topology.*

Proof. It suffices to show that for a single point $a \in k = \mathbb{A}_k^1$, $f^{-1}(a)$ is closed. Now, $f^{-1}(a)$ is closed if there is an open cover of X such that, for any open U in the cover, $f^{-1}(a) \cap U$ is closed in U. Thus, we can restrict to an open U along which $f = g/h$ for some polynomials g and h. But then $f^{-1}(a) \cap U = V(g - ah) \cap U$, which is closed in U. $\square$

We are already in fact familiar with many regular functions.

Proposition 3.2.3. *Let X be an affine variety. Then the set of regular functions on X is exactly $\mathcal{O}(X)$, the coordinate ring. Furthermore, if f is an element of $\mathcal{O}(X)$, then the regular function on $D(f)$ are $\mathcal{O}(X)_f$, the localization of $\mathcal{O}(X)$ at X.*[24]

Exercise 3.2.2. Describe the regular functions on an arbitrary open subset of $\mathbb{P}^1_k$. Hint: there are really 3 distinct cases.

On the one hand, it's clear that any element of the coordinate ring defines a regular function. That these are all the regular functions requires some more computations which we omit, see [**Hart**] chapter 1, section 3 for full details. Because of this fact, we will write $\mathcal{O}(U)$ for the regular functions on U.

Now that we have defined regular functions, we are in a position to define a morphism of varieties. Let X and Y be two quasiprojective varieties. A map $\varphi : X \to Y$ is a **morphism** if it's continuous, and for any open subset $U \subseteq Y$ and regular function f on U, $f \circ \varphi$ is a regular function on $\varphi^{-1}(U)$.

Initially this may look strange, so we will show that this agrees with our original definition of morphisms for affine varieties. We will use the following:

Proposition 3.2.4. *Let X be a quasiprojective variety and $Y \subseteq \mathbb{A}^n_k$ be an affine variety. A map $\varphi : X \to Y$ is a morphism iff $t_i \circ \varphi$ is a regular function on X for each i, where $t_i : Y \to \mathbb{A}^1_k$ denotes the i^{th} coordinate function.*

Proof. If φ is a morphism, then by definition $t_i \circ \varphi$ is regular. On the other hand, if $t_i \circ \varphi$ is regular for every i, then $f \circ \varphi$ is regular for any polynomial f, but then since regular functions are continuous and the topology on Y is

---

[24] Recall that the localization of a ring R at an element $f \in R$ is the ring obtained from R by "formally inverting f": its elements are of the form g/f^n for some $g \in R$, $n \geq 0$, and it comes with a map $l : R \to R_f$ sending g to $g/1$. For R an integral domain, the ring structure is given by the usual algebra of fractions. We note that R_f may be defined via the following universal property: the map $R \xrightarrow{l} R_f$ is initial among ring maps sending $f \in R$ to a unit, i.e. for any ring map $\varphi : R \to A$ sending f to a unit, there is a unique dotted arrow making the following triangle commute:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "l"'] & A. \\ R_f \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

induced by the topology of $\mathbb{A}_k^n$ which is given by the vanishing of polynomials, φ is continuous. Furthermore, since regular functions are locally of the form g/h for polynomials g and h, it follows that regular functions pull back to regular functions. $\square$

Important Exercise 3.2.3. Using the previous two propositions, show that the original definition of a morphism of algebraic sets and a morphism of quasiprojective varieties agree.

3.3 Big Takeaways

In the first place, algebraic geometry starts with just solving polynomial equations in n-space, and immediately then one can arrive at a powerful equivalence between certain rings and these spaces of solutions. Then, one can "glue" up affine varieties in projective space, which encompasses many important examples, and one works with these things by making use of the fact they locally resemble affine varieties. Furthermore, just as we can study affine varieties via their global functions, we can study more general objects by understanding regular functions on arbitrary opens. This important idea will lead us to the notion of a sheaf.

4 The Zariski Spectrum of a Ring

We now set out to make precise the motto that every commutative ring R should be interpretable as the ring of functions on some geometric space $\operatorname{Spec} R$, with rings of functions pulling back under continuous maps. Our goal is to ultimately make sense of $\operatorname{Spec} R$ in full generality, including for rings of an arithmetic nature such as $\mathbb{Z}$ or the ring of integers of a number field. We nonetheless begin by recalling the following classical story from topology for motivation:

4.1 Gelfand's theorem

Start life with a compact Hausdorff space X, and consider the associated commutative ring of continuous complex-valued functions $C^0(X) = \{X \xrightarrow{f} \mathbb{C}\}$, which is in fact a $\mathbb{C}$-algebra. In the early 20th century, Gelfand observed that the space X could be fully reconstructed starting only with the data of $C^0(X)$ - a sort of preamble to the later development of modern algebraic geometry.

The construction proceeds as follows: to each point $x \in X$, there corresponds an evaluation map

$$C^0(X) \xrightarrow{ev_x} \mathbb{C}$$

$$f \mapsto f(x),$$

which is necessarily surjective since constant functions are continuous. Thus, by the first isomorphism theorem, we are guaranteed that the kernel

$$\mathfrak{m}_x := \ker(ev_x) = \{f \in C^0(X) \mid f(x) = 0\}$$

is a *maximal ideal* of $C^0(X)$. Notice that distinct points $x \neq y$ will always produce distinct maximal ideals $\mathfrak{m}_x \neq \mathfrak{m}_y$ thanks to Urysohn's lemma from point-set topology. Thus, the set

$$\mathrm{mSpec}(C^0(X)) := \{\mathfrak{m} \subset C^0(X) \mid \mathfrak{m} \text{ is a maximal ideal}\}$$

admits a natural injection $X \hookrightarrow \mathrm{mSpec}(C^0(X))$, $x \mapsto \mathfrak{m}_x$. Next, we may equip $\mathrm{mSpec}(C^0(X))$ with the coarsest topology making all of the canonical functions

$$\begin{aligned} \mathrm{mSpec}(C^0(X)) &\xrightarrow{f} \mathbb{C}, \\ &\mathfrak{m}_x \mapsto f \pmod{\mathfrak{m}_x} \in C^0(X)/\mathfrak{m}_x \cong \mathbb{C}, \end{aligned}$$

continuous, as f ranges over $C^0(X)$. Notice that under the identification $ev_x : C^0(X)/\mathfrak{m}_x \xrightarrow{\cong} \mathbb{C}$, $f \pmod{\mathfrak{m}_x}$ is identified with $f(x)$.

Remark 1. Compare this with the remark made in section **??**, wherein the Zariski topology on affine n-space was defined analogously, only with $k[t_1, ..., t_n]$ playing the role of $C^0(X)$.

We are now ready to state Gelfand's theorem:

Theorem 4.1.1. *Let X be a compact Hausdorff space. Then the canonical map*

$$X \to \mathrm{mSpec}(C^0(X))$$

is a homeomorphism.

Another, more categorical, perspective on the fact that $C^0(X)$ "fully determines" X is the observation that the "ring of functions" functor

$$\begin{gathered}\mathrm{CpctHaus} \xrightarrow{C^0(-)} \mathrm{CAlg}_{\mathbb{C}}^{op} \\ X \mapsto C^0(X) \\ (X \xrightarrow{\varphi} Y) \mapsto (C^0(Y) \xrightarrow{\varphi^*} C^0(X))\end{gathered}$$

is *fully faithful*, meaning that it induces bijections between continuous maps $X \xrightarrow{\varphi} Y$ and the corresponding $\mathbb{C}$-algebra homomorphisms $C^0(Y) \xrightarrow{\varphi^*} C^0(X)$ obtained under pullback. Here and elsewhere, the "geometry to algebra" correspondence at the level of morphisms is given by the following:

Blueprint 4.1.2.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi^*(f)"'] & Y \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

It is furthermore straightforward to characterize those commutative $\mathbb{C}$-algebras which appear in the essential image of the above functor: notice that each ring of functions $C^0(X)$ comes equipped with an involution $C^0(X) \to C^0(X)$ induced from complex conjugation on $\mathbb{C}$. The pair $(C^0(X),)$ satisfies a collection of axioms making it a (commutative, unital) C^*-algebra. It then turns out that the functor $C^0(-)$ factors through an equivalence of categories:

$$C^0(-) : \mathrm{CpctHaus} \xrightarrow{\simeq} \big(\text{comm. unital } C^*\text{-algebras}\big)^{op}.$$

This story was one of the earliest instances of the (Geometry) $\simeq$ $(\text{Algebra})^{op}$ principle, which is at the heart of modern algebraic geometry.

The game we just played was to pass from geometry (compact Hausdorff spaces) to algebra (commutative C^*-algebras over $\mathbb{C}$). In section 2.3, we likewise passed from geometry (affine varieties defined over an algebraically closed field k) to algebra (commutative finitely-generated reduced k-algebras). We now aim to go the other way: starting with the category of commutative algebras over a fixed commutative ring k, we seek to build a geometric category AffSch_k admitting an equivalence

$$\mathrm{AffSch}_k \xrightarrow{\simeq} \mathrm{CAlg}_k^{op}$$

obtained by passing to "rings of functions". We further expect this equivalence to be retro-compatible with Theorem 2.3.2, in the sense that we should have a "commutative square":

$$\begin{tikzcd}
\mathrm{AffSch}_k \arrow[r, "\simeq"] & \mathrm{CAlg}_k^{op} \\
\mathrm{AffVar}_k \arrow[u] \arrow[r, "\simeq"] & \big(\text{comm. f.g. reduced } k\text{-algebras}\big)^{op} \arrow[u]
\end{tikzcd}$$

Remember our original motivation for this endeavor: the study of solution spaces for polynomial equations $f_1, ..., f_r \in k[t_1, ..., t_n]$ with coefficients in k is captured by looking at maps from the k-algebra $R := k[t_1, ..., t_n]/(f_1, ..., f_r)$ to various k-algebras A, thought of as the admissible values for the solutions. Conversely, any k-algebra may be interpreted as the quotient of some polynomial ring over k by picking generators. As such, our task is to **construct geometric avatars for spaces of solutions to polynomial equations**, keeping track of all spaces of solutions at once. As was discussed in section 1.3, this broadening of the algebro-geometric settings allows us to account for multiplicities, nilpotent information, and arithmetic phenomena.

4.2 From Gelfand to Zariski

Start life with a given commutative k-algebra R, thought of as some quotient $R = k[(x_i)_{i\in I}]/(S)$ corresponding to a system of polynomial equations $S = \{f_j\}_{j\in J} \subseteq k[(x_i)_{i\in I}]$. Recall from the first lecture that we were naturally led to consider the functor

$$V_R : \mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$$
$$A \mapsto V_R(A)$$

where

$$V_R(A) = \{\underline{a} \in A^I \mid f_j(\underline{a}) = 0 \ \forall j\} = \mathrm{Hom}_{\mathrm{CAlg}_k}(R, A)$$

and functoriality was given by substitution. Ideally, we would hope to be able to work with these functors directly as "geometric objects" in their own right, starting for instance by determining those (abstractly given) functors $\mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$ which "locally look like V_R for some commutative ring R". This is a workable approach known as the functor of points (FOP) perspective, but it is rather difficult to set up the theory along those lines (although not impossible, see for instance [**Raskin**] or [**DemazureGabriel**]). It is nonetheless good to keep the FOP perspective at hand, as it sheds light on certain notions which can seem complicated from the approach we are about to describe (the locally ringed space perspective).

Taking Gelfand's theorem as a starting point, let us try to "reconstruct" a geometric space from the data of R. In the Gelfand settings, we had $R = C^0(X)$, and the collection of evaluation maps $ev_x : R \to \mathbb{C}$ "witnessed the whole space", with a maximal ideal of R standing in for each point. In our settings, with $R = k[(x_i)_{i \in I}]/(f_j)$ we need to be a little more careful:

- Solutions for the system (f_j) can take values in any k-algebra A, and so a priori we would need to account for more general "evaluation maps" $R \to A$, for arbitrary $A \in \mathrm{CAlg}_k$ - we call these "A-points", to emphasize the intuition that such a ring map "singles out" a point of the A-valued solution space $V_R(A)$. Fortunately, it turns out to be enough for us to only keep track of field valued points

  $$R \to K, \text{ for } K \in \mathrm{CAlg}_k \text{ a field.}$$

  The idea is that, to each A-point $R \to A$, we can associate a family of field valued points

  $$\begin{tikzcd} & & A/\mathfrak{m}_1 \\ R \arrow[r] & A \arrow[ur] \arrow[r] \arrow[dr] & A/\mathfrak{m}_2, \\ & & \cdots \end{tikzcd}$$

  as the $\mathfrak{m}_i$'s range over all maximal ideals of A, and that together these field valued points determine the A-point $R \to A$ "set-theoretically".

- In the Gelfand set-up, all ring maps $C^0(X) \to \mathbb{C}$ were surjections due to the presence of constant maps for every $z \in \mathbb{C}$. This is no longer the case in our settings, as we wish to accommodate for arithmetic situations such as $R = \mathbb{Z}[x]/(x^2 - 2)$, admitting the $\mathbb{Q}$-point

  $$\mathbb{Z}[x]/(x^2 - 2) \to \mathbb{Q}, \; x \mapsto \sqrt{2},$$

  with image $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subsetneq \mathbb{Q}$. Nonetheless, for any field valued point $R \xrightarrow{\varphi} K$, we are guaranteed that $\mathrm{im}(\varphi) \subseteq K$ is a subring of a field, hence an integral domain, and since $\mathrm{im}(\varphi) \cong R/\ker(\varphi)$, it follows that $\ker(\varphi)$ will always be a prime ideal.

Exercise 4.2.1. Verify that, conversely, every prime ideal $\mathfrak{p}$ of R may be expressed as $\ker(\varphi)$ for some field valued point $R \xrightarrow{\varphi} K$.

Therefore, we arrive to a surjective map

$$\big\{\text{Field valued points } R \to K\big\} \twoheadrightarrow \big\{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal}\big\}$$

The analogous map in the Gelfand set-up was determined to be injective using Urysohn's lemma. In our set-up, since the target field may vary, we first need to make sure we don't double count points: a K-point will also correspond to an L-point for any field extension L/K, and we should make sure that these are identified. To this end, we introduce the following relation on the set of field valued points of R: identify $(R \to K) \sim (R \to K')$ whenever we can find a third field valued point $(R \to L)$ together with a commutative diagram:

$$\begin{tikzcd}
 & L & \\
K \arrow[ur] & & K' \arrow[ul] \\
 & R \arrow[ul] \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Exercise 4.2.2. (1) Check that this is an equivalence relation.
(2) Prove that the resulting map

$$\big\{\text{Field valued points } R \to K\big\}/\sim \longrightarrow \big\{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal}\big\}$$

is now a bijection.

Therefore, the set

$$\operatorname{Spec} R := \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal}\}$$

is in 1-1 correspondence with the set of all (singly counted) field valued points associated to the system of equations (f_j), and therefore appears to be the natural substitute for the set $\mathrm{mSpec}(C^0(X))$ appearing in Gelfand's theorem.

Next, let us put a topology on $\operatorname{Spec} R$, once again following the spirit of Gelfand's construction. There, each element $f \in C^0(X)$ was interpreted as a function on the set $\mathrm{mSpec}(C^0(X))$ via the assignment

$$\begin{aligned} \mathrm{mSpec}(C^0(X)) &\xrightarrow{f} \mathbb{C}, \\ \mathfrak{m}_x &\mapsto f \pmod{\mathfrak{m}_x} \in C^0(X)/\mathfrak{m}_x \cong \mathbb{C}, \end{aligned}$$

and we then gave $\mathrm{mSpec}(C^0(X))$ the coarsest topology making each such map continuous.

The same approach carries over to our settings, modulo a couple modifications. The picture is the following:

Blueprint 4.2.1. Any $f \in R$ may be interpreted as a function on $\operatorname{Spec} R$, namely

$$f : \operatorname{Spec} R \to \bigsqcup_{\mathfrak{p} \in \operatorname{Spec} R} \kappa(\mathfrak{p}). \tag{4}$$

To make sense of this, let us start with an element $f \in R$, to be "evaluated" at a point $\mathfrak{p} \in \operatorname{Spec} R$

- At first glance, we may want to set the "value" of f at $\mathfrak{p}$ to be $f \pmod{\mathfrak{p}} \in R/\mathfrak{p}$. However, for general $\mathfrak{p}$, the quotient $R/\mathfrak{p}$ is only guaranteed to be a domain. We can nonetheless always pass to the corresponding fraction field, $R/\mathfrak{p} \hookrightarrow \mathrm{Frac}(R/\mathfrak{p}) =: \kappa(\mathfrak{p})$, called the **residue field** of $\operatorname{Spec} R$ at $\mathfrak{p}$, and declare the value of f at $\mathfrak{p}$ to be

  $$f(\mathfrak{p}) := f \pmod{\mathfrak{p}} \in \kappa(\mathfrak{p}).$$

  This elucidates the notation in equation 4.

- Unlike $\mathbb{C}$, the field $\kappa(\mathfrak{p})$ may not have a natural topology, and it may vary from point to point (for instance, $\operatorname{Spec}(\mathbb{Z})$ has residue field $\mathbb{F}_p$ at each prime $p \in \operatorname{Spec}(\mathbb{Z})$, and $\mathbb{Q}$ at $0 \in \operatorname{Spec}(\mathbb{Z})$). So instead of imposing a continuity condition, we settle for the demand that $f^{-1}(0) =: V(f)$ be a closed subset of $\operatorname{Spec} R$ for each $f \in R$. Notice that, by definition,

  $$V(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f(\mathfrak{p}) = 0\} = \{\mathfrak{p} \ni f\}.$$

Thus, we define the **Zariski topology** on $\operatorname{Spec} R$ to be the coarsest topology for which each vanishing locus $V(f) \subseteq \operatorname{Spec} R$ is closed. This is equivalent to declaring that the collection of subsets $V(S)$ are the closed subsets for the Zariski topology on $\operatorname{Spec} R$, where, given a collection of elements $S \subseteq R$, we analogously define

$$V(S) := \{\mathfrak{p} \supset S\} = \bigcap_{f \in S} V(f).$$

By taking complements $D(f) := \operatorname{Spec} R - V(f)$ for each $f \in R$, we obtain a canonical basis for the Zariski topology of $\operatorname{Spec} R$, called the basis of **principal opens** or **basic opens**. Each $D(f) \subseteq \operatorname{Spec} R$ may be thought of as the set of points of $\operatorname{Spec} R$ along which f does not vanish, in the sense we have been employing above:

$$D(f) := \operatorname{Spec} R - V(f) = \{\mathfrak{p} \not\ni f\} = \{\mathfrak{p} \in \operatorname{Spec} R \mid f(\mathfrak{p}) \neq 0\}.$$

Notice that the assignment $R \mapsto \operatorname{Spec} R$ may be made contravariantly functorial in R. Namely, to any k-algebra map $\varphi : R \to A$, we may associate a continuous map:

$$F : \operatorname{Spec} A \to \operatorname{Spec} R$$
$$\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$$

Exercise 4.2.3. (1) Check that $F : \operatorname{Spec} A \to \operatorname{Spec} R$ is indeed continuous. (2) Show that any surjective k-algebra homomorphism $\varphi : R \twoheadrightarrow A$ induces an *injective* map of spaces $F : \operatorname{Spec} A \hookrightarrow \operatorname{Spec} R$ (this is starting to give some credence to the story told in the thought experiment 2 from the introduction).

Exercise 4.2.4. (Working with a basis)
(1) Check that the topology defined by the $V(S)$'s satisfies the necessary axioms, and that the family $\{D(f)\}_{f \in R}$ does indeed form a basis.
(2) Prove the following:

Proposition 4.2.2. *Given a family of elements $\{f_j\}_{j \in J} \subseteq R$, the basic opens $\{D(f_j)\}_{j \in J}$ form an open cover of* $\operatorname{Spec} R$ *iff* $1 \in (\{f_j\}_{j \in J})$.

Thus, *basic open covers correspond to partitions of unity.*
(3) Deduce that $\operatorname{Spec} R$ is always a quasi-compact topological space.

Extended Exercise 4.2.5. (Basic topological properties of $\operatorname{Spec} R$**)**

(1) Show that $\operatorname{Spec} R$ has the following property: given any two distinct points $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$, we may find an open set in $\operatorname{Spec} R$ containing one of the points but not the other[25].

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[25] This is called the Kolmogorov property, and guarantees for instance that every irreducible closed subset $Z \subset \operatorname{Spec} R$ admits a unique **generic point**, i.e. a unique point $\eta \in \operatorname{Spec} R$ such that $\overline{\{\eta\}} = Z$.

(2) Show that we have a bijection:

$$\operatorname{Spec}(R/I) \xrightarrow{\cong} V(I) \subseteq \operatorname{Spec} R$$

(in fact, a homeomorphism with respect to the subspace topology on $V(I)$) induced by the quotient map $R \to R/I$. The point η from the previous note may therefore be thought of as hovering over $V(I)$ as a "shadow" of the "closed subvariety" $\operatorname{Spec}(R/I) \hookrightarrow \operatorname{Spec} R$.

(3) Given $f \in R$, denote the *localization of* R *at* f by R_f. Show that we have a bijection:

$$\operatorname{Spec}(R_f) \xrightarrow{\cong} D(f) \subseteq \operatorname{Spec} R$$

(again, this is in fact a homeomorphism with respect to the subspace topology on $D(f)$) induced by the localization structure map $R \to R_f$.

(4) Let $\mathcal{N}(R) = \{f \in R \mid f^N = 0, \text{ some } N \geq 1\}$ denote the nilradical of R. Show that we have a homeomorphism $\operatorname{Spec}(R/\mathcal{N}(R)) \cong \operatorname{Spec} R$. That is, $\operatorname{Spec}(R)$ as a topological space is insensitive to reduced/infinitesimal information.

(5) Feel free to explore/discuss additional aspects of the topology of $\operatorname{Spec} R$ and its relationship to the algebraic properties of R, e.g. irreducibility/irreducible components, Noetherianness, the geometric significance of the Jacobson radical/nilradical, etc...

[Recap] - Looking back and looking forward

Let us specialize to the case $R = k[t_1, ..., t_n]/(f_1, ..., f_r)$ with algebraically closed ground field $k = \bar{k}$, and consider the set of k-points

$$\operatorname{Spec} R(k) = \operatorname{Hom}_{\mathrm{CAlg}_k}(R, k).$$

By definition, the classical affine algebraic set $V(f_1, ..., f_r)$ introduced in Section 2 is given by:

$$V(f_1, ..., f_r) = \{\underline{a} \in k^n \mid f_j(\underline{a}) = 0 \ \forall j\} \cong \operatorname{Hom}_{\mathrm{CAlg}_k}(R, k) = \operatorname{Spec} R(k).$$

Exercise 4.2.6. Using Zariski's lemma, show that we have a bijection

$$\operatorname{Spec} R(k) \xrightarrow{\cong} \{\text{Maximal ideals of } R\} =: \operatorname{mSpec}(R),$$

and check that a point of $\operatorname{Spec} R$ is closed iff the corresponding prime ideal is maximal.

Therefore, the classical affine algebraic set $V(f_1, ..., f_r)$ may be recovered precisely as the set of k-points $\operatorname{Spec} R(k)$, or equivalently as the set of *closed points* of the topological space $\operatorname{Spec} R$.

