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**“Discovering Beautiful Patterns
in Nature and Number”**



Fun Fibonacci Facts

1. The Fibonacci Numbers

Examine the following sequence of natural numbers.
Do you see a pattern?

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...

Each number in the sequence above is the sum of the previous two numbers in the sequence: $1+1=2$, $1+2=3$, etc. They are called the *Fibonacci numbers*, named after Leonardo Pisano Fibonacci, who studied them in the 13th century.



Leonardo Pisano
Fibonacci
(1170 – 1250)

One of the “greatest European mathematicians of the middle ages”, his real name was Leonardo of Pisa, or Leonardo Pisano in Italian since he was born in Pisa (Italy), the city with the famous Leaning Tower. He is now known as Fibonacci, short for “filius Bonacci”, which means “the son of Bonaccio”.

Surprisingly, these numbers are often found in nature. Pick up a pine cone and count the spirals going in each direction. You will find that there are 5 spirals in one direction, and 8 in the other. 5 and 8 are two consecutive Fibonacci numbers.

Similarly, count the spirals on the center part of a sunflower: there are 55 in one direction, and 89 in the other. The spines on a pineapple form 8 spirals in one direction, and 13 in the other.

2. Continued Fractions

What are the relative sizes of the Fibonacci numbers? To investigate this, we can take the quotients of consecutive Fibonacci numbers.

Ratio of Consecutive Fibonacci Numbers	Decimal Approximation
1/1	1.0
2/1	2.0
3/2	1.5
5/3	1.666...
8/5	1.6
13/8	1.625

Continue to fill in a few more rows. What do you notice?

The fractions appear to be oscillating: sometimes getting smaller, sometimes getting larger. However, they also seem to be approaching a certain value. In order to find out what that value is, let's examine the ratios in a slightly different way. Rewriting each Fibonacci number in the numerator as the sum of the previous two, we can rewrite each fraction like so:

$$\frac{1}{1} = 1$$

$$\frac{2}{1} = \frac{1+1}{1} = 1 + \frac{1}{1}$$

$$\frac{3}{2} = \frac{2+1}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{\frac{2}{1}} = 1 + \frac{1}{1 + \frac{1}{1}}$$

$$\frac{5}{3} = \frac{3+2}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{1}{\frac{3}{2}} = 1 + \frac{1}{1 + \frac{1}{\frac{2}{1}}}$$

This pattern of ones could keep going on forever if we rewrote more numbers, heading towards $1 + 1/(1 + 1/(1 + \dots))$ in which the denominator of

the fraction never ends. Let's call this number $\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$.

Now, look at the denominator of that first big fraction. Notice that it looks exactly like φ all over again. This fact means that we can rewrite φ as

$\varphi = 1 + \frac{1}{\varphi}$. Now all we have to do is solve for φ .

Subtract 1 from both sides of the equation, then multiply both sides by φ . Then subtract 1 from both sides of the equation again to get $\varphi^2 - \varphi - 1 = 0$.

The positive solution to this quadratic equation is $\varphi = \frac{1 + \sqrt{5}}{2}$.

Using a calculator, find the decimal approximation to this solution and compare it with the values in the "decimal approximation" column of your table. You will see that the ratios between two consecutive Fibonacci numbers appear to be heading towards φ .

This φ is a very interesting number, and it is called the *Golden Ratio*.

3. The Most Attractive Rectangle

When someone says rectangle, we think of a shape. What shape is it? It turns out that most people find a rectangle most aesthetically pleasing that has a certain length of the base relative to the length of the height – namely the Golden Ratio ϕ .

The precise mathematical definition of such a rectangle, which we call a *Golden Rectangle*: a rectangle having base b and height h such that

$$b/h = \phi = \frac{1 + \sqrt{5}}{2}.$$

The Golden Rectangle appears in a lot of unexpected places.

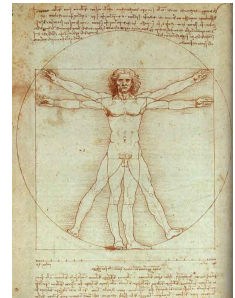
Parthenon, Acropolis, Athens

Once its ruined triangular pediment is restored, the ancient temple fits almost precisely into a golden rectangle.



Study of Human Proportions According to Vitruvius

Leonardo da Vinci made a close study of the human figure and showed how all its different parts were related by the golden ratio.