Exercise 4.2.7. Show that the resulting identification

$$V(f_1, ..., f_r) \xrightarrow{\cong} \mathrm{mSpec}(R)$$

is in fact a homeomorphism with respect to the classical Zariski topology on $V(f_1, ..., f_r)$ and the subspace topology on $\mathrm{mSpec}(R)$.

Now, recall that we set out to build a geometric object "Spec R" out of an arbitrary k-algebra R in such a way as to establish an equivalence of categories

$$\mathrm{CAlg}_k \xrightarrow{\text{"Spec"}} \text{(Geometric category)}.$$

Already by part (4) of this section's extended exercise, we saw that $\operatorname{Spec}(-)$ as a functor into topological spaces produces the same answer on a ring and its quotient by its nilradical, so that e.g. k and $k[t]/(t^2)$ are indistinguishable. Moreover, since any field has only one prime ideal (0), our current version of Spec is also unable to distinguish between k and any field extension L/k thereof. In order to amend for these shortcomings, we draw inspiration from manifold theory and further equip $\operatorname{Spec} R$ with the extra structure of regular functions on it, in a way that allows for local to global manipulations. This requires developing the language of sheaf theory[26], which we do in the next section.

Remark 2. The structure of the above section was particularly influenced by the introduction to the Springer edition of EGA I **[Groth1]** - the reader is invited to consult this reference for more details.

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[26] This is not quite true: we could get away simple-mindedly by declaring that one should view $\operatorname{Spec} R$ as a topological space together with R as its ring of global regular functions, and this would be enough data, essentially because global sections fully determine the structure sheaf in the affine settings. But as we shall see in the next section, if we are to use affine schemes as building blocks for our "brave new geometry", having access to local information is crucial in order to be able to carry out the necessary gluing procedures.

5 Some Sheaf Theory

5.1 Motivation: revisiting the notion of a smooth manifold

So far in our discussion, we have associated to any given commutative ring R a topological space $\operatorname{Spec} R$, which we wish to enhance into a geometric object admitting a notion of regular functions on it. To this end, let us draw inspiration from the classical notion of a smooth manifold. Recall that a smooth n-manifold consists of a (Hausdorff, second countable) topological space M^n, with the property that we can find an open cover $\{U_\alpha\}_\alpha$ of M, together with homeomorphisms

$$\varphi_\alpha : U_\alpha \xrightarrow{\cong} \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$$

onto open subsets of euclidean space, such that each transition function

$$\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \xrightarrow{\cong} \varphi_\beta(U_{\alpha\beta})[27]$$

is a diffeomorphism. We call each U_α an *affine chart* (or *affine open*) of M, and we think of it as canonically equipping M with a smooth structure along U_α. The condition on the transition functions ensures that these local smooth structures are compatible with one another.

When defining the notion of *smooth functions* on M, we make use of the following properties:

- Smoothness is a *local property*, i.e. it can be checked near each point.
- Locally near each point, M is canonically identified with an *open in Euclidean space*, a geometric space where smoothness is understood.
- A continuous function is uniquely determined by its restrictions to any open cover.

Exercise 5.1.1. Verify the third item of the above list. Namely, given two topological spaces X and Y and an open cover $\{U_\alpha\}_\alpha$ of X, show that any compatible family of continuous maps $f : U_\alpha \to Y$ (in the sense that $f_\alpha|_{U_{\alpha\beta}} =$

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[27] Here and elsewhere, we use the shorthand notation $U_{\alpha\beta} := U_\alpha \cap U_\beta$.

$f_\beta|_{U_{\alpha\beta}}$ for all α, β) glues uniquely to a continuous map $f : X \to Y$ (in the sense that $f|_{U_\alpha} = f_\alpha$ for all α)[28].

Together, these properties suggest that, in order to determine whether a given continuous function $f : M \to \mathbb{R}$ is smooth, it suffices to ensure that each of its restrictions $f|_{U_\alpha}$ to an affine open $U_\alpha \subseteq M$ is smooth *when considered as a continuous function between Euclidean spaces* via the canonical identification $\varphi_\alpha : U_\alpha \xrightarrow{\cong} \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$. Concretely, the restriction of f to U_α fits into the following triangle:

$$\begin{tikzcd}
U_\alpha \arrow[r, "\varphi_\alpha", "\cong"'] \arrow[dr, "f|_{U_\alpha}"'] & \varphi_\alpha(U_\alpha) \subseteq \mathbb{R} \arrow[d, "f \circ \varphi_\alpha^{-1}"] \\
& \mathbb{R}.
\end{tikzcd}$$

and we define f to be **smooth** if, and only if, for each affine open $U_\alpha \subseteq M$, the map between Euclidean spaces $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ corresponding to the restriction $f|_{U_\alpha}$ is smooth. We denote by $\mathcal{O}(M)$ the set of smooth functions on M.

Note 5.1.1. The definition we just gave applies to arbitrary smooth manifolds. In particular, we may view any open subset $U \subseteq M$ as a smooth manifold with the smooth structure induced from M, and consider the corresponding set of smooth functions $\mathcal{O}(U)$ on U.

Exercise 5.1.2. (1) Check that $\mathcal{O}(M)$ naturally comes equipped with an $\mathbb{R}$-algebra structure.
(2) Check that, locally on each affine open U_α, smooth functions may equivalently be defined as those continuous functions $f : U_\alpha \to \mathbb{R}$ which can be expressed as the *pullback of a smooth function under* φ_α. That is, we have a canonical bijection:

$$\varphi_\alpha^* : \mathcal{O}(\varphi_\alpha(U_\alpha)) \xrightarrow{\cong} \mathcal{O}(U_\alpha).$$

This local-to-global definition of smooth functions is a manifestation of the following, context-agnostic:

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[28] Throughout this section, we adopt the convention of denoting smooth manifolds by M, N, ..., while denoting general topological spaces by X, Y, ... when introducing notions/properties that hold beyond the smooth context - in particular, properties that will immediately translate to the algebro-geometric context).

Principle. Local notions on a geometric space can be formulated and worked with internally to the affine models.

Let us take a moment to formally define the mathematical structure instantiated by the various $\mathcal{O}(U)$'s as U ranges over various open subsets of M:

Exercise 5.1.3. Let $\mathrm{Op}(M)$ denote the *poset category* whose objects are the open subsets of M, and whose morphisms are given by

$$\mathrm{Hom}_{\mathrm{Op}(M)}(U,V) := \begin{cases} \{U \hookrightarrow V\}, & \text{if } U \subseteq V \\ \emptyset & , \text{ otherwise.} \end{cases}$$

(1) Check that the assignment $U \mapsto \mathcal{O}(U)$ assembles into a functor:

$$\mathcal{O} : \mathrm{Op}(M)^{op} \to \mathrm{CAlg}_{\mathbb{R}},$$

with functoriality given by restriction of functions. We call functors of this form **presheaves of $\mathbb{R}$-algebras on** M.
(2) Check that the definition of smooth functions we gave above satisfies the following categorical condition, which we will call the **unique gluing condition**: for any open cover $\{U_\alpha\}_\alpha$ of M, the following sequence of $\mathbb{R}$-vector spaces is exact[29]:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(M) \arrow[r] & \prod_\alpha \mathcal{O}(U_\alpha) \arrow[r] & \prod_{\alpha,\beta} \mathcal{O}(U_\alpha \cap U_\beta) \end{tikzcd} \tag{5}$$

Here, the first map is given by $f \mapsto (f|_{U_\alpha})_\alpha$, and the second by $(f_\alpha)_\alpha \mapsto (f_\alpha|_{U_{\alpha\beta}} - f_\beta|_{U_{\alpha\beta}})_{\alpha,\beta}$. Thus, for any such open cover, the first map in 5 induces a canonical bijection:

$$\mathcal{O}(M) \xrightarrow{\cong} \left\{ \begin{matrix} \text{Compatible families} \\ (f_\alpha)_\alpha \in \prod_\alpha \mathcal{O}(U_\alpha) \end{matrix} \right\}.$$

A presheaf of $\mathbb{R}$-algebras satisfying the unique gluing condition 5 for every choice of open cover $\{U_\alpha\}_\alpha$ of an arbitrary open subset $U \subseteq M$ is called a **sheaf of $\mathbb{R}$-algebras on** M.

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[29] Recall that, for any commutative ring R, a sequence of R-module maps

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is said to be **exact** if $\ker\psi = \mathrm{im}\varphi$. This in particular encapsulates injective maps by setting $M = 0$, and surjective maps by setting $P = 0$. Longer sequences are said to be exact if they are exact at each inner term.

The philosophy which we would like to turn into a formal statement is that the pair $(M, \mathcal{O})$, where M is regarded as a topological space equipped with the sheaf of $\mathbb{R}$-algebras $\mathcal{O}$, *completely characterizes the smooth structure on the smooth manifold* M. For this reason, we call $\mathcal{O}$ the **structure sheaf** of M.

To this end, let us see what the notion of a *smooth map* between smooth manifolds corresponds to in this language. Recall that a continuous map $F : M^n \to N^m$ is said to be *smooth* if it is "locally smooth with respect to the Euclidean structures on M and N", i.e. if, for each point $p \in M$, affine open neighborhood $(U_\alpha, \varphi_\alpha)$ containing p, and affine open neighborhood (V_β, ψ_β) containing $F(p)$, the composite

$$\psi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to \psi_\beta(V_\beta)$$

is smooth as a map between opens in Euclidean space.

Exercise 5.1.4. Show that we have a 1-1 correspondence between smooth maps $F : M \to N$ and continuous maps $F : M \to N$ with the property that, for each open $V \subseteq N$, pullback along F induces a valid map

$$F^* : \mathcal{O}_N(V) \to \mathcal{O}_M(F^{-1}(V)),$$

$$f \mapsto f \circ F.$$[30]

(Hint: recall that a map into an open subset of $\mathbb{R}^m$ is smooth iff each of its coordinate functions is smooth.)

This equivalent definition of the notion of a smooth map between manifolds may be readily formulated in sheaf-theoretic language:

Definition 5.1.2. Let $F : X \to Y$ be a continuous map between topological spaces, and let $\mathcal{F}$ be a sheaf on X. We define the **pushforward** of $\mathcal{F}$ under F to be the sheaf on Y given by the assignment

$$F_*\mathcal{F}(V) := \mathcal{F}(F^{-1}(V))$$

for each $V \in \mathrm{Op}(Y)$, and with restriction maps inherited from $\mathcal{F}$.

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[30] This is structurally identical to the definition of a regular function between quasiprojective varieties given in section 3.2!

Exercise 5.1.5. (1) Check that $F_*\mathcal{F}$ as defined above is indeed a sheaf on Y. (If it makes things more digestible, you could try to show this for the concrete case of the pushforward of the sheaf of smooth functions on M under some continuous map $M \to N$.)
(2) Check that the pullback maps $\{F^* : \mathcal{O}_N(V) \to \mathcal{O}_M(F^{-1}(V))\}_{V \in \mathrm{Op}(N)}$ assemble into a natural transformation of functors:

$$\mathcal{O}_N \xrightarrow{F^*} F_*\mathcal{O}_M.$$

Note 5.1.3. Since sheaves were defined as presheaves satisfying a certain property, and presheaves come to us as functors $\mathrm{Op}(M)^{op} \to \mathrm{CAlg}_{\mathbb{R}}$, it is natural for us to define a **sheaf morphism** to be a natural transformation between the corresponding functors. Thus, the previous exercise establishes that any smooth map $F : M \to N$ "decomposes" into a topological component, namely the map $F : M \to N$ viewed as a continuous map between topological spaces, and a smoothness component, namely the sheaf morphism $F^* : \mathcal{O}_N \to F_*\mathcal{O}_M$ given by pullback along F.

With the above example in mind, we may now introduce the following key notion:

Definition 5.1.4. The category $\mathcal{RS}$ of **ringed spaces** is the category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a topological space equipped with a sheaf of rings $\mathcal{O}_X$, and in which a morphism

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is given by the data of a continuous map $F : X \to Y$, together with a sheaf morphism $F^\# : \mathcal{O}_Y \to F_*\mathcal{O}_X$.

A few comments are in order. Unlike the sheaf of smooth functions on a smooth manifold, the "structure sheaf" $\mathcal{O}_X$ associated to a general ringed space X may not come to us as a subsheaf of the sheaf of continuous functions on X. As such, we cannot guarantee that the corresponding sheaf morphism $F^\#$ will actually be given by pullback along F - the notion of pullback may not even make sense at this level of generality. For our particular settings, this translates into the fact that the hom set between two smooth manifolds $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ viewed inside $\mathcal{RS}$ is *too large* - some ringed space morphisms do not correspond to any "geometric" smooth map $F : M \to N$. In order to adjust the target category accordingly, it is useful to keep track of the local behavior of *germs of smooth functions* near each point. These may be formally defined as follows:

Definition 5.1.5. Given a ringed space $(X, \mathcal{O}_X)$, define the **stalk**, or **ring of germs** of $\mathcal{O}_X$ at $x \in X$ to be the ring obtained as the direct limit

$$\mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U),$$

with structure maps given by restriction maps.

The ring $\mathcal{O}_{X,x}$ may be described more concretely as follows: we begin with the disjoint union $\bigsqcup_{U \ni x} \mathcal{O}_X(U)$, and identify $f_U \in \mathcal{O}_X(U)$ with $g_V \in \mathcal{O}_X(V)$ whenever they restrict to the same element in $\mathcal{O}_X(W)$ for some open subset $W \subseteq U \cap V$, i.e. $f_U|_W = g_V|_W$. A germ $f_x \in \mathcal{O}_{X,x}$ is then defined to be an equivalence class under this equivalence relation. In particular, we are always able to choose "local representatives" $f_U \in \mathcal{O}_X(U)$ for a given germ f_x, although they will be far from unique (e.g. any further restriction of f_U would also do). Notice that, conversely, any smooth function $f \in \mathcal{O}_X(U)$ gives rise to a germ $f_x \in \mathcal{O}_{X,x}$ for each $x \in U$. Conceptually speaking, the germ of f at x remembers the value of f at x as well as the "infinitesimal behavior of f" near x.

Let us focus our attention on the ring of germs of the structure sheaf $\mathcal{O}_M$ of a smooth manifold M at a given point $p \in M$. In this context, we have at our disposal an *evaluation map* at p, which makes sense as an $\mathbb{R}$-algebra map

$$\mathrm{ev}_p : \mathcal{O}_{M,p} \to \mathbb{R}$$

sending a germ f_x to the value $f(x)$ of some representative $f \in \mathcal{O}_M(U)$ of f_x. This map is well defined on germs since it only depends on the value of a given function at the point p.

Exercise 5.1.6. (1) Check that the ideal $\mathfrak{m}_p := \ker(\mathrm{ev}_p) \subseteq \mathcal{O}_{M,p}$ is maximal, and that it has the property that any $f_p \notin \mathfrak{m}_p$ is a *unit* (i.e. an invertible element) of the ring $\mathcal{O}_{M,p}$.
(2) Show that the above property implies that $\mathfrak{m}_p$ is the unique maximal ideal of $\mathcal{O}_{M,p}$.

We call commutative rings R with a unique maximal ideal $\mathfrak{m} \subseteq R$ **local rings**[31], and we usually denote by $\kappa := R/\mathfrak{m}$ the **residue field** to R. Thus,

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[31] This terminology reflects the fact that such rings are ubiquitous in the local theory of functions on geometric spaces. From the $\operatorname{Spec} R$ viewpoint, these rings have a *unique closed point*, which is a stand-in for the point at which we are studying germs of functions.

we see that *the stalk $\mathcal{O}_{M,p}$ of the structure sheaf of a smooth manifold at each point $p \in M$ is a local ring, with maximal ideal given by germs of functions that vanish at p.* This viewpoint provides meaningful geometric intuition for more general ringed spaces whose stalks are local rings:

Blueprint 5.1.6. Given a ringed space $(X, \mathcal{O}_X)$ whose stalk at $x \in X$ is a local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x, \kappa(x))$, we think of the following quotient map as *evaluating germs of functions at x*:

$$\mathcal{O}_{X,x} \to \kappa(x) \cong \mathcal{O}_{X,x}/\mathfrak{m}_x,$$

$$f_x \mapsto f(x) := f_x \pmod{\mathfrak{m}_x}.$$

Let us now come back to our original aim, which was to adequately restrict admissible morphisms of ringed spaces based on the behavior of pullbacks of smooth functions.

Exercise 5.1.7. (1) Let $(F, F^{\#}) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ be a morphism of ringed spaces. Show that, for each $p \in M$, the sheaf morphism $F^{\#}$ induces a well-defined map on stalks:

$$F_p^{\#} : \mathcal{O}_{N,F(p)} \to \mathcal{O}_{M,p}.$$

(Hint: use the fact that any germ $f_{F(p)} \in \mathcal{O}_{N,F(p)}$ may be represented by an "actual function" $f \in \mathcal{O}_N(V)$ on some neighborhood $V \ni F(p)$.)
(2) Check that, when $(F, F^*) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is the morphism of ringed spaces given by pullback under a smooth map F at the level of sheaves, the above maps on stalks each satisfy $F_p^*(\mathfrak{m}_{F(p)}) \subseteq \mathfrak{m}_p$.

Thus, the assignment $M \mapsto (M, \mathcal{O}_M)$, $F \mapsto (F, F^*)$ really produces a functor from the category of smooth manifolds to the subcategory $\mathcal{LRS} \subseteq \mathcal{RS}$ of **locally ringed spaces**, which is defined to have objects those ringed spaces $(X, \mathcal{O}_X)$ such that each stalk $\mathcal{O}_{X,x}$ is a local ring, and morphisms those ringed space morphisms $(F, F^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ with the additional property that the corresponding maps on stalks

$$F_x^{\#} : \mathcal{O}_{Y,F(x)} \to \mathcal{O}_{X,x}.$$

are *local ring maps* for each $x \in X$, i.e. that they satisfy $F_x^{\#}(\mathfrak{m}_{F(x)}) \subseteq \mathfrak{m}_x$. This condition may be thought of as requiring $F^{\#}$ to "respect vanishing of germs" at each point. The miracle is that this is a sufficient condition to impose on morphisms in order to restrict the admissible pairs $(F, F^{\#})$ to those that occur "geometrically":

Exercise 5.1.8. Show that *any* morphism $(F, F^{\#}) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between smooth manifolds viewed in the category of *locally ringed spaces* has the property that $F^{\#} = F^*$ is in fact given by pullback of smooth functions along F.
(Hint: It suffices to show that, for every smooth function $f \in \mathcal{O}_N(V)$ on some open $V \subseteq N$, and for every point $p \in F^{-1}(V)$, we have that $(F^{\#}f)(p) = f(F(p))$. Try to show this by passing to stalks!)

Hence, the assignment $M \mapsto (M, \mathcal{O}_M)$ produces a *fully faithful functor* into the category of locally ringed spaces $\mathcal{LRS}$, therefore establishing an equivalence of categories between the category Mfld of smooth manifolds and a full subcategory of $\mathcal{LRS}$.

Exercise 5.1.9. Show that the essential image of the functor

$$\text{Mfld} \to \mathcal{LRS}$$

$$M \mapsto (M, \mathcal{O}_M)$$

consists precisely of those locally ringed spaces $(X, \mathcal{O}_X)$ whose underlying topological space is Hausdorff and second countable, and which have the property that we can find an open cover $\{U_\alpha\}_\alpha$ together with isomorphisms of locally ringed spaces

$$(\varphi_\alpha, \varphi_\alpha^{\#}) : (U_\alpha, \mathcal{O}_X|_{U_\alpha}) \xrightarrow{\cong} (\varphi_\alpha(U_\alpha), \mathcal{O}_{\varphi_\alpha(U_\alpha)}),$$

where $(\varphi_\alpha(U_\alpha), \mathcal{O}_{\varphi_\alpha(U_\alpha)})$ denotes an open subset of Euclidean space equipped with its sheaf of smooth functions, and using the notation $\mathcal{O}_X|_{U_\alpha}$ to denote the **restriction of the sheaf** $\mathcal{O}_X$ to the open subset $U_\alpha \subseteq X$, which is the sheaf on U_α given by simply setting $\mathcal{O}_X|_{U_\alpha}(V) := \mathcal{O}_X(V)$ for each $V \in \text{Op}(U_\alpha)$.
(In particular, notice that smoothness of transition maps comes built in.)

Let us take stock of the above discussion. We have arrived at the conclusion that, in a precise (categorical) sense, the data of a smooth manifold M^n may be reorganized into the data of a locally ringed space $(M, \mathcal{O}_M)$, where now M is viewed simply as a topological space, equipped with a sheaf of rings $\mathcal{O}_M$ which fully encodes the smooth structure on M[32]. This suggests the following principle, where "smoothness" is thought of as a choice among a variety of possible "geometries" $\mathcal{G}$:

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[32] Strictly speaking, it would have been more "useful" to develop the locally ringed space

Principle. $\mathcal{G}$-geometry is concerned with the study of topological spaces X equipped with a "$\mathcal{G}$-structured" sheaf of rings $\mathcal{O}_X$.

This principle is inspired from [**Lurie**], wherein it is taken to its "logical conclusion" by serving to lay the foundations of $\mathcal{G}$-geometries beyond scheme theory[33]. For us, it will serve as a motivation for the development of scheme theory in parallel to manifold theory:

$\mathcal{G}$-Geometry	**$\mathcal{G}$-structured sheaf**	**$\mathcal{G}$-affine building blocks**	**$\mathcal{G}$-local theory**
Smooth manifolds M^n	Sheaf of smooth functions $\mathcal{O}_M$	Opens in Euclidean space $\Omega \subseteq \mathbb{R}^n$	Calculus
Complex manifolds X^n	Sheaf of holomorphic functions $\mathcal{O}_X$	Opens in complex n-space $\Omega \subseteq \mathbb{C}^n$	Complex analysis
Schemes X	Sheaf of regular functions $\mathcal{O}_X$	Affine schemes $\operatorname{Spec} R$	Commutative algebra

In this framework, which we will continue developing for the last row in the coming section, affine schemes will be to schemes as opens in Euclidean space are to smooth manifolds, suggesting the following picture[34]:

$$\begin{tikzcd}
\mathrm{Euclid} \arrow[d] \arrow[dr] & & \mathrm{CAlg}_k^{op} \arrow[dl] \arrow[d] \\
\mathrm{Mfld} \arrow[r] & \mathcal{LRS} & \mathrm{Sch}_k \arrow[l]
\end{tikzcd} \tag{6}$$

---

perspective on complex manifolds (the settings in which they arose in the first place), where the full data of a structure sheaf is actually needed - for smooth manifolds, existence of partitions of unity makes the theory "essentially affine", while this is not at all the case in complex geometry due to the identity theorem. We chose instead to operate on the assumption that the average reader is more familiar with smooth manifolds, and to keep this subtlety contained in a footnote.

[33] For instance, this language is used to develop the theories of derived algebraic geometry and derived differential geometry, both of which are in some sense "homotopical refinements" of their classical counterparts.

[34] Here again, complex manifolds would technically speaking present a stronger analogy, due to the fact for instance that every open ball in Euclidean n-space is diffeomorphic to $\mathbb{R}^n$, a fact which very much does not hold in $\mathbb{C}^n$, where open domains can be "genuinely distinct".

5.2 First principles

We begin by observing that, while the set-up of the previous section led us to consider presheaves, and eventually sheaves of $\mathbb{R}$-algebras on a topological space, we could equally well fruitfully study sheaves of abelian groups[35], sheaves of k-vector spaces, or even sheaves of sets[36]. For this reason, we shall develop the bits of sheaf theory that we will need in a "target-agnostic" language whenever it makes sense, working with simply a sheaf $\mathcal{F}$ viewed as an element of the category of sheaves $\mathrm{Sh}(X)$ valued in some target category $\mathcal{A}$ (throughout this section, X will denote some underlying topological space). The basic notions that we introduced in the previous sections make sense in any of the aforementioned contexts: sheaf morphisms, germs, pushforward of a sheaf along a continuous map, and restriction of a sheaf along an open subset $U \in \mathrm{Op}(X)$.

So far, we have defined $\mathrm{Sh}(X)$ as a full subcategory of the category $\mathrm{PSh}(X)$ of presheaves on X. Given a presheaf $\mathcal{F}$ on X, one may wonder whether there is a canonical way to associate to it a sheaf $\mathcal{F}^+$. Thinking of the inclusion

$$\mathrm{Sh}(X) \xrightarrow{i} \mathrm{PSh}(X)$$

as a "forgetful functor", experience suggests[37] that we could try looking for a left adjoint "free functor" - this turns out to be possible, and we call the corresponding functor the **sheafification** functor. By design of the free-forget adjunction, the sheaf $\mathcal{F}^+$ must come equipped with a presheaf morphism $\mathcal{F} \to \mathcal{F}^+$ which is initial among presheaf morphisms from $\mathcal{F}$ into a sheaf, i.e.

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[35] The primary motivation for this type of sheaf, which we will not develop for now, is that we would like to think of sheaves of abelian groups $\mathcal{A}$ on a space X as *coefficients* serving to define *cohomology groups of X with coefficients in $\mathcal{A}$*. This includes not only usual abelian groups as "constant coefficients", but also a variety of more exotic coefficients.

[36] In the latter case, the gluing diagram needs to be modified to account for the fact that "elements of a set can't always be substracted" - but that's fine, it suffices to consider an appropriate equalizer diagram instead (exercise: work out which one).

[37] Here, we have in mind for instance the free group construction taking a set S to the free group $F(S)$ with generators given by S, and which fits into an adjunction:

$$\begin{tikzcd}
\mathrm{Group} \arrow[r, bend right, "\mathrm{fgt}"'] \arrow[r, phantom, "\perp"] & \mathscr{S}\mathrm{et} \arrow[l, bend right, "F(-)"'] .
\end{tikzcd}$$

for any sheaf $\mathcal{G}$ and any presheaf morphism $\mathcal{F} \to \mathcal{G}$, there is a unique dotted arrow making the following triangle commute:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G}. \\ \mathcal{F}^+ \arrow[ur, dashed] & \end{tikzcd} \tag{7}$$

Let us now provide an explicit construction of $\mathcal{F}^+$, based on the following prototypical source of sheaves on a given space X:

Construction 5.2.1. (Sheaf of sections) Start life with a space E over X, i.e. with the data of a continuous map:

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ X. \end{tikzcd}$$

Then we may associate to this data a **sheaf of sections**

$$\begin{tikzcd} \mathcal{S}_E \arrow[d, dash] \\ X \end{tikzcd}$$

by assigning to each open $U \subseteq X$ the set of continuous sections of E over U:

$$\mathcal{S}_E(U) := \left\{ \begin{tikzcd} E|_U \arrow[d, "\pi"'] \\ U \arrow[u, bend right, "s"'] \end{tikzcd} \mid \pi \circ s = \mathrm{id}_U \right\}$$

That is, $\mathcal{S}_E(U)$ keeps track of all of the ways one may "continuously pick out a point from each fiber $E_x := \pi^{-1}(x)$ as x varies along U".

Exercise 5.2.1. (1) Verify that $\mathcal{S}_E$ is indeed a sheaf of sets over X. (Hint: you may find exercise 5.1.1 useful).
(2) Check that this construction is functorial in E, i.e. any continuous map between spaces over X:

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd}$$

induces a valid sheaf morphism $\mathcal{S}_E \to \mathcal{S}_{E'}$.
(3) Show that if each fiber E_x comes equipped with an abelian group structure, a k-vector space structure, or an R-algebra structure, then $\mathcal{S}_E$ can be viewed as a sheaf of abelian groups, k-vector spaces, or R-algebras.

Thus, given a presheaf $\mathcal{F}$ over X, if we can find a way to functorially view it as a "physical space" $\mathrm{Et}(\mathcal{F})$ over X, then we could pass to the sheaf of sections of $\mathrm{Et}(\mathcal{F})$ so as to obtain a sheaf functorially associated to $\mathcal{F}$. This will be the subject of our next:

Construction 5.2.2. (Espace étalé of a sheaf)[38] Start life with a presheaf $\mathcal{F}$ on X, viewed as a sheaf of sets (i.e. no need to keep track of the module structure or ring structure if there is any). Assemble the germs $\mathcal{F}_x$ collectively into a set $\mathrm{Et}(\mathcal{F}) := \bigsqcup_{x \in X} \mathcal{F}_x$ over X:

$$\begin{tikzcd}
\mathrm{Et}(\mathcal{F}) \arrow[d, "\pi"'] \\
X
\end{tikzcd}$$

via the canonical projection map π sending each $\mathcal{F}_x$ down to $x \in X$. Next, topologize $\mathrm{Et}(\mathcal{F})$ by declaring the following subsets to form a basis of open subsets, as $s \in \mathcal{F}(U)$ varies over all opens $U \in \mathrm{Op}(X)$ and all sections of $\mathcal{F}$ over U:

$$B(s, U) := \{s_x \mid x \in U\} \subseteq \mathrm{Et}(\mathcal{F}).$$

Note that $B(s, U)$ cuts out a "horizontal, U-shaped slice" across the fibers of $\mathrm{Et}(\mathcal{F})$ over X.

Exercise 5.2.2. (1) Show that the map $\pi : \mathrm{Et}(\mathcal{F}) \to X$ defined above is a *local homeomorphism*, i.e. for any $z \in \mathrm{Et}(\mathcal{F})$ we can find a neighborhood $V \ni z$ along which $\pi|_V$ is a homeomorphism onto its image.
(2) Show that the assignment $\mathcal{F} \mapsto \mathrm{Et}(\mathcal{F})$ is functorial in $\mathcal{F}$, i.e. that any sheaf morphism $\mathcal{F} \to \mathcal{G}$ induces a continuous map over X:

$$\begin{tikzcd}
\mathrm{Et}(\mathcal{F}) \arrow[rr] \arrow[dr] & & \mathrm{Et}(\mathcal{G}). \arrow[dl] \\
& X &
\end{tikzcd}$$

---

[38] As the accents may indicate, the name for this construction originated in French, and it roughly translates to the suggestive name "spread out space".

Definition 5.2.3. The **sheafification functor** is defined to be the following composite:

$$\begin{tikzcd}
\mathrm{PSh}(X) \arrow[r, "\mathrm{Et}(-)"] \arrow[rr, bend right, "(-)^+"'] & \mathrm{Top}_{/X} \arrow[r, "\mathcal{S}_-"] & \mathrm{Sh}(X).
\end{tikzcd}$$

Given a presheaf $\mathcal{F}$ over X, we denote its sheafification by $\mathcal{F}^+$.

Note that $\mathcal{F}^+$ comes equipped with a canonical map from $\mathcal{F}$ given on an open $U \in \mathrm{Op}(X)$ by:

$$\begin{gathered}
\mathcal{F}(U) \to \mathcal{F}^+(U) := \mathcal{S}_{\mathrm{Et}(\mathcal{F})}(U) \\
s \mapsto \big(x \mapsto s_x\big).
\end{gathered}$$

Exercise 5.2.3. (1) Show that, on an open $U \subseteq X$, the sheafification $\mathcal{F}^+ = \mathcal{S}_{\mathrm{Et}(\mathcal{F})}$ may be explicitly described by the following formula[39]:

$$U \longmapsto \left\{ \; s : U \to \coprod_{x \in X} \mathcal{F}_x \;\middle|\; \begin{array}{r} \forall x \in X, s(x) \in \mathcal{F}_x, \text{ and} \\ \exists\, W \ni x \text{ open }, \exists\, f \in \mathcal{F}(W) \text{ s.t} \\ s(y) = f_y \text{ for all } y \in W \end{array} \right\}.$$

(2) Check that the sheafification functor is indeed left adjoint to the inclusion of sheaves into presheaves, by showing that it satisfies the universal property stated in diagram 7 with respect to the canonical map $\mathcal{F} \to \mathcal{F}^+$ described above.

Exercise 5.2.4. Suppose we start with a sheaf $\mathcal{F}$ on X, viewed as a presheaf. Show that $\mathcal{F} \cong \mathcal{F}^+$ is isomorphic to its sheafification. Therefore, the composite

$$\mathrm{Sh}(X) \xrightarrow{i} \mathrm{PSh}(X) \xrightarrow{(-)^+} \mathrm{Sh}(X)$$

is naturally isomorphic to the identity - this suggests that we can think of the sheafification functor as a "categorical retract" onto $\mathrm{Sh}(X)$.[40]

---

[39] This formula is often given as the definition of the sheafification functor - we hope that the route chosen here provides some intuition as to why this is a reasonable thing to consider.

[40] Compare this situation with the notion of a retract in topology: given a subspace $Z \subseteq X$ of a topological space X, we call a continuous map $X \xrightarrow{r} Z$ a *retract* if the composite

$$Z \hookrightarrow X \xrightarrow{r} Z$$

is equal to the identity on Z.

Exercise 5.2.5. (Constant sheaves) Let A be an abelian group, and denote by A_X^{naive} the constant presheaf on X associated to A (taking values A everywhere, and sending every inclusion of opens in X to id_A).
(1) Show that A_X^{naive} is not generally a sheaf. (Hint: try looking at a space X with two connected components.)
(2) Denote the sheafification of A_X^{naive} by A_X. Show that, on a given open $U \in \mathrm{Op}(X)$,

$$A_X(U) = \{f : U \to A\}$$

is given by continuous functions from U to A viewed as a discrete set, i.e. by *locally constant A-valued functions* on U.

The espace étalé may be thought of as an alternative perspective on sheaves which is somewhat more geometric and "tangible" than the "functor on opens" perspective - a little like thinking of $\operatorname{Spec} R$ alternatively as a locally ringed space (a perspective we are currently building towards) or as a solution functor/"functor of points" $\operatorname{Spec} R = V_S(-) : \mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$. Formally, one can show that the local homeomorphism property of the structure map $\mathrm{Et}(\mathcal{F}) \xrightarrow{\pi} X$ fully characterizes the kinds of spaces over X that can be obtained via the espace étalé construction. That is, if we let $\mathrm{Et}_{/X} \subseteq \mathrm{Top}_{/X}$ denote the full subcategory of spaces over X whose structure map is a local homeomorphism, then the espace étalé functor restricts to an equivalence of categories:

$$\mathrm{Et}(-) : \mathrm{Sh}(X) \xrightarrow{\simeq} \mathrm{Et}_{/X},$$

with inverse given by the sheaf of sections construction. This equivalence is in fact obtained from a more general adjunction between presheaves on X and spaces over X given by the same two functors:

$$\begin{tikzcd}
\mathrm{PSh}(X) \arrow[r, bend right, "\mathrm{Et}(-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Top}_{/X} \arrow[l, bend right, "\mathcal{S}_-"'] \\
\mathrm{Sh}(X) \arrow[u, "i"] \arrow[r, bend right, "\mathrm{Et}(-)"'] \arrow[r, phantom, "\simeq"] & \mathrm{Et}_{/X} . \arrow[u, "i"'] \arrow[l, bend right, "\mathcal{S}_-"']
\end{tikzcd}$$

This pattern, whereby an adjunction pair restricts to an equivalence of categories onto the essential images of the functors involved, will appear again in the next section.

Let us give a concrete illustration of the value of having access to two different perspectives (other than the conceptually clear description of the sheafification functor we obtained above)[41]. Recall from section 5.1 that, given a continuous map $F : X \to Y$ and a sheaf $\mathcal{F}$ on X, we can define a sheaf $F_* \mathcal{F}$ on Y called the *pushforward of $\mathcal{F}$ along F*. This assignment is in fact functorial in $\mathcal{F}$, producing a functor:

$$\begin{tikzcd}\mathrm{Sh}(X) \arrow[r, "F_*"] & \mathrm{Sh}(Y).\end{tikzcd}$$

It turns out that this functor admits a left-adjoint[42], called the **inverse image functor**, and denoted by F^{-1}:

$$\begin{tikzcd}\mathrm{Sh}(X) \arrow[r, bend right, "F_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Sh}(Y). \arrow[l, bend right, "F^{-1}"']\end{tikzcd}$$

generalizing the restriction of a sheaf along an open subset $U \subseteq X$.

Now, suppose we start with a map $F : X \to Y$ and a sheaf $\mathcal{F}$ on Y. Under the equivalence $\mathrm{Sh}(X) \simeq \mathrm{Et}(X)$, $\mathcal{F}$ is identified with its espace étalé $\mathrm{Et}(\mathcal{F})$, and we may simply form the pullback in the category of topological spaces:

$$\begin{tikzcd}F^* \mathrm{Et}(\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Et}(\mathcal{F}) \arrow[d] \\ X \arrow[r, "F"] & Y\end{tikzcd} .$$

One may check that $F^* \mathrm{Et}(\mathcal{F})$ is again in $\mathrm{Et}(X)$ (similarly to how vector bundles behave well under pullbacks). We may therefore define the **inverse image** of $\mathcal{F}$ along F to be the unique sheaf $F^{-1} \mathcal{F}$ over X up to isomorphism such that

$$\mathrm{Et}(F^{-1} \mathcal{F}) \cong F^* \mathrm{Et}(\mathcal{F}).$$

---

[41] This construction will not be strictly needed for the rest of this chapter, so the reader eager to get back to the main storyline is welcome to skip ahead to 5.2.4.

[42] That is, for any $\mathcal{F} \in \mathrm{Sh}(X)$ and $\mathcal{G} \in \mathrm{Sh}(Y)$, we have a natural bijection:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(F^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{G}, F_*\mathcal{F}).$$

Exercise 5.2.6. (1) Using the espace étalé definition of the inverse image functor, show that we have an identification, for any sheaf $\mathcal{F}$ on X:

$$i_x^{-1}\mathcal{F} \cong \mathcal{F}_x,$$

where $i_x : \{x\} \hookrightarrow X$ denotes the inclusion of a point in X.
(2) Deduce that a presheaf and its sheafification have the same stalks (this can also be checked "by hand").
(3) Let $\pi : X \to \mathrm{pt}$ denote the projection of X down to a point, and let A be an abelian group viewed as a sheaf on pt. Show that the constant sheaf may be retrieved as the inverse image $A_X \cong \pi^{-1}A \in \mathrm{Sh}(X)$.

Exercise 5.2.7. (1) Show that if all of the stalks of an abelian sheaf $\mathcal{F}$ vanish, then $\mathcal{F} = 0$.
(2) Show that if two sheaf morphisms $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ induce equal maps on stalks $\varphi_x, \psi_x : \mathcal{F}_x \to \mathcal{G}_x$, then $\varphi = \psi$.

Compare the above definition of the inverse image functor with the usual definition: given the same data as above, one usually defines a presheaf $(F^{-1}\mathcal{F})^{\mathrm{pre}}$ on X which, to an open $U \in \mathrm{Op}(X)$, assigns

$$(F^{-1}\mathcal{F})^{\mathrm{pre}}(U) := \varinjlim_{V\in \mathrm{Op}(Y) | V \supset F(U)} \mathcal{F}(V)$$

with appropriate restriction maps. The inverse image of $\mathcal{F}$ along F is then defined to be the sheafification $F^{-1}\mathcal{F} := ((F^{-1}\mathcal{F})^{\mathrm{pre}})^+$. The interested reader may check that these two constructions do indeed agree.

Before specializing to the algebro-geometric settings, we end this theoretical section by stating a key lemma which will allow us to specify sheaves from a more manageable amount of data.

Definition 5.2.4. Let $\mathcal{B}$ be a basis of X, viewed as a full subcategory of $\mathrm{Op}(X)$, so that we may define $\mathcal{A}$-valued presheaves on $\mathcal{B}$ to be functors

$$\mathcal{F} : \mathcal{B}^{op} \to \mathcal{A}.$$

We say that $\mathcal{F}$ is a **$\mathcal{B}$-sheaf** if, for every open cover $B = \bigcup_{\alpha\in I} B_\alpha$ in $\mathcal{B}$, the following sequence is exact:

$$0 \longrightarrow \mathcal{F}(B) \longrightarrow \prod_\alpha \mathcal{F}(B_\alpha) \longrightarrow \prod_{V\in\mathcal{B} | V\subseteq B_\alpha\cap B_\beta} \mathcal{F}(V). \tag{8}$$

Here, the product on the RHS is taken over all basis elements $V \in \mathcal{B}$ contained in some intersection $B_\alpha \cap B_\beta$, $\alpha, \beta \in I$, the first map is given by restriction, and the second by sending

$$(f_\alpha)_\alpha \mapsto (f_\alpha|_V - f_\beta|_V)_V.^{43}$$

We therefore obtain a full subcategory of $\mathcal{B}$-sheaves:

$$\mathrm{Sh}_{\mathcal{B}}(X) \subseteq \mathrm{PSh}_{\mathcal{B}}(X) := \mathrm{Fun}(\mathcal{B}^{op}, \mathcal{A}).$$

Note that we are simply requiring the usual sheaf axiom to hold with respect to families of local sections which are "as compatible as possible" from the perspective of $\mathcal{B}$. Happily, we have the following:

Lemma 5.2.5. *Let $\mathcal{B}$ be a basis for a topological space X. Then every $\mathcal{B}$-sheaf extends uniquely to a sheaf on X. Furthermore, this assignment produces an equivalence of categories*

$$\mathrm{Sh}_{\mathcal{B}}(X) \to \mathrm{Sh}(X)$$

i.e. any sheaf morphism in $\mathrm{Sh}(X)$ *may be uniquely specified on a basis.*

Proof. (Sketch) Modulo checking that everything works out, the proof may be given in one line, by setting:

$$\mathcal{F}(U) := \varinjlim_{B \in \mathcal{B} | B \subseteq U} \mathcal{F}(B).$$

□

Most relevantly, what this means for us is that in order to define a sheaf on $\operatorname{Spec} R$, it suffices to provide an assignment on principal opens $D(f)$ for each $f \in R$, satisfying the gluing criterion from equation 8.

5.3 The structure sheaf of an affine scheme

Recall that we have so far constructed a topological space $X = \operatorname{Spec} R$ out of any k-algebra R, and we would like to enhance it into a locally ringed space so as to be able to fully distinguish $\operatorname{Spec} R$ from $\operatorname{Spec} A$ as geometric objects whenever R and A are non-isomorphic. We expect the structure sheaves $\mathcal{O}_X$ to satisfy the following two basic properties:

---

[43] Here, we are once again writing out the gluing condition with the implicit assumption that $\mathcal{A}$ admits things such as an abelian group structure on its hom sets etc... However, as for the sheaf axiom, this diagram may be rewritten equivalently as an equalizer diagram that makes sense for sheaves of sets as well.

- (1) The ring of global regular functions on an affine scheme $X = \operatorname{Spec} R$ should be given by $\mathcal{O}_X(X) = R$.
- (2) For each $f \in R$, the homeomorphism $i : \operatorname{Spec} R_f \xrightarrow{\cong} D(f) \subseteq X$ defined in exercise 4.2.5 should induce an isomorphism of ringed spaces via pullback:

$$(i, i^*) : (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f}) \xrightarrow{\cong} (D(f), \mathcal{O}_X|_{D(f)}).$$

This short list of expected properties turns out to fully pin down the structure sheaf of $\operatorname{Spec} R$, in a way which is compatible with the notion of regular functions on a classical algebraic variety. Indeed, item (2) tells us that we must have an isomorphism on global sections:

$$i^*_{D(f)} : \mathcal{O}_X|_{D(f)}(D(f)) = \mathcal{O}_X(D(f)) \xrightarrow{\cong} i_*\mathcal{O}_{\operatorname{Spec} R_f}(D(f)) = \mathcal{O}_{\operatorname{Spec} R_f}(\operatorname{Spec} R_f)$$

By item (1) applied to the affine scheme $\operatorname{Spec} R_f$, this translates into a ring isomorphism:

$$\mathcal{O}_X(D(f)) \xrightarrow{\cong} R_f.$$

That is, *the regular functions of* $\operatorname{Spec} R$ *over the principal open subset* $D(f)$ *should be given by the localization of* R *at* f[44].

Exercise 5.3.1. Let $f \in R$ be a nilpotent element. Show that $D(f) = \emptyset$ and that $R_f = 0$ - this is consistent: the empty set cannot support any non-trivial function!

Next, suppose we are given an inclusion of principal opens $D(g) \subseteq D(f)$.

Exercise 5.3.2. Show that $D(g) \subseteq D(f)$ iff $g \in \sqrt{f}$, where $\sqrt{f}$ denotes the radical of the ideal generated by f

In particular, we may write $g^n = fh$ for some $n \geq 1$, $h \in R$. This means that f is a *unit* inside R_g[45], since we have that

$$f\frac{h}{g^n} = 1.$$

---

[44] Note that this assignment is reasonable for another, perhaps more geometric reason: we should be able to invert the function f away from its vanishing locus, which is precisely what is enabled by setting $\mathcal{O}_X(D(f)) := R_f$.

[45] This is something that also makes a lot of sense intuitively: if f doesn't vanish along all of $D(f)$, then in particular it won't vanish on any subset $D(g) \subseteq D(f)$, and so we also expect f to be invertible when restricted to this subset.

Now, we may invoke the UP of localization to obtain an associated dotted ring map:

$$\begin{tikzcd}
R \arrow[r, "f\mapsto \text{unit}"] \arrow[d] & R_g \\
R_f \arrow[ur, dashed] &
\end{tikzcd}$$

We therefore obtain a natural restriction map

$$\mathcal{O}_X(D(f)) \cong R_f \to R_g \cong \mathcal{O}_X(D(g))$$

sending $1/f$ to h/g^n. One may then check that this assignment is compatible with iterated inclusions of principal opens, so that altogether, this data fits into a *presheaf* on the basis $\mathcal{B}$ of principal opens in $\operatorname{Spec} R$.

Therefore, as long as the assignment $D(F) \mapsto R_f$ with restriction maps defined above satisfies the sheaf condition on a basis 8, we may invoke 5.2.5 to obtain a unique extension of this assignment into a sheaf of rings $\mathcal{O}_X$ on $\operatorname{Spec} R$, which will necessarily have to be the **structure sheaf** of $\operatorname{Spec} R$. Verifying that the sheaf condition on a basis is satisfied is possible, although it is a bit delicate - we refer the reader to proposition 1.18 in [**EH**] for a proof.

Thus, we have arrived at a way to assign to each k-algebra R a ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. This assignment is naturally contravariantly functorial in R: in the previous chapter, we have already described a way to assign to any k-algebra map $\varphi : R \to A$ a continuous map $F : \operatorname{Spec} A \to \operatorname{Spec} R$, setting $F(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$. It remains to specify an appropriate sheaf morphism:

$$F^* : \mathcal{O}_{\operatorname{Spec} R} \to F_* \mathcal{O}_{\operatorname{Spec} A}. \tag{9}$$

By 5.2.5, it suffices to describe this sheaf morphism on the basis of principal opens. At the level of global sections, we simply take

$$F^*_{\operatorname{Spec} R} := \varphi : \mathcal{O}(\operatorname{Spec} R) = R \to A = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A).$$

Next, given $f \in R$, we need to specify a k-algebra map:

$$F^*_{D(f)} : \mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \to A_{\varphi(f)} = \mathcal{O}_{\operatorname{Spec} A}(F^{-1}(D(f))) = \mathcal{O}_{\operatorname{Spec} A}(D(\varphi(f))).$$

To do this, we invoke the defining UP of localization once again to obtain a dotted arrow completing the following triangle:

$$\begin{tikzcd} R \arrow[r, "\varphi", "f\mapsto \text{unit}"'] \arrow[d] & A_{\varphi(f)}. \\ R_f \arrow[ur, dashed] & \end{tikzcd}$$

The maps we have exhibited may be seen to be compatible with restriction maps on principal opens, giving rise to a unique sheaf morphism as in 9.

All in all, we have constructed a functor:

$$\mathrm{Spec} : \mathrm{CAlg}_k^{op} \to \mathcal{RS}$$

$$R \mapsto (\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\, R}),$$

$$(R \xrightarrow{\varphi} A) \mapsto ((\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A}) \xrightarrow{(F,F^*)} (\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\, R})$$

Let us verify that this functor in fact factors through locally ringed spaces (a subcategory of $\mathcal{RS}$ in which one can always "evaluate regular functions at a point").

Exercise 5.3.3. (1) Show that we have an isomorphism

$$\varinjlim_{f \notin \mathfrak{p}} R_f \cong R_{\mathfrak{p}},$$

where the direct system structure maps are given by restriction of regular functions, and where we denote by $R_{\mathfrak{p}} := (R - \mathfrak{p})^{-1}R$ the **localization of R at $\mathfrak{p}$**, obtained by localizing at the multiplicatively closed subset $R - \mathfrak{p}$. Thus, the stalk of $X = \mathrm{Spec}\, R$ at $\mathfrak{p}$ is given by $\mathcal{O}_{X,\mathfrak{p}} \cong R_{\mathfrak{p}}$.
(2) Check that $R_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$, and that we can identify:

$$\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \cong \mathrm{Frac}(R/\mathfrak{p}).$$

Thus, the stalks of $X = \mathrm{Spec}\, R$ are all local rings, and the residue field at a given point $\mathfrak{p}$ may be unambiguously extracted pre- or post-localization.
(3) Finally, check that, if $F : \mathrm{Spec}\, A \to \mathrm{Spec}\, R$ is the morphism associated to a ring map $R \to A$, then $F^* : \mathcal{O}_{\mathrm{Spec}\, R} \to F_*\mathcal{O}_{\mathrm{Spec}\, A}$ induces a local ring morphism on stalks at each point $\mathfrak{p} \in \mathrm{Spec}\, A$:

$$F^*_{\mathfrak{p}} : \mathcal{O}_{\mathrm{Spec}\, R, F(\mathfrak{p})} \to \mathcal{O}_{\mathrm{Spec}\, A, \mathfrak{p}}.$$

Therefore, the Spec construction really defines a functor:

$$\mathrm{Spec} : \mathrm{CAlg}_k^{op} \to \mathcal{LRS}.$$

In line with the smooth manifold discussion from section 5.1, we define the category AffSch_k of **affine k-schemes** to be the full subcategory of $\mathcal{LRS}_k$ whose objects lie in the essential image of the Spec functor. To complete our "geometrization" process, it remains to verify that the Spec functor does indeed define an *equivalence of categories* between CAlg_k^{op} and AffSch_k.

As with the $\mathrm{Sh}(X) \simeq \mathrm{Et}_{/X}$ discussion from the previous section, this equivalence of categories is best understood as arising from an ambient adjunction of functors. Namely, we may define a **global sections functor** in the other direction:

$$\Gamma : \mathcal{LRS} \to \mathrm{CAlg}_k^{op}$$

sending a locally ringed space $(X, \mathcal{O}_X)$ to its k-algebra of global sections $\Gamma(X, \mathcal{O}_X) := \mathcal{O}_X(X)$[46], and sending a morphism of locally ringed spaces $F : X \to Y$ to the corresponding pullback

$$F_Y^{\#} : \Gamma(Y, \mathcal{O}_Y) = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) = \Gamma(X, \mathcal{O}_X)$$

at the level of global sections. The backbone of this section comes down to the following:

Theorem 5.3.1. *The Spec and global sections functors fit into an adjunction:*

$$\begin{tikzcd} \mathcal{LRS}_k \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathrm{CAlg}_k^{op}, \arrow[l, bend left, "\mathrm{Spec}"] \end{tikzcd}$$

i.e. for every locally ringed space $(X, \mathcal{O}_X)$ and every k-algebra R, we have a natural bijection:

$$\mathrm{Hom}_{\mathcal{LRS}_k}(X, \mathrm{Spec}\, R) \cong \mathrm{Hom}_{\mathrm{CAlg}_k^{op}}(R, \mathcal{O}_X(X)).$$

---

[46] As the notation suggests, $\Gamma(X, -)$ may also be viewed as a functor out of some category of sheaves on X. Studying exactness properties of the global sections functor

$$\Gamma(X, -) : \mathrm{Sh}(X, \mathcal{A}\mathrm{b}) \to \mathcal{A}\mathrm{b}$$

out of the category of abelian sheaves on a topological space X leads to the study of *sheaf cohomology*.

Proof. (Sketch) We use the unit/counit approach to exhibiting an adjunction. Thus, it suffices to exhibit natural transformations of functors:

$$\eta : \mathrm{Id}_{\mathcal{LRS}_k} \to \mathrm{Spec} \circ \Gamma \quad \text{and} \quad \varepsilon : \Gamma \circ \mathrm{Spec} \to \mathrm{Id}_{\mathrm{CAlg}_k^{op}}$$

satisfying certain compatibility conditions. The latter map is simply given by the canonical identification, for each k-algebra A:

$$\varepsilon_A : \Gamma(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A}) \xrightarrow{\cong} A.$$

The second map is very much analogous to Gelfand's map from section 4.1. Given a locally ringed space $(X, \mathcal{O}_X)$, we use the fact that we have "evaluation maps", for every $x \in X$:

$$\mathrm{ev}_x : \mathcal{O}_X(X) \to \kappa(x)$$
$$f \mapsto f(x) := f_x \pmod{\mathfrak{m}_x}$$

where $(\mathcal{O}_{X,x}, \mathfrak{m}_x, \kappa(x))$ is the local ring of germs at x. We therefore obtain a map:

$$\eta_X : X \to \mathrm{Spec}(\mathcal{O}_X(X))$$
$$x \mapsto \ker(_x).$$

This map is continuous as a result of the identity $\eta_X^{-1}(D(f)) = X - V_X(f)$ for any given principal open $D(f) \subseteq \mathrm{Spec}(\mathcal{O}_X(X))$, where

$$V_X(f) := \{x \in X \mid \mathrm{ev}_x(f) = 0\} \subseteq X$$

is closed. Finally, the corresponding sheaf morphism

$$\eta_X^* : \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_X(X))} \to (\eta_X)_* \mathcal{O}_X$$

is specified on each principal open $D(f) \subseteq \mathrm{Spec}(\mathcal{O}_X(X))$ as the unique map

$$\mathcal{O}_X(X)_f \to \mathcal{O}_X(X - V_X(f))$$

obtained via the UP of localization applied to the following restriction map:

$$\begin{tikzcd}
\mathcal{O}_X(X) \arrow[r, "f \mapsto \text{unit}"] \arrow[d] & \mathcal{O}_X(X - V_X(f)). \\
\mathcal{O}_X(X)_f \arrow[ur, dashed] &
\end{tikzcd}$$

□

In particular, the data of the adjunction provides us with the following analogue of theorem 2.3.2:

Corollary 5.3.2. *Let* $\operatorname{Spec} A$ *and* $\operatorname{Spec} R$ *be two affine schemes over* k. *Then passing to the rings of global regular functions induces a canonical bijection:*

$$\mathrm{Hom}_{\mathrm{AffSch}_k}(X, Y) \overset{\cong}{\to} \mathrm{Hom}_{\mathrm{CAlg}_k}(R, A)$$

$$(X \overset{\varphi}{\to} Y) \mapsto (R = \mathcal{O}_Y(Y) \xrightarrow{\varphi^*} \mathcal{O}_X(X) = A).$$

Proof. Using the Γ-Spec adjunction, we have the following chain of bijections:

$$\mathrm{Hom}_{\mathrm{AffSch}_k}(\operatorname{Spec} A, \operatorname{Spec} R) \cong \mathrm{Hom}_{\mathrm{CAlg}_k^{op}}(\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}), R) \cong \mathrm{Hom}_{\mathrm{CAlg}_k}(R, A).$$

□

Passing to essential images, we therefore obtain the following diagram:

$$\begin{tikzcd}
\mathcal{LRS}_k \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathrm{CAlg}_k^{op} \arrow[l, bend left, "\mathrm{Spec}"] \\
\mathrm{AffSch}_k \arrow[u, "i"] \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\simeq"] & \mathrm{CAlg}_k^{op} . \arrow[l, bend left, "\mathrm{Spec}"] \arrow[u, "||"']
\end{tikzcd}$$

Exercise 5.3.4. (1) Using this identification, check that

$$\mathrm{Hom}_{\mathrm{AffSch}_k}(\operatorname{Spec} R, \mathbb{A}^1_k) \cong R,$$

i.e. morphisms from $\operatorname{Spec} R$ into the affine line are given by global regular functions on $\operatorname{Spec} R$.

(2) Start with a morphism of affine schemes $F : \operatorname{Spec} A \to \operatorname{Spec} R$, corresponding to a ring map $\varphi : R \to A$. Under the identification given in part (1), show that φ is "literally" given by pullback of morphisms to $\mathbb{A}^1_k$, in the sense that if we identify $f \in R$ with the corresponding morphism $f : \operatorname{Spec} R \to \mathbb{A}^1_k$, resp. $\varphi(f) : \operatorname{Spec} A \to \mathbb{A}^1_k$, then we have a commutative triangle:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "F"] \arrow[dr, dashed, "\varphi(f)"'] & \operatorname{Spec} R \arrow[d, "f"] \\
& \mathbb{A}^1_k .
\end{tikzcd}$$

Although Spec R has now been defined for an arbitrary commutative ring, this geometric language should first and foremost be thought of as an "organizational principle", as suggested by the following quote by Paul Balmer [**Balmer**]:

> "In commutative algebra, the Zariski spectrum is not meant to be explicitly computed for every single commutative ring in the universe; instead, it serves as a stepping stone towards the geometric reasonings of algebraic geometry".

To illustrate this principle, here is a list of "spaces one may wish to have access to in geometric settings" and the corresponding affine spectra (working over a fixed commutative ring k):

Geometric object	**Corresponding affine scheme**
Point	$\operatorname{Spec} k$, or $\operatorname{Spec} L$ for any field extension L/k
Line	$\mathbb{A}^1_k := \operatorname{Spec} k[t]$
Punctured line	$\mathbb{G}_m := \operatorname{Spec} k[t, t^-1]$
Affine n-space	$\mathbb{A}^n_k := \operatorname{Spec} k[t_1, ..., t_n]$
Space of solutions of $f_1, ..., f_r \in k[t_1, ..., t_n]$	$\operatorname{Spec} k[t_1, ..., t_n]/(f_1, ..., f_r)$
First order tangential direction at a point	$\operatorname{Spec} k[t]/(t^2)$
n^{th} order tangential direction at a point	$\operatorname{Spec} k[t]/(t^n)$
Formal disk around a point	$\operatorname{Spec} k[[t]]$[47]
Punctured formal disk around a point	$\operatorname{Spec} k((t))$

Notice in particular that remembering nilpotent data allows us to have infinitesimal information directly accessible within our theory, through objects such as $\operatorname{Spec} k[t]/(t^2)$ (or, more generally, $\operatorname{Spec} k[t_1, ..., t_n]/(t_1^2, ..., t_n^2)$). This leads to a particularly simple definition of the "tangent space of a variety/scheme at a point", which we will introduce at a later time.

Let us also comment on how this language retrospectively agrees with

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[47] More precisely, we should really be considering the space Spf $k[[t]]$, which is a geometric object from formal algebraic geometric that realizes the "infinite union" $\varinjlim_n \operatorname{Spec}(k[t]/(t^n))$ that we are after (and likewise for the following line of the table). The space $\operatorname{Spec} k[[t]]$ is "slightly larger", in that it contains both a closed point and a generic point; but we will not be doing anything that requires us to elucidate the situation further in these introductory notes.

the "functor of points" perspective from earlier: geometrically, a point $x \in X$ may be encoded by a map $\mathrm{pt} \xrightarrow{x} X$ selecting x. Now, starting with a scheme $X = \operatorname{Spec} k[t_1, ..., t_n]/(f_1, ..., f_r)$ thought of as the "universal solution space" for the system of equations $\{f_1, ..., f_r\}$, a solution $\underline{a} \in L^n$ with values in a field extension L of k may be encoded by the data of an "L-point"

$$\operatorname{Spec} L \to X = \operatorname{Spec} k[t_1, ..., t_n]/(f_1, ..., f_r),$$

which corresponds under pullback to a k-algebra homomorphism

$$k[t_1, ..., t_n]/(f_1, ..., f_r) \to L$$

selecting out the solution $\underline{a} = (a_1, ..., a_n) \in L^n$: this precisely recovers the naive perspective with which we opened this course.

With this language in place, the definition of a general scheme is now just a matter of adapting the definition of a smooth manifold as given in exercise 5.1.9 to our context, wherein our affine building blocks are affine schemes rather than opens in Euclidean space:

Definition 5.3.3. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ with the property that we can find an open cover $\{U_i\}_i$ of X together with isomorphisms of locally ringed spaces onto affine schemes for each i:

$$(U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{\cong} (\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i}).$$

Exercise 5.3.5. Show that a scheme may be equivalently defined to be a locally ringed space $(X, \mathcal{O}_X)$ such that, for any $x \in X$, we can find an open neighborhood $U \ni x$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to some affine scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

Note 5.3.4. Notice that the definition of a general scheme does not make any assumptions on the underlying topological space X, unlike the definition of a smooth manifold which requires the space to be Hausdorff and 2nd countable. We learned in section 2.1 that Hausdorffness does not play well with the Zariski topology. There is nonetheless a viable alternative in the world of schemes, namely the notion of *separatedness*. The idea is to start with the fact that a topological space X is Hausdorff iff the diagonal map $\Delta : X \to X \times X$ is closed, to translate the RHS statement in the category of schemes, and to take that as a definition for separatedness. This rules out spaces such as the affine line with a double origin. [48]

Heuristically, schemes are geometric spaces which "locally look like the universal space of solutions of some system of polynomial equations". General schemes may also be studied from the functor of points perspective: any k-scheme $(X, \mathcal{O}_X)$ may be thought of as "locally representable functor"

$$X : \mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$$

$$X(A) := \mathrm{Hom}_{\mathrm{Sch}_k}(\mathrm{Spec}\, A, X).$$

A precise characterization of those functors $\mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$ which arise from schemes may be found in e.g. section VI.2.1 of [**EH**].

Exercise 5.3.6. Verify that, for $X = \mathrm{Spec}\, R$ the affine scheme associated to a system of polynomial equations $S \subseteq k[(t_i)_{i \in I}]$, the definition given above agrees with the solution functor

$$V_S(-) : \mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$$

defined in the introduction.

6 Sheaves of Modules and Vector Bundles

Our next objective is to introduce a geometric way to think about modules over a ring R as certain kinds of sheaves over $\mathrm{Spec}\, R$ called *quasi-coherent sheaves*, a category which encompasses the notion of *algebraic vector bundles* as a special case. This globalizes into a central linearization procedure in algebraic geometry, whereby one studies a scheme X via its abelian category of quasi-coherent sheaves $\mathrm{QCoh}(X)$.

6.1 Motivation: revisiting the notion of a topological vector bundle

We begin once again by drawing inspiration from the corresponding classical notion from topology. Start life with a compact Hausdorff topological space

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[48] For a time, all schemes were assumed to be separated, and non-separated schemes were called pre-schemes. This is no longer the case.

X. Recall that a *topological real vector bundle*[49] of rank n over X consists of the data of a space E over X

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ X \end{tikzcd}$$

such that every fiber $E_x := \pi^{-1}(x)$ is a real vector space, with the additional property that, locally near any point $x \in X$, we are guaranteed the existence of a neighborhood $U \ni x$ over which we have a commutative triangle:

$$\begin{tikzcd} E|_U \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

where φ is a homeomorphism which, along the fiber associated to each $y \in U$, φ, restricts to a linear isomorphism:

$$E_y \xrightarrow{\cong} \mathbb{R}^n = \{y\} \times \mathbb{R}^n.^{50}$$

Also recall that a *morphism* between two vector bundles E and E' consists of the data of a continuous map over X

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd}$$

which restricts to a linear map on each fiber.

Now, the notion of a vector bundle may alternatively be expressed internally to the language of sheaf theory, using the sheaf of sections construction introduced in section 5.2. Recall that we may associate to E a sheaf of sets

$$\begin{tikzcd} \mathcal{S}_E \arrow[d, dash] \\ X \end{tikzcd}$$

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[49] Here as elsewhere, one may freely choose to consider real or complex vector bundles.

[50] In a sense that can be made precise, the study of topological vector bundles over X is the study of "parametrized linear algebra" with parameter space X. The categorically-inclined reader may want to try the exercise of formally defining the notion of an $\mathbb{R}$-vector space object in the category of topological spaces over X, and of reformulating the notion of a real topological vector bundle in this language.

which is furthermore a sheaf of $\mathbb{R}$-vector spaces thanks to the vector space structure on the fibers of E.

Exercise 6.1.1. (1) Check that each $\mathcal{S}_E(U)$ may furthermore be equipped with a module structure over the ring $C^0(U)$ of continuous functions on U, and that this structure is compatible with the restriction maps associated to the sheaves $\mathcal{S}_E$ and C^0. That is, for every inclusion of open sets $V \subseteq U$, we have a commutative square:

$$\begin{tikzcd}
C^0(U) \times \mathcal{S}_E(U) \arrow[r, "\mathrm{act}"] \arrow[d] & \mathcal{S}_E(U) \arrow[d] \\
C^0(V) \times \mathcal{S}_E(V) \arrow[r, "\mathrm{act}"'] & \mathcal{S}_E(V).
\end{tikzcd}$$

(2) Show that the assignment $E \mapsto \mathcal{S}_E$ is functorial in E, in the sense that any map of vector bundles over X induces a sheaf morphism on the corresponding sheaves of sections respecting the module structures.

Due to part (1) of the above exercise, we call $\mathcal{S}_E$ a **sheaf of modules**[51] over the sheaf C^0 of continuous functions on X. Altogether, we have obtained a functorial assignment:

$$\mathcal{S}_{-} : \left\{ \begin{matrix} \text{Topological vector bundles} \\ \text{over } X \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Sheaves of } C^0\text{-modules} \\ \text{over } X \end{matrix} \right\}.$$

As usual when presented with such a translation process, we are led to asking what properties characterize the sheaves of modules lying in the essential image of this functor. To this end, let us re-examine the local triviality condition from the perspective of the sheaf of sections: over a local trivialization $U \subseteq X$ of a given vector bundle E, we have the following sequence of equivalences:

$$\mathcal{S}_E(U) := \left\{ \begin{tikzcd} E|_U \arrow[d, "\pi"'] \\ U \arrow[u, bend right, "s"'] \end{tikzcd} \mid \pi \circ s = \mathrm{id}_U \right\} \xrightarrow{\varphi\circ -} \left\{ \begin{tikzcd} U \times \mathbb{R}^n \arrow[d, "\pi"'] \\ U \arrow[u, bend right, "s'"'] \end{tikzcd} \mid \pi \circ s' = \mathrm{id}_U \right\} \cong \{U \xrightarrow{\mathrm{cts}} \mathbb{R}^n\} \cong C^0(U)^{\oplus n},$$

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[51] In general, a **sheaf of modules** over a ringed space $(X, \mathcal{O}_X)$ is an abelian sheaf $\mathcal{F}$ on X such that each $\mathcal{F}(U)$ may be equipped with an $\mathcal{O}_X(U)$-module structure, compatibly with restriction maps in the sense of exercise 6.1.1.

where s' denotes the composite $\varphi \circ s$ for each $s \in \mathcal{S}_E(U)$. That is, near each point $x \in X$, we are guaranteed the existence of a neighborhood $U \ni x$ over which we have an isomorphism

$$\mathcal{S}_E(U) \xrightarrow{\cong} C^0(U)^{\oplus n}.$$

In fact, since restricting further to opens inside of U preserves the trivialization property, we observe that we actually have gathered the data of a *sheaf isomorphism*:

$$\mathcal{S}_E|_U \xrightarrow{\cong} C^0|_U^{\oplus n}.$$

Accordingly, we say that the sheaf of modules $\mathcal{S}_E$ is **locally free of rank** n. Thus, our original functor is seen to factor into:

$$\mathcal{S}_{-} : \left\{ \begin{matrix} \text{Topological vector bundles} \\ \text{over } X \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Locally free sheaves of } C^0\text{-modules} \\ \text{of finite rank over } X \end{matrix} \right\}.$$

Exercise 6.1.2. (1) Check that the above functor is now an equivalence of categories. (Hint: use the characterization of vector bundles in terms of cocycle data to exhibit an inverse functor.)
(2) Convince yourself that there was nothing special about the adjective "continuous" in these settings, i.e. that one may have worked throughout with *smooth* vector bundles over a fixed smooth manifold M, and arrived at an analogous equivalence of categories with respect to locally free sheaves of $\mathcal{O}_M$-modules of finite rank, where $\mathcal{O}_M$ denotes the sheaf of smooth functions on M.

In the spirit of part (2) of this exercise, the above equivalence may be summed up in the following, "context agnostic" principle, which we will next investigate in the algebro-geometric settings:

Principle. The data of a vector bundle E over X is equivalent to the data of its associated sheaf of sections $\mathcal{S}_E$ over X, viewed as a sheaf of $\mathcal{O}_X$-modules.

Before leaving the topological settings, we conclude this section by stating a fundamental theorem which provides a third perspective on topological vector bundles on compact Hausdorff spaces, highlighting the "affineness" properties of these spaces (a feature which ultimately comes down to Urysohn's lemma):

Theorem 6.1.1. ***(Swan's theorem)*** *Let X be a compact Hausdorff space, and denote by $C^0(X)$ its ring of continuous $\mathbb{K}$-valued functions, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there is an equivalence of categories:*

$$\mathrm{Vect}_{\mathbb{K}}^{top}(X) \xrightarrow{\simeq} \mathrm{PMod}_{C^0(X)}^{fg},$$

where $\mathrm{PMod}_{C^0(X)}^{fg}$ denotes the category of finitely-generated projective $C^0(X)$-modules[52]*. This equivalence is given by sending a vector bundle E over X to its $C^0(X)$-module of global sections $\Gamma(X, E) := \mathcal{S}_E(X)$.*

That is, when working with topological vector bundles algebraically, we don't need to carry around the whole apparatus of a sheaf of sections around: the data of the sheaf of global sections, viewed as a module over $C^0(X)$, suffices. We should note that Hausdorff spaces should be thought of as "affine spaces", in the sense that Gelfand's theorem tells us that X may be presented as $\mathrm{mSpec}(C^0(X))$. As such, we can only reasonably expect Swan's theorem to admit an analogue for affine schemes, whereas vector bundles on more general schemes will in general not be determined by their module of global sections.

6.2 Modules are quasi-coherent sheaves

Start life with a module M over a commutative ring R. We aim to think of M as some sort of "sheaf of $\mathcal{O}_X$-modules" over $X = \operatorname{Spec} R$, taking as our clue the fact that M itself is already a module over the ring of global regular functions $R = \mathcal{O}_X(X)$.

Given an element $f \in R$, we may localize M at f to obtain an associated R_f-module

$$M_f \cong M \otimes_R R_f.$$

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[52] Recall that an R-module M is said to be **projective** if it may be expressed as a direct summand $R^{\oplus n} \cong M \oplus N$ of a free module, which happens iff the hom functor $\mathrm{Hom}_R(M, -)$ is exact, i.e. iff a dotted arrow exists in any situation of the following form:

$$\begin{tikzcd}
 & M \arrow[dl, dashed] \arrow[d] & \\
N \arrow[r, two heads] & P \arrow[r] & 0.
\end{tikzcd}$$

This module is characterized by a similar universal property to that R_f. Namely, it comes with a structure map $M \to M_f$, $m \mapsto m/1$, and for any R-module N such that f acts on N via an *automorphism* $f\cdot : N \xrightarrow{\cong} N$ and any R-module map $M \to N$, there exists a unique dotted arrow making the following triangle commute:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & N \arrow[loop right, "f\text{: unit}"] \\
M_f \arrow[ur, dotted] &
\end{tikzcd}$$

One may analogously localize M at a more general multiplicatively closed subset $S \subseteq R$:[53]

$$S^{-1}M \cong M \otimes_R S^{-1}R.$$

Exercise 6.2.1. Show that, given a multiplicatively closed subset $S \subseteq R$, the localization functor

$$\mathrm{Mod}_R \xrightarrow{S^{-1}(-)} \mathrm{Mod}_{S^{-1}R}$$

$$M \mapsto S^{-1}M := M \otimes_R S^{-1}R$$

is *exact*, meaning that it takes every short exact sequence in Mod_R to a short exact sequence in $\mathrm{Mod}_{S^{-1}R}$.

Now, localizing at various elements of R leads us to an assignment

$$\widetilde{M} : D(f) \mapsto M_f,$$

and one may prove that this is indeed a valid sheaf of abelian groups on the basis B of principal opens, via a similar argument to the one used to build the structure sheaf. Notice that each $\widetilde{M}(D(f)) = M_f$ comes equipped with an action of $R_f = \mathcal{O}_X(D(f))$ by construction, compatibly with restrictions in $\mathcal{B}$ - i.e. the assignment $\widetilde{M}$ is a *$\mathcal{B}$-sheaf of $\mathcal{O}_X$-modules*, where $\mathcal{O}_X$ is viewed as a $\mathcal{B}$-sheaf. Now, lemma 5.2.5 admits an analogue for $\mathcal{B}$-sheaves of modules, which in our situation implies that $\tilde{M}$ extends to a unique sheaf of $\mathcal{O}_X$-modules on $X = \operatorname{Spec} R$:

$$\widetilde{M} : \mathrm{Op}(X)^{op} \to \mathcal{A}b$$

which we call the **quasi-coherent sheaf** on $\operatorname{Spec} R$ associated to M.

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[53] As in the previous chapter, the main example of localization "at more than one element" which we will be concerned with is localization at the complement $S = R - \mathfrak{p}$ of a prime ideal in R.

Exercise 6.2.2. Check that the assignment $M \mapsto \tilde{M}$ defines a functor:

$$\mathrm{Mod}_R \xrightarrow{\tilde{\ }} \left\{ \begin{matrix} \text{Sheaves of } \mathcal{O}_X\text{-modules} \\ \text{on } X = \operatorname{Spec} R \end{matrix} \right\},$$

called the **localization functor** on R-modules.

Definition 6.2.1. The category of **quasi-coherent sheaves** on an affine scheme $X = \operatorname{Spec} R$, denoted by $\mathrm{QCoh}(X)$, is defined to be the essential image of the localization functor on Mod_R. We also define the category of **coherent sheaves** on X to be the fully subcategory $\mathrm{Coh}(X) \subseteq \mathrm{QCoh}(X)$ corresponding to *finitely generated R-modules* under the localization functor.

Exercise 6.2.3. Show that the assignment $M \mapsto \widetilde{M}$ is *fully-faithful*. By definition of $\mathrm{QCoh}(X)$ as the essential image of this functor, we therefore obtain an equivalence of categories:

$$\mathrm{Mod}_R \simeq \mathrm{QCoh}(\operatorname{Spec} R).$$

(Hint: by lemma 5.2.5, this can be verified on the basis of principal opens!)

Note 6.2.2. The above equivalence further restricts to an equivalence of categories:

$$\mathrm{Mod}_R^{\mathrm{fg}} \simeq \mathrm{Coh}(\operatorname{Spec} R).$$

Although full-faithfulness could be checked by hand, the “raison d'être” for this equivalence of categories once again comes from an ambient adjunction which restricts to an equivalence of categories on “those sheaves of $\mathcal{O}_{\operatorname{Spec} R}$-modules that come from R-modules”, in the same way that the global sections-Spec adjunction restricts to an equivalence of categories on “those locally ringed spaces that come from k-algebras”. Namely, we have a “global sections functor”:

$$\Gamma : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_R$$

$$\mathcal{M} \mapsto \Gamma(X, \mathcal{M}) := \mathcal{M}(X),$$

which is right adjoint to the localization functor obtained in exercise 6.2.3. With this adjunction in hand, full-faithfulness reduces to the following chain of bijections, for any two R-modules M and N:

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_X}}(\widetilde{M}, \widetilde{N}) \cong \mathrm{Hom}_{\mathrm{Mod}_R}(M, \widetilde{N}(\operatorname{Spec} R)) \cong \mathrm{Hom}_{\mathrm{Mod}_R}(M, N).$$

We thus arrive at the following diagram:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathcal{O}_X} \arrow[r, bend right, "\Gamma"'] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_R \arrow[l, bend right, "\simeq"'] \\
\mathrm{QCoh}(X) \arrow[u, "i"] \arrow[r, bend right, "\Gamma"'] \arrow[r, phantom, "\simeq"] & \mathrm{Mod}_R \, . \arrow[l, bend right, "\simeq"'] \arrow[u, "||"']
\end{tikzcd}$$

As such, one may think of quasi-coherent sheaves as a sort of *geometrization* of the notion of a module over a ring, in the same way that affine schemes provide a geometrization of the notion of a k-algebra. This viewpoint can help provide geometric intuition when doing module theory, and the sheaf theoretic language allows one to introduce the notion of a quasi-coherent sheaf on a general scheme X, a geometric object of independent interest. Namely, we define a **quasi-coherent sheaf** on a scheme X to be a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ with the property that we can find an open cover of X by affine opens $U_i = \operatorname{Spec} R_i$ such that the restriction of $\mathcal{F}$ to each U_i is isomorphic to the localization of some R_i-module M_i. We further say that $\mathcal{F}$ is **coherent** if each M_i can be taken to be finitely-generated.

In practice, it can be useful to have explicit criteria for determining whether a given sheaf of $\mathcal{O}_X$-modules is (quasi-)coherent, other than the fact that it abstractly "lies in the essential image of the localization functor". The following equivalence, in terms of a "local presentation", provides one such criterion:

Proposition 6.2.3. *Let X be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. Then the following are equivalent:*
(1) $\mathcal{F}$ is a quasi-coherent sheaf on X;
(2) There exists an open cover $\{U_i\}_i$ of X such that the restriction of $\mathcal{F}$ to each U_i fits into some exact sequence:

$$\mathcal{O}_X^{\oplus J} \to \mathcal{O}_X^{\oplus I} \to \mathcal{F} \to 0.$$

[54]

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[54] Compare this with the fact that any R-module M may be

Let us deduce some functoriality properties at the level of quasi-coherent sheaves from familiar concepts in module theory. Start with a morphism $F : \operatorname{Spec} A \to \operatorname{Spec} R$, associated to a ring map $\varphi : R \to A$.

On the one hand, we can think of φ as equipping A with an R-module structure, leading to a "change of scalars" functor at the level of modules:

$$\mathrm{Mod}_R \xrightarrow{-\otimes_R A} \mathrm{Mod}_A$$

$$M \mapsto M \otimes_R A.$$

Notice that in the special case of the localization structure map $R \to R_f$ associated to an element $f \in R$, this functor takes a module M to its localization M_f.

In the other direction, given an A-module N, we may use φ to view N as an R-module (namely by setting $r \cdot n := \varphi(r) \cdot n$). Notice that, tautologically, we have an identification $N \cong \operatorname{Hom}_A(A, N)$, functorially in N, which automatically guarantees that the above two functors fit in a tensor-hom adjunction (associated to the (R, A)-bimodule A):

$$\begin{tikzcd}
\mathrm{Mod}_A \arrow[r, bend right, "\operatorname{Hom}_A(A,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_R . \arrow[l, bend right, "-\otimes_R A"']
\end{tikzcd}$$

From there, we may fall back on the equivalence $\mathrm{Mod}_{(-)} \simeq \mathrm{QCoh}(\operatorname{Spec} -)$ to obtain an adjoint pair of functors in the geometric settings:

$$\begin{tikzcd}
\mathrm{QCoh}(\operatorname{Spec} A) \arrow[r, bend right, "F_*"'] \arrow[r, phantom, "\perp"] & \mathrm{QCoh}(\operatorname{Spec} R) \arrow[l, bend right, "F^*"'] \\
\mathrm{Mod}_A \arrow[u, "\simeq"] \arrow[r, bend right, "\operatorname{Hom}_A(A,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_R . \arrow[l, bend right, "-\otimes_R A"'] \arrow[u, "\simeq"']
\end{tikzcd}$$

Exercise 6.2.4. (1) Verify that the functor F_* in the above diagram is precisely the pushforward functor associated to the continuous map F.

(2) Find a concrete example that shows that F^* is *not* equal to the inverse image functor F^{-1} defined in the previous chapter.

The left adjoint F^* to the pushforward functor is called the **pullback functor** on quasi-coherent sheaves. It is subtler to define formally than the pushforward functor on general schemes (see e.g. section II.5 of [**Hart**] for details), but for our purposes it will suffice to remember that it fits into an adjunction pair (F^*, F_*), and that it is locally given by "change of scalars":

$$\begin{tikzcd}
F^*\widetilde{M} \cong \widetilde{M \otimes_R A} \arrow[d, dash] & \tilde{M} \arrow[d, dash] \\
\operatorname{Spec} A \arrow[r, "F"] & \operatorname{Spec} R.
\end{tikzcd}$$

In particular, given the inclusion of a point $i_x : \operatorname{Spec} \kappa(\mathfrak{p}) \hookrightarrow X$, we may define the **fiber** of a quasi-coherent sheaf $\mathcal{F}$ at $\mathfrak{p}$ to be the $\kappa(\mathfrak{p})$-vector space $\mathcal{F}|_x := i_x^*\mathcal{F}$ obtained by pullback under i_x. Affine locally on some $U \cong \operatorname{Spec} R$ containing x, we may always identify $\mathcal{F}|_U \simeq \tilde{M}$ with some R-module M, so that we may express:

$$i_x^*\mathcal{F} \cong M \otimes_R \kappa(\mathfrak{p}) = M/\kappa(\mathfrak{p})M.$$

Notice that this object is distinct from the *stalk* of $\mathcal{F}$ at $\mathfrak{p}$, which can be expressed as the inverse image $\mathcal{F}_x = i_x^{-1}\mathcal{F}$, and which keeps track of "germs of sections near x". To illustrate this, take $\mathcal{F} = \mathcal{O}_X$ to be the structure sheaf. Given a function $f \in \mathcal{O}_X(U)$ near x, one should think of the stalk $f_x \in \mathcal{O}_{X,x}$ as keeping track of the whole "Taylor series expansion" of f at x, while the fiber $\mathcal{O}_X|_x$ only remembers the "value" $f(x) \in \kappa(x)$ of f at x.

Exercise 6.2.5. (1) Check that the regular functions $f(x) = x$ and $f(x) = x^2$ on $\mathbb{A}^1_k$ are distinct as elements of the stalk $\mathcal{O}_{\mathbb{A}^1_k,0}$, but equal as elements of the fiber of $\mathcal{O}_{\mathbb{A}^1_k}|_0$.
(2) Let $k = \bar{k}$ be an algebraically closed field. Check that the quasi-coherent sheaf over $\mathbb{A}^1_k$ associated to $k[t]/(t - \lambda)$ has fiber equal to k at $(t - \lambda)$ and zero at every other closed point.

6.3 Revisiting the structure theorem for f.g. modules over a PID

Let us use the language we have developed so far to understand a classical theorem from abstract algebra in geometric terms - namely, the structure theorem for finitely generated modules over a PID R. We begin by recalling the statement:

Theorem 6.3.1. *Let R be a PID, and let M be a finitely generated A-module. Then M admits a unique decomposition (up to shuffling) into a free summand followed by direct sums of quotients of R by powers of prime ideals:*

$$M \cong R^{\oplus n} \oplus R/(p_1)^{s_1} \oplus \ldots \oplus R/(p_m)^{s_m}. \tag{10}$$

Thinking of M as a quasi-coherent sheaf over $\operatorname{Spec} R$, this is saying that M looks like a direct sum of n copies of the structure sheaf, plus some pointwise fuzz of order s_i over a finite number of points $\mathfrak{p}_i := (p_i) \in \operatorname{Spec} R$, given by pushforwards of structure sheaves of the corresponding closed subschemes. In particular, M looks like a free sheaf away from a finite set of points: $\widetilde{M}|_{D(f)} \cong \mathcal{O}^{\oplus n}_{\operatorname{Spec} R_f}$ whenever $D(f) \subseteq \operatorname{Spec} R$ avoids the points $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ (i.e whenever. f vanishes at each $\mathfrak{p}_i$). This picture may be formalized in algebraic language as follows:

Definition 6.3.2. Say that an R-module M is **supported** at $\mathfrak{p} \in \operatorname{Spec} R$ if $M_{\mathfrak{p}} \neq 0$, and define the **set-theoretic support** of M to be the subset

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \operatorname{Spec} R \mid M \text{ is supported at } \mathfrak{p}\}.$$

Geometrically, the condition $M_{\mathfrak{p}} \neq 0$ means that the sheaf $\tilde{M}$ admits non-zero germs of sections near $\mathfrak{p}$.

Exercise 6.3.1. (1) Show that, for finitely-generated M,

$$\operatorname{supp}(M) = V(\operatorname{ann}_R(M)),$$

where $\operatorname{ann}_R(M) := \{f \in R \mid f \cdot M = 0\}$. Thus, the set-theoretic support of a coherent sheaf over an affine scheme is always closed.
(2) Show that if M decomposes as in 10 with no free part (i.e. $n = 0$), then

$$\operatorname{supp}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}.$$

(3) Show that if two R-modules M, N have disjoint supports, then the *only* R-module map $M \to N$ is the zero map. (Hint: you may find exercise 5.2.7 useful.)

Note 6.3.3. The set-theoretic support is insensitive to nilpotent thickenings (whence its name!). As such, this notion is closer to a property of the *fibers* of M. To amend this, one may instead consider the **scheme-theoretic support** of M, which is defined to be the closed subscheme:

$$\operatorname{Spec}(R/\operatorname{ann}_R(M)) \hookrightarrow \operatorname{Spec} R.$$

Exercise 6.3.2. Exhibit two abelian groups which have equal set-theoretic support but different scheme-theoretic support when viewed as quasi-coherent sheaves over $\operatorname{Spec}\mathbb{Z}$.

Specializing our discussion to the case $R = \mathbb{C}[t]$ provides a new lens on a classical result from linear algebra:

Extended Exercise 6.3.3. (The Jordan normal form from the geometric viewpoint)

Let $A : V \to V$ denote a linear endomorphism of a finite dimensional $\mathbb{C}$-vector space V.

(1) Explain how this data can be interpreted as a coherent sheaf $\tilde{V}$ over $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[t])$.

(2) Using the structure theorem for f.g. modules over PIDs, express $\tilde{V}$ as a direct sum of pushforwards of structure sheaves over irreducible closed subschemes of $\mathbb{A}^1_{\mathbb{C}}$ (note: these closed subschemes may not be reduced!).

(3) Show that elements of the support of V as a $\mathbb{C}[t]$-module correspond to eigenvalues of A, and that the direct summands from (2) correspond to the generalized eigenspaces of A.

(4) Formulate an interpretation of the geometric multiplicity of a given eigenvalue in terms of a geometric property of the coherent sheaf $\tilde{V}$.

(5) You may now recover the Jordan normal form of A by picking an appropriate basis for each direct summand!

6.4 Algebraic vector bundles and the Serre-Swan theorem

Having developed some tools and intuition for working with modules as sheaves over affine schemes, let us come back to our original goal, which was to study algebraic vector bundles over a scheme. Motivated by our discussion in section 6.1, we define an **algebraic vector bundle** over a scheme X to be a locally free sheaf of $\mathcal{O}_X$-modules of finite rank over X.

Affine locally over $X = \operatorname{Spec} R$, via the QCoh-Mod correspondence, these correspond to R-modules M with the property that we can find an open cover $X = \bigcup_i D(f_i)$ such that each M_{f_i} is isomorphic to a free R_{f_i}-module of finite rank[55]. Algebraic vector bundles may be viewed as a full subcategory $\mathrm{Vect}^{\mathrm{alg}}(X) \subseteq \mathrm{QCoh}(X)$, and in fact of $\mathrm{Coh}(X)$ since the finite rank hypothesis guarantees that the corresponding modules are finitely generated.

This definition turns out to be both sensible and tractable: sensible because it is possible to show that algebraic vector bundles as defined are in 1-1 correspondence with "physical vector bundles" over $\operatorname{Spec} R$ which are locally free with respect to the Zariski topology (more on this later), and tractable because it is possible to prove the following algebraic analogue of Swan's theorem:

Theorem 6.4.1. ***(Serre-Swan theorem)***[56] *Let R be a Noetherian ring. Then there is an equivalence of categories:*

$$\mathrm{Vect}^{alg}(\operatorname{Spec} R) \xrightarrow{\simeq} \mathrm{PMod}_R^{fg},$$

where PMod_R^{fg} denotes the category of finitely-generated projective R-modules. This equivalence is given by sending a vector bundle $\widetilde{M}$ over X to its R-module of global sections M.

Let us move towards a proof of the Serre-Swan theorem, as this will be a good occasion to illustrate how geometric intuition can blend with algebraic arguments. Since we already know that the assignment $\widetilde{M} \mapsto M$ is fully faithful at the level of $\mathrm{Coh}(\operatorname{Spec} R)$, it suffices to check that the essential image of the restriction of this functor to $\mathrm{Vect}^{\mathrm{alg}}(\operatorname{Spec} R)$ is indeed given precisely by finitely generated projective modules.

We start by looking more carefully at the relationship between the stalk $M_{\mathfrak{p}}$ of a sheaf of modules at a point and the fiber $M|_{\mathfrak{p}} = M \otimes_R \kappa(\mathfrak{p})$ at that point. We would like to say that, even though the stalk contains more information than the fiber, the latter should control the former in some capacity.

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[55] Recall that $\{D(f_i)\}_{i=1}^r$ forms an open cover of $\operatorname{Spec} R$ iff we can use the f_i's to construct a "partition of unity" $1 = \sum_{i=1}^r f_i g_i$, for some choice of $g_1, ..., g_r \in R$.

[56] In the spirit of diagram 6, it is actually possible to prove the Serre-Swan theorem, Swan's theorem, and the analogue of Swan's theorem for smooth vector bundles "in one fell swoop" by working at the appropriate level of generality in the category $\mathcal{RS}$ of ringed spaces. See [**Morye**] for more details.

This is essentially the content of Nakayama's lemma, which may be stated as follows:

Lemma 6.4.2. ***(Nakayama's lemma)*** *Let $(R, \mathfrak{p})$ be a local ring, and let M be a finitely-generated R-module. Then any basis of $M|_{\mathfrak{p}} = M \otimes_R \kappa(\mathfrak{p})$ as a $\kappa(\mathfrak{p})$-vector space lifts to a minimal generating set of M.*

Proof. Let us prove the equivalent statement that $M \otimes_R \kappa(\mathfrak{p}) = 0$ implies that $M = 0$. To this end, choose a generating set $s_1, ..., s_n \in M$, and let us induct on n. The case $n = 0$ is trivially true. Now, $M \otimes_R \kappa(\mathfrak{p}) = M \otimes_R R/\mathfrak{p} = M/\mathfrak{p}M = 0$ holds iff $M = \mathfrak{p}M$, so we may view $s_1 \in M$ as an element of $\mathfrak{p}M$ and express it in the form

$$s_1 = \sum_{i=1}^{n} f_i s_i, \text{ each } f_i \in \mathfrak{p}.$$

It follows that $(1 - f_1)s_1 = \sum_{i=2}^{n} f_i s_i$. But since $f_1 \in \mathfrak{p}$ and R is a local ring, $1 - f_1$ is a unit, and we may invert it to get that

$$s_1 = (1 - f_1)^{-1} \sum_{i=2}^{n} f_i s_i,$$

contradicting minimality of the generating set unless $M = 0$, as needed. □

Exercise 6.4.1. (1) Check that the statement we just proved is indeed equivalent to Nakayama's lemma as stated.
(2) Using that M is finitely generated, show that Nakayama's lemma may be interpreted as saying that *if the fiber of a coherent sheaf $\widetilde{M}$ vanishes at a point $\mathfrak{p}$, then we can find an open neighborhood $U \ni \mathfrak{p}$ along which $\widetilde{M}|_U = 0$.*
(3) Interpreting the Jacobson radical $J \subseteq R$ geometrically, try to deduce Nakayama's lemma in its classical form - namely, that for M a f.g. module over an arbitrary ring, $JM = M$ implies that $M = 0$. (Hint: you may find it useful to adapt exercise 5.2.7.)

With Nakayama's lemma in hand, we are in a position to prove the following proposition, which will in particular imply the Serre-Swan theorem:

Proposition 6.4.3. *Let M be a finitely-generated module over a Noetherian ring R. Then the following are equivalent:*

- *(1) M is projective.*
- *(2) The stalks of the quasi-coherent sheaf $\tilde{M}$ are all free.*
- *(3) The quasi-coherent sheaf $\tilde{M}$ is locally free in the Zariski topology.*

Proof. We sketch the proof that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2): Localizing at a prime $\mathfrak{p}$ (and using that localization is exact), we are left to show that if M is a finitely-generated projective module over a Noetherian local ring $(R, \mathfrak{p}, \kappa)$, then M is necessarily free. We may pick a basis for the unique fiber $M|_{\mathfrak{p}} = M \otimes_R \kappa$, and invoke Nakayama's lemma to extend it to a generating set of M, corresponding to a surjection $R^{\oplus n} \twoheadrightarrow M$ for some n. This map fits into a short exact sequence

$$0 \to K \to R^{\oplus n} \to M \to 0,$$

and it suffices to show that $K = 0$. Now, notice that the above sequence is split-exact since M is projective, hence it is taken to an exact sequence upon tensoring with κ over R, so that $K|_{\mathfrak{p}} = 0$, and therefore $K = 0$ by Nakayama's lemma, as needed.

(2) $\implies$ (3): Let us look for a trivializing Zariski open around a given $\mathfrak{p} \in \operatorname{Spec} R$. By assumption, $M_{\mathfrak{p}}$ is free, and we may choose a basis $s_1, ..., s_n$. Since this is a finite set, and $M_{\mathfrak{p}} = \varinjlim_{g \notin \mathfrak{p}} M_g$, we may view the s_i's collectively as sections of M over a principal open $D(g) \ni \mathfrak{p}$. We wish to show that, possibly upon restricting to a further open $D(f) \subseteq D(g)$, the s_i's form a basis of M_f. To this end, let L be the submodule of M_g generated by the s_i's and set $Q := M_g/L$, with $Q_{\mathfrak{p}} = 0$ by exactness of localization. By a similar finiteness argument to the above, we may find a principal open $\mathfrak{p} \in D(h) \subseteq D(g)$ along which $Q_h = 0$, so that $L_h = M_h$, and therefore the $s_i|_{D(h)}$'s lead to an exact sequence:

$$0 \to K \to R_h^{\oplus n} \to M_h \to 0.$$

We may conclude by showing that $K_f = 0$ for some $\mathfrak{p} \in D(f) \subseteq D(h)$ using Nakayama's lemma.

(3) $\implies$ (1) Assume $\tilde{M}$ is locally free with respect to an open cover $\{D(f_i)\}$, and suppose we are given a SES of R-modules

$$0 \to L \to N \to P \to 0.$$

We wish to show that the sequence that results upon applying $\mathrm{Hom}_R(M,-)$ is exact. We may equivalently prove that it is exact inside $\mathrm{QCoh}(\operatorname{Spec} R)$, where it suffices to check exactness of the sequences

$$0 \to \mathrm{Hom}_R(M,L)_{f_i} \to \mathrm{Hom}_R(M,N)_{f_i} \to \mathrm{Hom}_R(M,P)_{f_i} \to 0$$

for each i. Since M is finitely-generated, we have that $\mathrm{Hom}_R(M,L)_f \cong \mathrm{Hom}_{R_f}(M_f,N_f)$ for any R-module N, so that exactness of the above sequences follows from projectivity of the free modules M_{f_i}. $\square$

Exercise 6.4.2. Complete the above proof by showing that, given R-modules M and N such that M is finitely-generated, then for any $f \in R$,

$$\mathrm{Hom}_R(M,N)_f \cong \mathrm{Hom}_{R_f}(M_f,N_f).$$[57]

Putting everything together, we have arrived at the following picture:

Geometry	**Vector bundles as sheaves**	**Vector bundles as spaces**	**Stalks**	**Fibers**
Compact Hausdorff space X	Locally free sheaf of C^0_X-modules $\mathcal{E}$	Topological vector bundle E	$\mathcal{E}_x = i_x^{-1}\mathcal{E}$	$E_x \cong \mathbb{R}^n$
Smooth manifold M	Locally free sheaf of $\mathcal{O}_M$-modules $\mathcal{E}$	Smooth vector bundle E	$\mathcal{E}_x = i_x^{-1}\mathcal{E}$	$E_x \cong \mathbb{R}^n$
Affine scheme $\operatorname{Spec} R$	Finitely-generated projective R-module M	Relative Spec $\mathrm{Spec}_{\operatorname{Spec} R}(\mathrm{Sym}^*_R(M^\vee))$	$M_\mathfrak{p} \cong M \otimes_R R_\mathfrak{p}$	$M \otimes_R \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})^{\oplus n}$
Scheme X	Locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$	Relative Spec $\mathrm{Spec}_X(\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E}^\vee))$	$\mathcal{E}_x = i_x^{-1}\mathcal{E}$	$i_x^*\mathcal{E} \cong \kappa(\mathfrak{p})^{\oplus n}$

We should note that the lower two entries of the third column involve a construction which we have not discussed, namely the *relative Spec*, or *global Spec* construction. We only make a few comments to provide some intuition

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[57] This means that, for finitely-generated M, *the internal hom in the category* $\mathrm{QCoh}(\operatorname{Spec} R)$ *coincides with the R-module* $\mathrm{Hom}_R(M,N)$ *interpreted as a quasi-coherent sheaf*, a fact which is not true in general in the category $\mathrm{QCoh}(X)$.

for this construction; for more details, see section 1.3.3 of [**EH**]. First, note that, given an n-dimensional k-vector space V, the polynomial ring

$$k[t_1, ..., t_n] \cong \mathrm{Sym}_k^*(V)$$

may be interpreted as the algebra of regular functions on V - so that V may be encoded algebro-geometrically as $\mathrm{Spec}\,\mathrm{Sym}_k^*(V^\vee)$, where we only pass to duals to ensure the right directionality of functoriality. Next, given a scheme X and a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$, the *relative Spec construction over* X provides us with a way of "spreading the sheaf $\mathcal{A}$ into a physical space over X", denoted by $\mathrm{Spec}_X(\mathcal{A})$, with fiber over each point $x \in X$ given by $\mathrm{Spec}\, i_x^* \mathcal{A}$:

$$\begin{tikzcd}
\mathrm{Spec}\, i_x^*\mathcal{A} \arrow[r] \arrow[d] & \mathrm{Spec}_X(\mathcal{A}) \arrow[d] \\
\mathrm{Spec}\, \kappa(x) \arrow[r, "i_x"] & X.
\end{tikzcd}$$

6.5 A word on algebraic K-theory

Using the Serre-Swan theorem, we see that the structure theorem for f.g. modules over a PID has the following geometric corollary:

Corollary 6.5.1. *Let R be a PID. Then every algebraic vector bundle over* $\mathrm{Spec}\, R$ *is free.*

In particular, we see that the affine line $\mathbb{A}_k^1 = \mathrm{Spec}\, k[t]$ over a field k "behaves like a contractible space" as far as vector bundles over it are concerned. Finding analogues of this result for other algebro-geometric spaces that "should be contractible" leads to a lot of interesting mathematics. For instance, Serre conjectured in 1957 and Quillen-Suslin proved in 1976 that every algebraic vector bundle over $\mathbb{A}_k^n$ is free for arbitrary $n \geq 1$.

Taking this observation as a starting point, we take a moment to touch on the vast landscape of algebraic K-theory, and to indicate some parallels with its topological analogue. For a wonderfully readable treatment on more early results in this direction, we invite the reader to consult [**Dugger**].

In algebraic topology, one may associate to a space X an algebraic invariant built out of the monoid of topological vector bundles on X, known as the *topological K-theory group of X*. Importing this idea to the present settings leads to the following:

Definition 6.5.2. Given a commutative ring R, we define its 0^{th} **algebraic K-theory group** $K_0(R)$ to be the Grothendieck group associated to the monoid formed by isomorphism classes of finitely-generated projective R-modules under direct sum. Explicitly,

$$K_0(R) := \mathbb{Z}\left\langle \begin{matrix} \text{iso. classes of f.g.} \\ \text{projective } R\text{-modules} \end{matrix} \right\rangle \Big/\!\!\Big/ \Big([P \oplus P'] = [P] + [P'] \Big).$$

More generally, given a scheme X, we define

$$K_0(X) := \mathbb{Z}\left\langle \begin{matrix} \text{iso. classes of algebraic} \\ \text{vector bundles on } X \end{matrix} \right\rangle \Big/\!\!\Big/ \Big([P \oplus P'] = [P] + [P'] \Big).$$

Note 6.5.3. (1) For "non-singular" affine schemes Spec R, which algebraically translates to R being a *regular* ring, one could do away with the projectivity assumption. Namely, for regular R, one can show that $K_0(R)$ may equivalently be obtained as follows:

$$G_0(R) := \mathbb{Z}\left\langle \begin{matrix} \text{iso. classes of f.g.} \\ R\text{-modules} \end{matrix} \right\rangle \Big/\!\!\Big/ \left(\begin{matrix} [M'] + [M''] = [M] \text{ for every SES} \\ 0 \to M' \to M \to M'' \to 0 \end{matrix} \right).$$

However, for non-regular schemes, the group $G_0(R)$ will in general differ from $K_0(R)$ - for instance, this is the case for the affine plane with a double origin[58].
(2) Unlike projectivity, finite-generation is crucial for this definition to even make sense, due to the so-called "Eilenberg swindle": if we didn't require our modules to be finitely-generated, then given any R-module M, we could build a short exact sequence:

$$0 \to M \to M^{\oplus \mathbb{N}} \to M^{\oplus \mathbb{N}} \to 0,$$

where the second map is given by "shifting right by one". This would then imply that $[M] = 0$ for every M.

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[58] To first approximation, one may think of $G_0(R)$ as a "homology theory", of $K_0(R)$ as a "cohomology theory", and of the isomorphism between them for regular R as a kind of "Poincaré duality".

Exercise 6.5.1. (1) Show that $K_0(k) = \mathbb{Z}$ for any field k.
(2) Let G be a finite group, and let $\mathbb{C}G$ denotes its group algebra over $\mathbb{C}$. Using Maschke's theorem, show that $K_0(\mathbb{C}G) \cong \mathbb{Z}^r$, where r denotes the number of irreducible representations of G.

Theorem 6.5.4. *($\mathbb{A}^1_k$-**homotopy invariance of algebraic** K_0) Let R be a Noetherian k-algebra. Then we have an isomorphism:*

$$K_0(R[t]) \cong K_0(R).$$[59]

Let us elucidate the name of this theorem, which is a good occasion to introduce the following:

Definition 6.5.5. Given two k-schemes X and Y, their **product** $X \times Y \in \mathrm{Sch}_k$ is defined to be the categorical product in Sch_k, i.e.

$$X \times Y := \varprojlim \left(\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} k \end{tikzcd} \right)$$

Exercise 6.5.2. Given an affine k-scheme $\operatorname{Spec} R$, show that

$$\operatorname{Spec} R \times \mathbb{A}^1_k \cong \operatorname{Spec} R[t].$$

Hence, in geometric terms, the above theorem posits that for any affine k-scheme X, we have an isomorphism:

$$K_0(X \times \mathbb{A}^1_k) \cong K_0(X),$$

so that algebraic K_0 is "insensitive to crossing with the algebro-geometric unit interval".

Exercise 6.5.3. Let k be a field. Show that we can identify

$$K_0(\operatorname{Spec} k[t]/(t^2)) \cong K_0(\operatorname{Spec} k).$$

This hints at the more general fact that algebraic K-theory "is insensitive to infinitesimal thickenings".

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[59] This is one of the ways in which $\mathbb{A}^1_k$ behaves "like a unit interval" when trying to apply methods from algebraic topology to algebraic geometry over k - taking this principle seriously has led to a vibrant new field of study known as motivic homotopy theory, an overview of which may be found in [**Levine**].

We end this section with a comment which may be of interest to topologically-minded readers. Starting with a compact Hausdorff space X, we may form the *complex topological K-theory group* $KU(X)$ by taking the Grothendieck group of the category $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{top}}(X)$ of topological complex vector bundles on X. By Swan's theorem, we have an equivalence:

$$\mathrm{Vect}_{\mathbb{C}}^{\mathrm{top}}(X) \simeq \mathrm{PMod}_{C^0(X)}^{\mathrm{fg}}.$$

So that, passing to Grothendieck groups on the monoids of isomorphism classes of objects on either end, we obtain an isomorphism, for any compact Hausdorff space X:

$$KU(X) \simeq K_0(C^0(X)).$$[60]

Thus, the topological K-theory groups of any compact Hausdorff space X may be expressed internally to the language of algebraic geometry.[61]

7 Tangent Spaces and Algebraic Differential Forms

The basic goal of this section, and the following, is to demonstrate the ways in which one can talk about differential geometric notions through entirely algebraic means. As usual, the game we are about to engage in consists in taking definitions from manifold theory, interpreting them in an equivalent form which can be phrased without explicit reference to Euclidian space, and then importing the notions into an algebro-geometric context. For this particular section we will work with varieties over an algebraically closed field k. By point $x \in X$, we always mean a *closed point*, which necessarily has residue field $\kappa(x) = k$ since k is algebraically closed. However, the notions we introduce can be defined in much more general settings. Much of material for this section is drawn from [Mum99].

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[60] We could just as well have obtained an isomorphism $KO(X) \simeq K_0(C^0(X;\mathbb{R}))$ for the real topological K-theory group of X by working with the ring of real-valued continuous functions on the algebraic side.

[61] This might be as fitting a place as any to mention David Mumford's amusing comment about algebraic geometry: "Algebraic geometry seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics. In one respect this last point is accurate."

7.1 Starting point: tangent vectors on manifolds as derivations

Our first goal will be to develop the notion of the tangent space to a scheme. We start by reviewing some ideas from the study of manifolds. Given a smooth manifold M^n, recall that one may define the *tangent space* T_xM to M at a point x to be the collection of curves $\gamma : (-1,1) \to M$ with $\gamma(0) = x$, under the following equivalence relation: two curves are identified if they have the same derivative at 0 when interpreted as maps to Euclidean space with respect to an affine chart U around x. In local coordinates $x_1, ..., x_n$ on U, any tangent vector $[\gamma] \in T_xM$ may then be expressed uniquely as a linear combination of the form:

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}|_x, \tag{11}$$

where $a_i \in \mathbb{R}$ is the i^{th} entry of $\gamma'(0)$ in local coordinates. Thus, T_xM is an n-dimensional vector space with basis given by "partial derivatives at x" $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$.

The above expression suggests another approach: define a **derivation** on the ring $\mathcal{O}(M)$ of smooth functions on M at x to be an $\mathbb{R}$-linear map $\delta : \mathcal{O}(M) \to \mathbb{R}$ satisfying the following "Leibniz rule" for every $f, g \in \mathcal{O}(M)$:

$$\delta(fg) = f(x)\delta(g) + g(x)\delta(f).$$

Denote the set of derivations of $\mathcal{O}(M)$ at x by $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}(M), \mathbb{R}_x)$. Note that we are able to add and scale derivations, so we obtain a vector space structure on $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}(M), \mathbb{R}_x)$.

Exercise 7.1.1. Verify that any derivation vanishes on constant functions. Deduce that δ is fully determined by its behavior on the maximal ideal $\mathfrak{m}_x \ \ker(\mathrm{ev}_x) \subseteq \mathcal{O}(M)$ of smooth functions vanishing at x.

Now, there is a map from our previous notion of tangent vectors to derivations by taking a curve γ and defining the derivation $\delta(f) := (f \circ \gamma)'(0)$. In the local coordinate notation of equation 11, this expression takes the following form:

$$\delta(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(x). \tag{12}$$

Exercise 7.1.2. (1) Show that the above map defines an $\mathbb{R}$-linear isomorphism:

$$T_xM \xrightarrow{\cong} \mathrm{Der}_{\mathbb{R}}(\mathcal{O}(M), \mathbb{R}_x).$$

(2) Let $\mathcal{O}_{M,x}$ denote the ring of germs of smooth functions at x. Using equation 12 to interpret derivations explicitly, show that we have an isomorphism:

$$\mathrm{Der}_{\mathbb{R}}(\mathcal{O}(M), \mathbb{R}_x) \cong \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{M,x}, \mathbb{R}_x),$$

where $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{M,x}, \mathbb{R}_x)$ denotes the $\mathbb{R}$-vector space of derivations of germs at x, i.e. $\mathbb{R}$-linear maps out of $\mathcal{O}_{M,x}$ satisfying the same Leibniz rule as above. Thus, derivations at x only depend on infinitesimal information near x.

The perspective on tangent vectors as derivations admits an equivalent form that will be most useful to us. By exercise 7.1.1, any derivation $\delta : \mathcal{O}(M) \to \mathbb{R}$ is fully determined by its restriction to the maximal ideal $\mathfrak{m}_x = \ker(\mathrm{ev}_x) \subseteq \mathcal{O}(M)$. Furthermore, this map factors through $\mathfrak{m}_x/\mathfrak{m}_x^2$, because if $f, g \in \mathfrak{m}_x$ then $\delta(fg) = 0$. Conversely, if we are given a map $\alpha : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{R}$, then by setting

$$\delta(f) := \alpha(f - f(x) \pmod{\mathfrak{m}_x^2})$$

we get a valid derivation of $\mathcal{O}(M)$ at x. Thus, the tangent space of M at x may have been equivalently defined to be the vector space $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R})$.[62]

With the above motivation in mind, we define the tangent space to an algebraic variety as follows:

Definition 7.1.1. The **Zariski cotangent space** of a variety X at a point $x \in X$ is defined to be the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the maximal ideal of the stalk $\mathcal{O}_{X,x}$. The **Zariski tangent space** is defined to be the dual vector space $\mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$. We will denote the tangent space at x by T_xX and the cotangent space by T_x^*X.

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[62] There is a precise sense in which the correspondence

$$\mathrm{Der}_{\mathbb{R}}(\mathcal{O}(X), \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R})$$

boils down to the fact that a derivation on smooth functions at a point x only depends the *first-order jet of smooth functions at* x, coupled with the fact that derivations vanish on constant functions.

Example 7.1.2. Let $X = \mathbb{A}_k^n = \operatorname{Spec} k[t_1, \dots t_n]$. Then the cotangent space at the origin is given by $T_0^*\mathbb{A}_k^n = (t_1, \dots t_n)/(t_i t_j)_{i,j=1,n}$, which has basis $t_1, \dots t_n$, and thus the tangent space is isomorphic to k^n, as expected.

Example 7.1.3. Let $X = \operatorname{Spec} k[t_1, t_2]/(t_1^2 - t_2 + t_1^3) \subseteq \mathbb{A}_k^2$. If $x \in X$ denotes the origin, we have that

$$T_x^*M = (t_1, t_2)/(t_1^2 - t_2^2 + t_1^3, t_1^2, t_1 t_2, t_2^2) = (t_1, t_2)/(t_1^2, t_1 t_2, t_2^2) \cong k^2,$$

so $T_xM \cong k^2$. However, for any other point on the curve, the tangent space is 1 dimensional! The latter is the expected dimension given that X is a curve. This example suggests there is a connection between the dimension of the Zariski tangent space and smoothness.

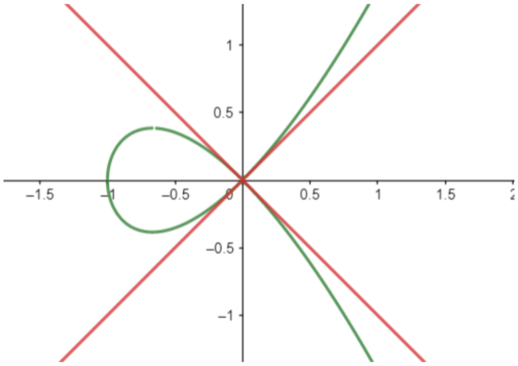

Figure 2: Visualizing $X = \operatorname{Spec} k[t_1, t_2]/(t_1^2 - t_2 + t_1^3)$ and its tangents at the origin.

We also comment that the "tangent space at x as derivations out of the stalk of $\mathcal{O}_{X,x}$" perspective obtained for smooth manifolds in exercise 7.1.2 is also something that can be formulated in the algebro-geometric settings, producing an equivalent definition once again:

Exercise 7.1.3. (1) Let $\operatorname{Der}_k(\mathcal{O}_{X,x}, k_x)$ denote the k-vector space of derivations of the stalk $\mathcal{O}_{X,x}$ of regular functions at $x \in X$. These are the k-linear maps $\delta : \mathcal{O}_{X,x} \to k$ satisfying the Leibniz rule:

$$\delta(fg) = f(x)\delta(g) + g(x)\delta(f),$$

where we write $f(x) := f \pmod{\mathfrak{m}_x} \in \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k_x$ Show that the assignment sending a functional $\alpha : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ to the map

$$\delta : \mathcal{O}_{X,x} \to k$$

$$f \mapsto \alpha\big(f - f(x) \pmod{\mathfrak{m}_x^2}\big)$$

gives a well-defined map:

$$T_xX := \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k) \to \mathrm{Der}_k(\mathcal{O}_{X,x}, k_x).,$$

and show that this map is in fact a k-linear isomorphism.
(2) Suppose that $X = \operatorname{Spec} R$ is affine, and that $x \in X$ corresponds to the maximal ideal $\mathfrak{m} \subseteq R$. Show that have a k-linear isomorphism:

$$\mathrm{Der}_k(R, k_{\mathfrak{m}}) \cong \mathrm{Der}_k(R_{\mathfrak{m}}, k_{\mathfrak{m}}).$$

(This is the formal analogue of part (2) of exercise 7.1.2.)

One last initial observation is that, just as smooth maps between manifolds have derivatives, morphisms between varieties induce maps on tangent spaces. Given a morphism $\varphi : X \to Y$ and a point $x \in X$, recall that we have an associated map on stalks given by “pulling back representatives” (or alternatively by the UP of colimits):

$$\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}.$$

This is furthermore a map of local rings, so it induces a map

$$\mathfrak{m}_{\varphi(x)}/\mathfrak{m}_{\varphi(x)}^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2,$$

which produces the desired map $d_x\varphi : T_xX \to T_{\varphi(x)}Y$ by passing to duals.

Exercise 7.1.4. Show that, under the identification of tangent spaces as spaces of derivations obtained in exercise 7.1.3, the map $d_x\varphi : T_xX \to T_{\varphi(x)}Y$ could have equivalently been defined via “pullback of derivations” along φ_x^*:

$$\begin{tikzcd}
\mathcal{O}_{Y,\varphi(x)} \arrow[r, "\varphi_x^*"] \arrow[dr, dashed] & \mathcal{O}_{X,x} \arrow[d, "\delta"] \\
 & k.
\end{tikzcd}$$

(Note that the Leibniz rules for derivations out of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,\varphi(x)}$ differ!)

Exercise 7.1.5. Let $\varphi : X \to Y$, $X \subseteq \mathbb{A}^n_k$, $Y \subseteq \mathbb{A}^m_k$ be a map of varieties given by polynomials $\varphi_1, \dots \varphi_m \in k[t_1, \dots t_n]$. Show that $d_x\varphi : T_xX \to T_{\varphi(x)}Y$ the linear map corresponding to the Jacobian matrix evaluated at x:

$$\big(\frac{d\varphi_i}{dt_j}\big|_x\big)_{i,j}.$$

7.2 The Zariski tangent space and smoothness

We will now investigate what smoothness might mean in algebro-geometric context. For a d-dimensional subset, X, of $\mathbb{R}^n$ defined by equations $f_1 = \cdots f_r = 0$ one way of checking if X is smooth at a point x is to verify that the rank of the following matrix is $n - d$:

$$\text{Rank}\,\big(\frac{df_i}{dt_j}\big|_x\big)_{i=1,n\ j=1,r} = n - d$$

So for a general variety d-dimensional variety $X = \operatorname{Spec} k[t_1, \dots t_n]/(f_1 \dots f_r)$ we say that X is **smooth at** x if the rank of the above matrix is $n-d$, where all derivatives are just taken as formal derivatives of polynomials.

Example 7.2.1. Let $X = \operatorname{Spec} k[t_1, t_2]/(t_1^2 - t_2 + t_1^3)$, then if x is the origin we have that:

$$\big(\frac{df_i}{dt_j}\big|_x\big)_{i=1,n\ j=1,r} = \begin{pmatrix} 2t_1 + 3t_1^2 & -2t_2 \end{pmatrix}\big|_{(0,0)} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

And X is one dimensional, and inside of 2 dimensional space, so X is *not* smooth at the origin.

Furthermore one can show that the Zariski tangent space at a point x is given by the equations:

$$\sum_{i=1}^n \frac{\partial f_1}{\partial t_j}\big|_x t_j = \cdots \sum_{i=1}^n \frac{\partial f_r}{\partial t_j}\big|_x t_j = 0$$

which is exactly the kernel of the matrix:

$$\big(\frac{df_i}{dt_j}\big|_x\big)_{i=1,n\ j=1,r}$$

Therefore the condition of being smooth is the same as $\dim T_xX = \dim X$.

Exercise 7.2.1. Consider the variety $X = \operatorname{Spec} k[t_1, \dots t_n]/(f_1, \dots f_r) \subseteq \mathbb{A}^n_k$. Show that the tangent space to a point $x \in X$ is given by the equations:

$$\sum_{i=1}^{n} \frac{\partial f_1}{\partial t_j}\Big|_x t_j = \cdots = \sum_{i=1}^{n} \frac{\partial f_r}{\partial t_j}\Big|_x t_j = 0$$

7.3 The co/tangent sheaves and the sheaf of relative differential forms

We now seek a more conceptual understanding of tangent spaces, in the hope of introducing some notion of a tangent or cotangent bundle. To do this, we start by introducing the algebraic analogue of differential 1-forms - classically, the latter can be understood as sections of the cotangent bundle of a manifold M, or equivalently as smooth choices of linear functionals on the tangent space of M at each point.

Definition 7.3.1. Let $A \to B$ be a map of rings. We define the B-module of **relative differentials** $\Omega_{B/A}$ to be the following quotient of the free B-module generated by the set $\{df \mid f \in B\}$:

$$\Omega_{B/A} := B\big\langle df \;|f \in B\big\rangle \Big/ \left(\begin{array}{c} d(f+g) = df + dg \\ d(fg) = fdg + gdf \\ da = 0 \;\text{ if }\; a \in A \end{array} \right),$$

where we identify $a \in A$ with its image under $A \to B$.

Note 7.3.2. (1) Note that $\Omega_{B/A}$ comes equipped with a canonical A-module homomorphism:

$$d : B \to \Omega_{B/A}$$

defined by $f \mapsto df$. By definition of $\Omega_{B/A}$, this map is an A-linear derivation. We shall return to this shortly.

(2) Let us provide some geometric intuition about the B-module $\Omega_{B/A}$, thought of as a quasi-coherent sheaf over $\operatorname{Spec} B$: one can think of the ring map $A \to B$ geometrically as a "bundle map"

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} A. \end{tikzcd}$$

The module of relative differentials $\Omega_{B/A}$ consists of those algebraic differential forms on $\operatorname{Spec} B$ which, as functionals on tangent vectors, are only allowed to be non-trivial along the fibers of this map. Under this viewpoint, the map $d : B = \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \to \Omega_{B/A}$ may be thought of as an algebraic analogue of the de Rham differential, taking a regular function f to its differential df.

Example 7.3.3. The central example is the case when $A = k$ and $B = k[t_1, \dots t_n]/(f_1, \dots f_r)$ - i.e. the "absolute case" of a variety over a point. In this case, $\Omega_{B/k}$ is generated as a B-module by elements dt_i, with relations $df_i = 0$, and all elements df are of the form:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} dt_i.$$

Note that if $f : X \to \mathbb{A}^1_k$ is a morphism, thought of as an algebraic function on X, then df does give the derivative of f at any point x, so these differential can be viewed as giving linear functions on the tangent space.

We next introduce a more algebraic description of the module of relative differentials, which generalizes well to the non-affine settings. We start by observing that, under the $\mathrm{CAlg}_k^{op} \simeq \mathrm{AffSch}_k$ equivalence, the ring map $A \to B$ gives rise to a "diagonal map" $\Delta : \operatorname{Spec} B \to \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B$ via the following picture:

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[dr, dashed, "\exists!\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \operatorname{Spec}(B \otimes_A B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
& \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A.
\end{tikzcd}$$

This map corresponds to the ring map $\delta : B \otimes_A B \to B$ given by $b_1 \otimes b_2 \mapsto b_1 b_2$. The kernel of δ is an ideal of $B \otimes_A B$ cutting out a closed subscheme of $\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B$ which we will denote by $\Delta(\operatorname{Spec} B)$.

Proposition 7.3.4. *Let I be the kernel of the map $\delta : B \otimes_A B \to B$ corresponding to the diagonal map $\Delta : \operatorname{Spec} B \to \operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B$. Then*

$$\Omega_{B/A} \cong I/I^2.$$

Proof. See theorem 4 of chapter 3, section 1 in [Mum99]. ☐

Similarly, for a general scheme morphism $F : X \to Y$, one may define a diagonal map $\Delta : X \to X \times_Y X$. In most reasonable situations[63], Δ will be a closed immersion (an isomorphism onto a closed subset of $X \times_Y X$), so that its image $\Delta(X)$ corresponds to a *quasi-coherent sheaf of ideals* $\mathcal{I}$ (this is simply the sheaf-theoretic analogue of ideals in a ring one obtains upon gluing affine schemes together). We may then associate to $\mathcal{I}$ a quasi-coherent sheaf on $X \times_Y X$ given by $\mathcal{I}/\mathcal{I}^2$. Finally, we define **the sheaf of relative differentials** $\Omega_{X/Y}$ to be the pullback this sheaf back along Δ:

$$\Omega_{X/Y} := \Delta^* \mathcal{I}/\mathcal{I}^2.$$

One may initially be mostly interested in the "absolute" case when Y is just a point $\operatorname{Spec} k$, producing the **cotangent sheaf** $\Omega_{X/k}$ of X, but the relative notion we just introduced provides the ability to talk readily about smooth morphisms, i.e. about families of smooth schemes parametrized by a general base Y. This principle of working "relative to an arbitrary base" is often a powerful technical framework, to which we will return in chapter 9.

Let us now illustrate how this discussion connects back to our starting point. To this end, we begin by observing that the sheaf of relative differentials $\Omega_{B/A}$ associated to a ring map $A \to B$ enjoys very pleasant categorical properties. First, notice that the notion of an "A-linear derivation out of B" can be defined with target any B-module M: the Leibniz rule

$$\delta(fg) = f\delta(g) + g\delta(f)$$

only requires a notion of addition and scaling by elements of B. Thus, for any B-module M, we obtain an associated B-module of derivations $\operatorname{Der}_A(B, M)$ with values in M. This is something we have already been doing "behind the scene" throughout this chapter:

Example 7.3.5. Given a smooth manifold M, derivations of $\mathcal{O}(M)$ at $x \in M$ are the same data as $\mathbb{R}$-linear derivations of $\mathcal{O}(M)$ with values in the $\mathcal{O}(M)$-module $\mathbb{R}_x$, whose underlying abelian group is $\mathbb{R}$, and whose $\mathcal{O}(M)$-module structure is given by the canonical projection map:

$$\operatorname{ev}_x : \mathcal{O}(M) \to \mathcal{O}(M)/\mathfrak{m}_x \cong \mathbb{R}_x.$$

---

[63] Namely, when the morphism f is *separated*, an analogue of Hausdorffness in algebraic geometry that is often expected to hold.

That is, given $f \in \mathcal{O}(M)$ and $\lambda \in \mathbb{R}_x$, we set $f \cdot \lambda := f(x)\lambda$. The same remark applies for derivations of the ring $\mathcal{O}(X)$ of regular functions of a variety at a point $x \in X$[64].

Thus, in the general set-up of a ring map $A \to B$, we are naturally led to considering the functor:

$$\operatorname{Der}_A(B, -) : \operatorname{Mod}_B \to \operatorname{Mod}_B .$$

The "miracle" is that this functor turns out to be co-representable, with representing B-module given precisely by the module of relative differentials $\Omega_{B/A}$. That is, for any B-module M, we have the data of a bijection, naturally in M:

$$\operatorname{Der}_A(B, M) \cong \operatorname{Hom}_B(\Omega_{B/A}, M).$$

This bijection may be described explicitly as follows: recall from note 7.3.2 that $\Omega_{B/A}$ comes equipped with a canonical A-linear derivation (a "de Rham differential"):

$$d : B \to \Omega_{B/A}.$$

Then, the above correspondence is given by sending a B-module homomorphism $\varphi : \Omega_{B/A} \to M$ to the composite $\delta := \varphi \circ d : B \to M$, which may be checked to be a derivation. This is often encapsulated by the statement that *any A-linear derivation out of B factors uniquely through the de Rham differential*:

$$\begin{tikzcd}
B \arrow[r, "\forall\delta"] \arrow[d, "d"'] & M \\
\Omega_{B/A} \arrow[ur, dashed, "\exists!\varphi"']
\end{tikzcd}$$

The above diagram may be taken as the defining universal property of $\Omega_{B/A}$, uniquely characterizing it as a B-module up to canonical isomorphism. Although we do not do it here, one could also define the sheaf of relative differentials $\Omega_{X/Y}$ associated to a scheme morphism $F : X \to Y$ via an analogous universal property, characterizing $\Omega_{X/Y}$ as the unique sheaf of $\mathcal{O}_X$-modules co-representing a suitable "sheaf-theoretic derivation functor":

$$\mathcal{D}\mathrm{er}_{\mathcal{O}_Y}(F_*\mathcal{O}_X, -) : \operatorname{Mod}_{\mathcal{O}_X} \to \operatorname{Mod}_{\mathcal{O}_X} .$$

With this universal property in hand, we may readily prove the following:

---

[64] Here, $A = k$, $B = \mathcal{O}(X)$ or equivalently $\mathcal{O}_{X,x}$, and $M = k_x \cong \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Theorem 7.3.6. *Let X be a variety over k, and let $x \in X$. Then we have a k-linear isomorphism:*

$$\operatorname{Hom}_{\mathcal{O}_{X,x}}((\Omega_{X/k})_x, k_x) \cong T_x X,$$

where $(\Omega_{X/k})_x$ denotes the stalk of the cotangent sheaf at x, and k_x is viewed as an $\mathcal{O}_{X,x}$-module via the projection map $ev_x : \mathcal{O}_{X,x} \to k_x \cong \mathcal{O}_{X,x}/\mathfrak{m}_x$.

Proof. We saw in exercise 7.1.3 that we can identify

$$T_x X \cong \operatorname{Der}_k(\mathcal{O}_{X,x}, k_x).$$

Thus, it suffices to identify the RHS with $\operatorname{Hom}_{\mathcal{O}_{X,x}}((\Omega_{X/k})_x, k_x)$. Since this is a local problem, we may assume that $X = \operatorname{Spec} R$, so that, on the one hand, $\operatorname{Der}_k(\mathcal{O}_{X,x}, k_x) \cong \operatorname{Der}_k(R_\mathfrak{m}, k_\mathfrak{m})$, where $\mathfrak{m} \subseteq R$ is the maximal ideal corresponding to x, and on the other hand:

$$\operatorname{Hom}_{\mathcal{O}_{X,x}}((\Omega_{X/k})_x, k_x) \cong \operatorname{Hom}_{R_\mathfrak{m}}((\Omega_{R/k})_\mathfrak{m}, k_\mathfrak{m}) \cong \operatorname{Hom}_{R_\mathfrak{m}}(\Omega_{R_\mathfrak{m}/k}, k_\mathfrak{m}).$$

The desired identification thus follows automatically from the universal property of $\Omega_{R_\mathfrak{m}/k}$ applied to the $R_\mathfrak{m}$-module $k_\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. ☐

Thus, we can sensibly define the **tangent sheaf** of a variety X to be the hom sheaf:

$$T_X := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

By the above theorem, one may recover the tangent space of X at any point $x \in X$ by passing to the fiber at x. Affine locally near x,

$$T_{X,x} \cong \operatorname{Hom}_R(\Omega_{R/k}, R) \otimes_R k_x \cong T_x X.$$

It may appear surprising that the initially separate vector spaces $T_x X$'s all come together under this sheaf. One benefit to this construction is that it gives a criterion for smoothness which is that X if and only if the cotangent sheaf is locally free, furthermore one can also show that for any variety the set smooth points is dense. See [Mum99] Chapter 3 section 4 for more details.

Warning!. When working over non-algebraically closed fields, some of the aforementioned conditions about smoothness are not equivalent. In particular the condition on the Jacobian may be satisfied even if the tangent space has the correct dimension: `https://mathoverflow.net/questions/12688/nonsingular-normal-schemes`.

8 Analytic Methods for Complex Varieties

The main objective for today will to continue our exploration of analytic ideas appearing in algebraic geometry, in particular we will be trying to see how complex analysis and complex geometry relate to complex algebraic varieties.The basic idea that drives us today if that often algebraic varieties over $\mathbb{C}$ look like relatively nice subsets of $\mathbb{C}^n$, we even expect some to be manifolds. We will first introduce complex geometry using the language of sheaves and then have a brief discussion of Serre's GAGA theorems, and Chow's theorem. Some of the theorems from this section are more difficult to prove so we will omit proofs in general here. One can find many of these ideas in [**GAGA**], and one can find an overview of much of this content in Appendix B of [Har77]. A thorough development of the material of this section from first principles may also be found in [**Neeman**].

8.1 Analytic spaces

Similar to section 5.1 we will use sheaves of holomorphic functions to define our notion of a complex analytic space.

Definition 8.1.1. An **analytic subset** $U \subseteq \mathbb{C}^n$ is one such that each point $x \in U$ has a neighborhood W in $\mathbb{C}^n$ where $U \cap W$ is equal to the points in W where a finite collection of holomorphic functions $f_1, \dots f_r$ all vanish. We such sets the subspace topology from $\mathbb{C}^n$.

Example 8.1.2.

Any affine variety is analytic (though with a different topology!).

Any open subset or closed subset of $\mathbb{C}^n$ is analytic.

Any discrete set is analytic.

We will now construct a sheaf of functions on an analytic subset $U \subseteq \mathbb{C}^n$. In general we will write $\mathcal{C}(X)$ to denote the set of complex valued functions on a set X. Let $\mathscr{H}$ denote the sheaf of holomorphic functions on $\mathbb{C}^n$, this is a subsheaf of $\mathcal{C}(X)$. For each $x \in U$ there is a map on stalks:

$$r_x : \mathcal{C}(\mathbb{C}^n)_x \to \mathcal{C}(U)_x$$

given by restricting a function to U. Let the image of $\mathscr{H}_x$ under this map be denoted by $\mathscr{H}_{U,x}$. This defines a sheaf $\mathscr{H}_U$[65] on U which we will call the sheaf of holomorphic functions on U, and we denote it by $\mathscr{H}_U$.

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[65] using the notion of compatible germs as in definition 2.4.3 of[Vak].

Now we define a morphism of analytic spaces to be a map $U \to V$ such that it's continuous and for any $f \in \mathscr{H}_{V,\phi(x)}$ we have that $f \circ \phi \in \mathscr{H}_{U,x}$. Clearly these morphisms are closed under composition and the identity is a morphism of analytic sets. Recall that we can define smooth maps of manifolds in effectively the exact same way!

Exercise 8.1.1. Show that a morphism of analytic spaces from $\mathbb{C} \to \mathbb{C}$ is the same thing as a holomorphic map.

We now give a general a general definition of the spaces of interest.

Definition 8.1.3. An **analytic space** X is a locally ringed space whose sheaf of functions $\mathscr{H}_X$ is a subsheaf of $\mathcal{C}(X)$ and we have that X has an open cover of sets isomorphic to analytic subsets of $\mathbb{C}^n$, and X is Hausdorff. A morphism of analytic spaces, $\phi : X \to Y$ is a continuous map such that for any $f \in \mathscr{H}_{V,\phi(x)}$ we have that $f \circ \phi \in \mathscr{H}_{U,x}$.

We now want to associate to a variety X over the complex numbers a canonical complex analytic space. First notice that the Zariski topology is a coarser topology than the analytic topology on $\mathbb{C}^n$. Furthermore any morphism of varieties is a holomorphic map as it's defined by polynomials, so if U is an affine algebraic variety then it uniquely inherits a complex analytic topology and sheaf of holomorphic functions, and if X is a general variety then take an open affine cover of X and by putting the analytic topology on the open affines and gluing their sheaves of holomorphic functions X inherits the structure of an analytic space. We denote this space by X^{an}. Note that there is a canonical map: $X^{\text{an}} \to X$ given by the identity on the underlying set, and including polynomial functions into the sheaf of holomorphic functions.

Exercise 8.1.2. Check that the above construction of an analytic structure on a variety is well-defined. In particular show that it doesn't depend on choice of affine cover for X.

Exercise 8.1.3. Note that while passing from a variety X to the analytic space X^{an} is functorial because polynomial maps are holomorphic, it's not an equivalence of cateogries. Give an example of variety of a map of analytic spaces $X^{\text{an}} \to Y^{\text{an}}$, where X and Y are varieties, that isn't given by a polynomial map.

Exercise 8.1.4. Show that the unit disc in $\mathbb{C}$ is not the analytic space associated to any variety.

We would now like to analyze the local structure of function on analytic spaces, and compare it to the algebraic case. In general for any $x \in X$ there is a map $\theta_x : \mathcal{O}_{X,x} \to \mathscr{H}_{X^{\mathrm{an}},x}$ given by simply viewing polynomial functions as holomorphic functions. Note that this is a map of local rings, and thus it defines a map on the completion of the local rings:

$$\widehat{\theta_x} : \widehat{\mathcal{O}_{X,x}} \to \widehat{\mathscr{H}_{X^{\mathrm{an}},x}}$$

in fact this map is an isomorphism!

Example 8.1.4. Consider the case when x is the origin on the affine line $\mathbb{A}^1_{\mathbb{C}}$ the map including rational functions of the form $g(z)/f(z)$, $f(z)$ such that $f(0) \neq 0$ into the collection of holomorphic functions defined at the origin induces an isomorphism $\mathbb{C}[[t]] \to \mathscr{H}_{\mathbb{C},0}$.

8.2 Sheaves on analytic spaces

During this next section we will concern ourselves with the study of sheaves on spaces. In particular coherent sheaves. Coherent sheaves are of special interest because they both satisfy convenient algebraic properties but also contain many sheaves of interest (under mild hypotheses) on a scheme, such as sheaves corresponding to vector bundles, and ideal sheaves which are in bijection with closed subschemes.

Definition 8.2.1. Given a ringed space $(X, \mathcal{O}_X)$, a sheaf of $\mathcal{O}_X$-modules, $\mathcal{F}$ is **coherent** if $\mathcal{F}$ is of **finite type** every point of X has a neighborhood U such that $\mathcal{F}$ is finitely generated (there's a surjection $\mathcal{O}_X|_U^{\oplus n} \to \mathcal{F}$ for some finite n), *and* if for any open set $U \subseteq X$, any natural number n, and any morphism $\phi : \mathcal{O}_X|_U^{\oplus n} \to \mathcal{F}$ the kernel is finite type.

Remark 3. In the case of locally Noetherian schemes this is equivalent to $\mathcal{F}$ being quasi-coherent with modules in the relevant short exact sequence being finitely generated.

Definition 8.2.2. An **analytic sheaf** is any sheaf of modules for an analytic space $(X, \mathscr{H}_X)$.

One example of a coherent analytic sheaf is the sheaf of holomorphic functions $\mathscr{H}_X$ itself. This is actually a hard theorem from complex geometry known as *Oka's coherence theorem*, and it is one of the key technical inputs that go into the GAGA dictionary.

We will now construct an analytic sheaf on X^{an} given a sheaf of $\mathcal{O}_X$-modules on a variety X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on a variety X, let $\mathcal{F}' := \theta^{-1}\mathcal{F}$ be the inverse image of $\mathcal{F}$ along the natural map $\theta : X^{\text{an}} \to X$. Then, we define the **analytification** of $\mathcal{F}$ to be the following coherent analytic sheaf:

$$\mathcal{F}^{\text{an}} := \mathcal{F}' \otimes \mathscr{H}_{X^{\text{an}}}.$$

This construction is functorial since passing to the inverse image and tensoring are functorial operations. It is in fact an exact functor which preserves coherent sheaves.

A major component of Serre's GAGA theorem then is that this map defines an equivalence of categories when X is a projective scheme over $\mathbb{C}$:

$$\text{Coh}(X) \xrightarrow{\simeq} \text{Coh}(X^{\text{an}}).$$

8.3 When are complex analytic spaces algebraic?

We have seen that we can naturally make algebraic objects into analytic ones, and this gives us access to the tools of complex analysis and complex function theory, but can we go back? One might hope so due to the fact that ultimately algebraic functions and Zariski open sets are compartively simple against general holomorphic functions and open sets in $\mathbb{C}^n$. There are cases where this is possible.

Theorem 8.3.1 (Chow's Theorem)**.** *Any compact analyticc subspace of* $\mathbb{P}^n_{\mathbb{C}}$ *is of the form* X^{an} *for some subscheme* $X \subseteq \mathbb{P}^n_{\mathbb{C}}$.

Furthermore in dimension 1 we have the following hard theorem:

Theorem 8.3.2 (Riemann)**.** *Every compact 1 dimensional complex manifold is of the form* C^{an} *for a smooth projective algebraic curve* C.

9 Working in Families

9.1 Relative algebraic geometry

At the start of this course, we fixed a ground commutative ring k, and set out to study systems of polynomial equations S with coefficients in k. We quickly realized that this problem was tightly related to k-algebras of the form $R = k[(t_i)_{i \in I}]/(S)$ and the ways in which they interacted with other k-algebras - i.e. to the study of the category CAlg_k. As such, every commutative ring in sight came equipped with a k-algebra structure map $k \to A$, and every homomorphism was required to respect the structure maps:

$$\begin{tikzcd}
R \arrow[rr] & & A. \\
& k \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Following the geometrization procedure $\mathrm{AffSch}_k \simeq \mathrm{CAlg}_k^{op}$ and its "globalization" Sch_k, arrows were reversed, so that every k-scheme came with a canonical structure map down to $\operatorname{Spec} k$:

$$\begin{tikzcd}
X \arrow[d] \\
\operatorname{Spec} k,
\end{tikzcd}$$

and every scheme morphism was required to respect structure maps:

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr] & & Y. \arrow[dl] \\
& \operatorname{Spec} k &
\end{tikzcd}$$

In the case where k is a field, the above diagrams admit a straightforward interpretation: the structure map $X \to \operatorname{Spec} k$ indicates that all rings of regular functions $\mathcal{O}_X(U)$ over opens $U \subseteq X$ are k-algebras (so that, in particular, X is cut out locally by affine schemes corresponding to k-algebras), and the commutative triangle only imposes a condition at the level of sheaves, namely all ring maps $\mathcal{O}_Y(V) \to \mathcal{O}_X(F^{-1}(V))$ are required to be k-algebra homomorphisms.

When the base ring k is no longer a field, the situation becomes a little more delicate. The affine scheme $S = \operatorname{Spec} k$ that we are working over now consists of more than one point, so that a fiber-preserving condition is introduced at the level of topological spaces, and the algebra structure provided on the ring of regular functions may vary from point to point. There is nonetheless a sense in which algebraic geometry over a field is the "building block" upon which relative algebraic geometry relies. Namely, let us attempt to give an impressionistic overview of the following:

Principle. Algebraic geometry over a base scheme S studies S-parametrized families of schemes.

We begin by formally defining the category we are working with:

Definition 9.1.1. Given a base scheme S, define the category Sch_S of **schemes over** S to be the category whose objects are schemes X equipped with a structure map

$$\begin{tikzcd}
X \arrow[d] \\
S,
\end{tikzcd}$$

and whose morphism sets $\mathrm{Hom}_{\mathrm{Sch}_S}(X, Y)$ are given by scheme morphisms $X \xrightarrow{F} Y$ respecting the structure maps:

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr] & & Y. \arrow[dl] \\
& S &
\end{tikzcd}$$

We sometimes use the shorthand X/S to mean that X is an object of Sch_S.

Whenever we are handed an S-scheme X/S, it is usually fruitful to think of X as a "coherent amalgamation" of the various fibers of its structure map $\{X_s\}_{s \in S}$ along points of S. That is, for each point $s \in S$ with residue field $\kappa(s)$, we may extract the fiber of X over s as filling in the following pullback square in the category of schemes[66]:

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \kappa(s) \arrow[r, "s"'] & S.
\end{tikzcd}$$

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[66] Recall that, via the $\mathrm{CAlg}^{op} \simeq \mathrm{AffSch}$ equivalence of categories, fibered products of

Likewise, a morphism of S-schemes can be thought of as a "continuous family of morphisms between the respective fibers":

$$\begin{tikzcd}
 & Y_s \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\
X_s \arrow[ur] \arrow[rr] \arrow[dr] & & X \arrow[ur] \arrow[dr] & \\
 & \operatorname{Spec} \kappa(s) \arrow[rr, "s"'] & & S.
\end{tikzcd}$$

Then, to say that a morphism $X \xrightarrow{\pi} S$ has property P (e.g. that π is smooth, proper, etc...) often more or less boils down to saying that *each fiber X_s has property P as a $\kappa(s)$-scheme*, plus possibly some coherence condition about the way the fibers fit together. That property P is shared by all fibers is formally true as soon as P is **preserved under base change**, a commonly appearing condition that states that whenever X/S has property P and $S' \to S$ is an arbitrary scheme morphism, the LHS vertical map in the following diagram also has property P:

$$\begin{tikzcd}
X_{S'} \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\
S' \arrow[r, "s"'] & S.
\end{tikzcd}$$

For instance, we mentioned at the end of chapter 7 that a smooth morphism $X \to S$ could be thought of as an "S-parametrized family of smooth schemes". Indeed, smoothness is one of the properties that is preserved under base change, so that a smooth morphism $X \to S$ automatically yields smooth schemes $X_s \to \operatorname{Spec} \kappa(s)$ along each fiber. Furthermore, there is a

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affine schemes are given by the following pattern:

$$\begin{tikzcd}
\operatorname{Spec}(A \otimes_R B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} R.
\end{tikzcd}$$

General pullbacks of schemes may then be obtained by working affine locally and gluing back the pieces - see e.g. theorem II.3.3 of [**Hart**] for details.

sense in which an S-parametrized family $X \to S$ is smooth iff all of its geometric fibers[67] are smooth schemes - see e.g. theorem III.10.2 of [**Hart**].

We make one last comment about the relative framework, namely that "not all fibers are created equal". By this, we mean that oftentimes the fiber of a morphism $X \to S$ over a non-closed point η *governs the generic behavior of the fibers over its specializations*, where we say that $s \in S$ is a specialization of η if $s \in \overline{\{\eta\}}$. That is, if X_η has property P, it will often follow that almost every X_s has property P for s a specialization of η. This has interesting consequences in arithmetic start with a scheme X over Spec $\mathbb{Z}$ locally of finite type[68]. We say that X has *bad reduction* at a prime $p \in \mathbb{Z}$ if $X_{\mathbb{F}_p}$ is not smooth over $\mathbb{F}_p$. Now, suppose that the base change $X_{\mathbb{Q}}$ of X to the generic fiber $\operatorname{Spec}\mathbb{Q}$ is smooth. Then one may prove that such an X can only have bad reduction at finitely many primes, using a theorem of Grothendieck stating that the set of points of Spec $\mathbb{Z}$ at which $X \to \operatorname{Spec}\mathbb{Z}$ is smooth is Zariski open, non-empty in our case since it contains the generic point, and thus its complement is Zariski closed in Spec $\mathbb{Z}$ hence finite.

Note 9.1.2. Oftentimes in this chapter, when talking about "S-parametrized families" X/S, there will be an implicit assumption that the admissible morphisms are **flat**, which is an algebraic condition that geometrically translates to the fact that "the fibers of X/S vary continuously". However, it takes some work to elucidate why this algebraic condition guarantees such continuous variation, and we choose to omit it from our discussion.

Having introduced the relative viewpoint abstractly, let us discuss some of the ways in which the relative viewpoint arises "in practice":

- **(Dimension reduction)** Naively, it is natural to expect algebro-geometric objects to appear in families, for the elementary reason that the vanishing set of a polynomial equation $f \in k[t_1, ..., t_{n+1}]$ may be written as a

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[67] By **geometric fiber** of a morphism $X \to S$ at a point $s \in S$, we mean that we first take the fiber X_s over $\operatorname{Spec}\kappa(s)$, then further base change X_s to a choice of algebraic closure of $\kappa(s)$. The terminology aligns with the notion that algebraic geometry over an algebraically closed field is "maximally geometric", in the sense that the arithmetic input is minimal.

[68] Read: locally cut out by a family of polynomial equations in finitely many variables defined over $\mathbb{Z}$

union of vanishing sets of polynomial equations $\{f_\lambda \in k[t_1, ..., t_n]\}_{\lambda \in \mathbb{A}^1_k}$, where we define $f_\lambda(t_1, ..., t_n) := f(t_1, ..., t_n, \lambda)$, so that:

$$V(f) = \bigcup_{\lambda \in \mathbb{A}^1_k} V(f_\lambda) \times \{\lambda\}.$$

As such, $V(f)$ decomposes into an $\mathbb{A}^1_k$-parametrized family of varieties $V(f_\lambda)$ that will typically be one dimension lower. More generally, the relative viewpoint gives a language through which one can attempt to probe higher dimensional structures "one fiber at a time", with each fiber being of a lower dimension. For instance, the study of *ruled surfaces* is concerned with projective surfaces S (two dimensional closed subschemes of projective spaces) admitting a surjective smooth morphism down to a curve C, such that each fiber is isomorphic to a projective line:

$$\begin{tikzcd}
\mathbb{P}^1_k \arrow[r] \arrow[d] & S \arrow[d] \\
\operatorname{Spec} k \arrow[r] & C.
\end{tikzcd}$$

Using the language of relative algebraic geometry, one may show for instance that any ruled surface S/C admits a section $C \to S$.

- **(Degeneration phenomena)** Keeping track of varieties in families can provide insight into the ways that singular varieties arise as "degenerations" of families of smooth varieties. The value of such an approach has classical roots in differential topology, such as Morse theory or the theory of Milnor numbers.

Example 9.1.3. [69] Let k be an algebraically closed field, and consider the polynomial equation $f(x, y, t) := xy - t \in k[x, y, z]$. Let $X = \operatorname{Spec} k[x, y, t]/(xy - t)$. Then we may think of this situation relative to $\mathbb{A}^1_k := \operatorname{Spec} k[t]$ via the "projection map onto the t-coordinate":

$$\begin{tikzcd}
\operatorname{Spec} k[x, y, t]/(xy - t) \arrow[d] \\
\operatorname{Spec} k[t]
\end{tikzcd}$$

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[69] The picture was graphed on desmos.com.

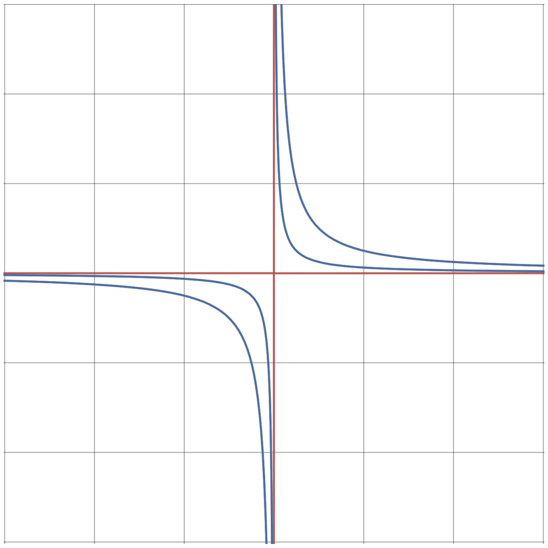

Figure 3: A family of curves parametrized $\mathbb{A}^1_k$ degenerating at the origin.

Exercise 9.1.1. (1) Verify that the above projection map corresponds to the k-algebra map $k[t] \to k[x, y, t]/(xy - t)$, $t \mapsto t$.
(2) Compute the fiber X_λ over a given closed point $[\lambda] \in \mathbb{A}^1_k$, and show that X_λ is smooth for every $\lambda \neq 0$, while X_0 is singular.

Accounting for degeneration phenomena is also central to the proper study of *intersection theory*, which is roughly concerned with "counting solutions to geometric problems" (quote taken from the excellent reference [**3264**]). For instance, given two subschemes Z, Z' of complementary dimension in some ambient scheme X, we expect the intersection $Z \cap Z'$ to generically[70] consist of a finite number of points, and we may be interested in counting the number of intersection points with multiplicities. Readers with experience in differential topology may be familiar with the protocol: one should first "deform" Z and/or Z' so that the two varieties meet *transversely*[71], and then count intersec-

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[70] The word **generic** is frequently used in algebraic geometry to mean that something holds "almost everywhere", or more precisely that it holds "on a dense open subset" of the space of possible positions. In this particular example, we may expect e.g. that the intersection of two lines in $\mathbb{P}^2_k$ consists of a single point, except in the degenerate situation where the two lines are equal.

[71] This is the condition that, at every point of intersection $x \in Z \cap Z'$, we have an

tion points in the transverse situation. This perturbation is needed to account for "incorrect" intersection counts one may obtain from degenerate situations such as $Z = V(y)$, $Z' = V(y - x^2) \subseteq \mathbb{A}^2_k$.[72]

One way to formally define perturbations is as follows: we say that two subschemes $Z, Z' \subseteq X$ are *rationally equivalent* if there exists a flat $\mathbb{P}^1_k$-parametrized family $\mathcal{Z} \subseteq X \times \mathbb{P}^1_k$ of cycles in X such that Z and Z' are each isomorphic to some fiber of that family over a closed point of $\mathbb{P}^1_k$:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] & Z' \arrow[l] \arrow[d] \\
\operatorname{Spec} k \arrow[r, "i_x"'] & \mathbb{P}^1_k & \operatorname{Spec} k. \arrow[l, "i_y"]
\end{tikzcd}$$

Thus, the $\mathbb{P}^1_k$-scheme $\mathcal{Z}$ provides a way to interpolate between Z and Z' as subschemes of X.

- **(Moduli problems)** Recall that, prior to developing the language of schemes inside the category of locally ringed spaces, the solutions spaces to a given system $S \subseteq k[(t_i)_{i \in I}]$ assembled into a functor:

  $$V_S(-) : \mathrm{CAlg}_k \to \mathscr{S}\mathrm{et}$$

  which is co-represented by $R = k[(t_i)_{i\in I}]/(S)$, hence is equivalent to the functor

  $$\operatorname{Spec} R : \mathrm{AffSch}_k^{op} \to \mathscr{S}\mathrm{et}\,.$$

  With perhaps the hindsight of scheme theory, we can reasonably think of the set of A-points of $\operatorname{Spec} R$[73] in the following way:

  $$\operatorname{Spec} R(A) = \left\{ \begin{matrix} \operatorname{Spec} A\text{-parametrized families of} \\ \text{field-valued points in } \operatorname{Spec} R \end{matrix} \right\}.$$

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equality $T_xZ + T_xZ' = T_xX$, i.e. the tangent spaces of Z and Z' at x "span every possible direction away from x".

[72] The language of scheme theory is robust enough to allow one to read off the correct intersection multiplicity from the sheaf theoretic information contained in $Z \cap Z'$ in low dimensional examples such as the one given, but this is not always the case. See e.g. [**Dugger**], example 1.1.4 for an example where a perturbation (or derived algebraic geometry!) is needed.

[73] When working with functors $\mathcal{X} : \mathrm{AffSch}_k \to \mathscr{S}\mathrm{et}$, we adopt the convention of writing $\mathcal{X}(A) := \mathcal{X}(\operatorname{Spec} A)$ and calling $\mathcal{X}(A)$ the set of A-points of $\mathcal{X}$.

Indeed, given an A-point $\operatorname{Spec} A \xrightarrow{\varphi} \operatorname{Spec} R$, each point $x \in \operatorname{Spec} A$ produces a $\kappa(x)$-valued point of $\operatorname{Spec} R$ by composing:

$$\operatorname{Spec} \kappa(x) \to \operatorname{Spec} A \xrightarrow{\varphi} \operatorname{Spec} R.$$

Next, functoriality with respect to a given affine scheme morphism $\operatorname{Spec} B \xrightarrow{F} \operatorname{Spec} A$ is given by "pulling back families". Pictorially, given an A-point $\varphi : \operatorname{Spec} A \to \operatorname{Spec} R$ thought of as a $\operatorname{Spec} A$-parametrized family of field-valued points of $\operatorname{Spec} R$, we may obtain an induced $\operatorname{Spec} B$-parametrized family as follows:

$$\begin{tikzcd}
\{\operatorname{Spec} \kappa(y) \xrightarrow{\varphi \circ F} \operatorname{Spec} R\}_{y \in \operatorname{Spec} B} \arrow[d, dash] & \{\operatorname{Spec} \kappa(x) \xrightarrow{\varphi} \operatorname{Spec} R\}_{x \in \operatorname{Spec} A} \arrow[d, dash] \\
\operatorname{Spec} B \arrow[r, "F"] & \operatorname{Spec} A.
\end{tikzcd}$$

The starting point of moduli theory is to realize that one could replace the phrase "field-valued point" above by "$\mathcal{B}$-object", where $\mathcal{B}$-object stands for any kind of algebro-geometric structure one is interested in studying in families, and setting out to try to geometrically understand the functor:

$$\mathcal{M}_{\mathcal{B}} : \mathrm{AffSch}_k^{op} \to \mathscr{S}\mathrm{et}$$

$$\mathcal{M}_{\mathcal{B}}(A) := \left\{ \begin{matrix} \operatorname{Spec} A\text{-parametrized families} \\ \text{of } \mathcal{B}\text{-objects} \end{matrix} \right\}.$$

with an appropriate notion of pullback of families of $\mathcal{B}$-objects providing functoriality: given an affine scheme morphism $\operatorname{Spec} B \xrightarrow{F} \operatorname{Spec} A$ and an A-point $\mathcal{Y} \in \mathcal{M}_{\mathcal{B}}(B)$, we expect to be able to obtain an A-point $F^*\mathcal{Y}$ by "pulling back the family $\mathcal{Y}$ along F:

$$\begin{tikzcd}
F^*\mathcal{Y} \arrow[r] \arrow[d, dash] & \mathcal{Y} \arrow[d, dash] \\
\operatorname{Spec} B \arrow[r, "F"] & \operatorname{Spec} A.
\end{tikzcd}$$

9.2 A primer on deformation theory

We would now like to tell a story that will combine some of the ideas of the previous section with the notion of co/tangent spaces developed in chapter

7. The big picture is as follows: when studying degenerations of families of $\mathcal{B}$-objects, one is naturally led to considering "infinitesimal perturbations" thereof. The attempt to classify such infinitesimal perturbations begins with the study of the tangent space of the associated moduli functor $\mathcal{M}_{\mathcal{B}}$ at a point, and, provided $\mathcal{M}_{\mathcal{B}}$ is represented by an actual scheme $\mathcal{X}$, eventually leads to the following:

Principle. The cotangent sheaf $\Omega_{\mathcal{X},k}$ controls the first-order deformation theory of $\mathcal{X}$.

Note 9.2.1. In what follows, we work with an algebraically closed field k to make life simpler. In particular, as in chapter 7, "closed point" means k-point unless otherwise specified.

Let us begin with an explicit example of a moduli functor to provide some motivation. Given a fixed scheme X, the **Hilbert scheme** of X is defined to be the following functor:

$$\mathrm{Hilb}_X : \mathrm{AffSch}_k^{op} \to \mathscr{S}\mathrm{et}$$

$$\mathrm{Hilb}_X(A) := \left\{ \begin{matrix} \operatorname{Spec} A\text{-parametrized families of} \\ \text{closed subschemes of } X \end{matrix} \right\},$$[74]

and with functoriality given as follows: for any morphism of affine schemes $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ and any family $\mathcal{Y} \in \mathrm{Hilb}_X(B)$, viewed as a subscheme $\mathcal{Y} \subseteq X \times \operatorname{Spec} B$ with certain properties, we may pull back the family along φ to obtain a family $\varphi^*\mathcal{Y}$ over $\operatorname{Spec} A$:

$$\begin{tikzcd}
\varphi^*\mathcal{Y} := \mathcal{Y} \times_{\operatorname{Spec} B} \operatorname{Spec} A \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\operatorname{Spec} A \arrow[r, "\varphi"'] & \operatorname{Spec} B,
\end{tikzcd}$$

thereby obtaining a map $\mathrm{Hilb}_X(B) \to \mathrm{Hilb}_X(A)$.

In particular, the closed points $Y \in \mathrm{Hilb}_X(k)$ correspond to individual closed subschemes of X. The idea that a given closed subscheme Y arises as a

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[74] This is one of the places in which a flatness condition should be imposed on the families involved. For us, all will be relevant is that flatness is preserved under base change, guaranteeing that one can pull back families along arbitrary morphisms.

"degeneration" of a Spec A-parametrized family of closed subschemes can now be made precise: such a degeneration consists of an A-point $\mathcal{Y} \in \mathrm{Hilb}_X(A)$ together with the inclusion of a closed point $\operatorname{Spec} k \xrightarrow{i_x} \operatorname{Spec} A$ such that Y is recovered from $\mathcal{Y}$ by taking the fiber over x:

$$\begin{tikzcd}
\varphi^*\mathcal{Y} \cong Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\operatorname{Spec} k \arrow[r, "i_x"'] & \operatorname{Spec} A.
\end{tikzcd}$$

We would now like to focus our attention on the "limiting event" of such a degeneration - what happens "right before the family collapses to Y", or, equivalently, "right as Y is about to be deformed into other closed subschemes of X". This is one of the situations where having access to infinitesimal data in our geometric theory (in the form of nilpotent elements in our sheaves of functions) will prove to be extremely valuable. We begin with the following crucial:

Exercise 9.2.1. Recall that the tangent space to a variety X at a closed point $x \in X$ was defined to be the following k-vector space:

$$T_xX = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k) \cong \mathrm{Der}_k(\mathcal{O}_{X,x}, k_x),$$

where $\mathfrak{m}_x$ denotes the maximal ideal of the stalk $\mathcal{O}_{X,x}$. Show that the following data are equivalent:

- (a) Scheme morphisms $\operatorname{Spec} k[\varepsilon] \to X$;[75]
- (b) A choice of closed point $x \in X$ together with a tangent vector in T_xX.

(Hint: note that you may assume that $X = \operatorname{Spec} A$ is an affine scheme, because the issue is local.)

Accordingly, if we wish to single out tangent vectors at a closed point $x \in X$ from the functor of points perspective, we are led to looking for those

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[75] We use the notation $k[\varepsilon] := k[t]/(t^2)$ to denote the ring of dual numbers.

$k[\varepsilon]$-points of X which *lift* the given k-point x, i.e. for dotted arrows making the following diagram commute:

$$\begin{tikzcd} & X(k[\varepsilon]) \arrow[d] \\ \{x\} \arrow[ur, dashed] \arrow[r] & X(k), \end{tikzcd}$$

where the vertical map is the restriction map induced by the inclusion $\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]$, dual to the k-algebra map $k[\varepsilon] \to k$, $\varepsilon \mapsto 0$. Such dotted arrows in turn correspond precisely to elements of the fibered product:

$$\begin{tikzcd} X(k[\varepsilon]) \times_{X(k)} \{x\} \arrow[r] \arrow[d] & X(k[\varepsilon]) \arrow[d] \\ \{x\} \arrow[r] & X(k) \end{tikzcd}$$

We therefore arrive at the following equivalent definition of the tangent space of X at x, this time purely from the functor of points perspective:

$$T_x X \cong X(k[\varepsilon]) \times_{X(k)} \{x\}.$$

We may therefore formulate the following definition, suitable to more general moduli functors as well:

Definition 9.2.2. Let $\mathcal{X} : \mathrm{AffSch}_k^{op} \to \mathscr{S}\mathrm{et}$ be a functor, and let $Y \in \mathcal{X}(k)$ denote a given k-point of $\mathcal{X}$. We define the **tangent space** of $\mathcal{X}$ at Y to be the following set:

$$T_Y \mathcal{X} \cong \mathcal{X}(k[\varepsilon]) \times_{\mathcal{X}(k)} \{Y\}.$$

It is worthwhile to examine what this perspective translates into in the special case of the functor of points associated to an affine scheme $X = \operatorname{Spec} R$ (this is, of course, closely related to exercise 9.2.1). In this case, a k-point is given by the data of a k-algebra map $\mathrm{ev}_x : R \to k$ (i.e. an evaluation map, à la Gelfand!), and to find a $k[\varepsilon]$-point in the corresponding fibered product is to find a lift in the following situation:

$$\begin{tikzcd} & k[\varepsilon] \arrow[d, "\varepsilon\mapsto 0"] \\ R \arrow[ur, dashed] \arrow[r, "\mathrm{ev}_x"'] & k \end{tikzcd}$$

Let us return back to the example of the Hilbert scheme $\mathcal{X} = \mathrm{Hilb}_X$. Starting with a closed subscheme $Y \subseteq X$ viewed as a closed point $Y \in \mathrm{Hilb}_X(k)$, a tangent vector to Hilb_X at Y corresponds to a (flat) family $\mathcal{Y}$ over $\operatorname{Spec} k[\varepsilon]$ which restricts to Y over the unique closed point:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon].
\end{tikzcd}$$

Notice that if we had started with e.g. a family $\widetilde{Y}$ over the whole of $\mathbb{A}^1_k$ whose fiber over the origin is Y, we could extract an element of $T_Y \mathrm{Hilb}_X$ via the following two-step pullback:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[r] \arrow[d] & \widetilde{\mathcal{Y}} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon] \arrow[r] & \mathbb{A}^1_k.
\end{tikzcd}$$

This picture provides us with a source of tangent vectors:

$$\mathrm{Hilb}_X(k[t]) \times_{\mathrm{Hilb}_X(k)} \{Y\} \to T_Y \mathrm{Hilb}_X .$$

For instance, example 9.1.3 provided us with a tangent vector to $\mathrm{Hilb}_{\mathbb{A}^2_k}$ at the singular subscheme $\operatorname{Spec} k[x,y]/(xy)$ pointing in the direction of a family of smooth subschemes.

Exercise 9.2.2. Think about similar ways to build tangent vectors to the closed subscheme $\operatorname{Spec} k[t]/(t^3)$ thought of as a closed point of $\mathrm{Hilb}_{\mathbb{A}^1_k}$.

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