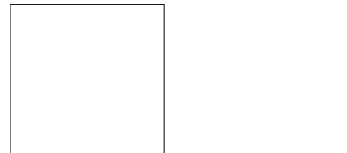


Here is how to **construct your own Golden Rectangle** by a simple geometric procedure:

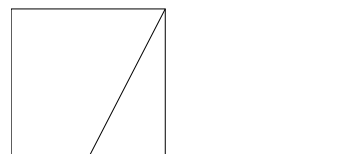
- First we build a square.



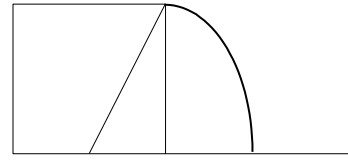
- We extend the base of the square with a straight line segment off to the east.



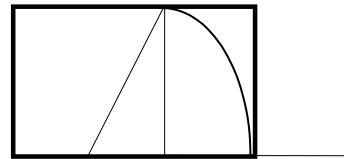
- Next, we draw a line from the midpoint of one side of the square to an opposite corner, for example we connect the midpoint of the base of the square to the northeast corner of the square with a straight line segment.



- Now we use that line as the radius to draw part of a circle whose center is the midpoint of the base and whose radius extends to the northeastern corner of the square. We note where the circle portion hits the extended base. (The line segment drawn inside the square from the midpoint to the northeastern corner is actually the radius of the circle arc drawn.)

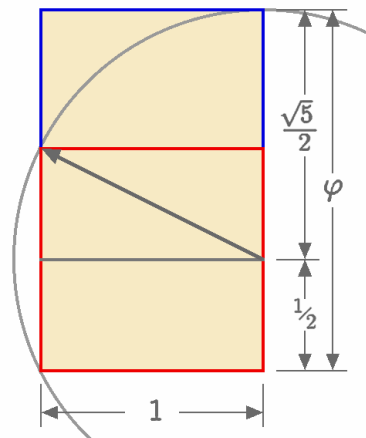


- Next, we construct a line perpendicular to the extended base and passing through the point where the circle hits the extended base. We then extend the top edge of the square to the right with a straight line until it hits the perpendicular line just drawn.



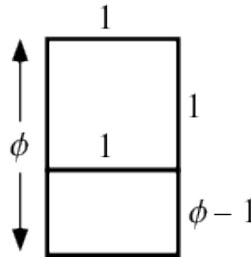
Voilà – that big rectangle we just constructed is a perfectly precise Golden Rectangle! Its resulting dimensions are in the ratio $1:\phi$, the Golden Ratio.

Challenge: Use Pythagoras' Theorem and the figure below to prove that this procedure does indeed produce a Golden Rectangle!



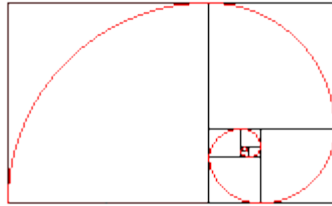
Some Fun Facts:

- If a Golden Rectangle is divided into a square and a smaller rectangle, then the small rectangle is another Golden Rectangle. To see this, start with a rectangle whose ratio of base and height resembles the Golden Ratio. All you need to remember now is the equation $\phi = 1 + \frac{1}{\phi}$ from above. It can be rewritten as $\phi - 1 = \frac{1}{\phi}$ or $\frac{\phi}{1} = \frac{1}{\phi - 1}$ which is exactly the relation of base to height of the new smaller rectangle.



Thus, we can create a sequence of smaller and smaller Golden Rectangles.

- If we continue this construction, that is start drawing successive squares in the smaller and smaller Golden Rectangles, and then draw a quarter circle in each square having radius equal to the side of the square we get a spiral.



This spiral is very similar to the famous spiral called the *logarithmic spiral*, and it occurs in nature in various forms, such as the nautilus sea shell.

4. Zeckendorf's Theorem

Zeckendorf's Theorem, named after the Belgian medical doctor, army officer and amateur mathematician Edouard Zeckendorf (1901 – 1983), is a theorem about the representation of integers as sums of Fibonacci numbers.

Zeckendorf proved that any positive integer N can be expressed in a unique way as the sum of one or more distinct Fibonacci numbers, no two of which are consecutive. The sequence of Fibonacci numbers which add up to N and meets these conditions is called the *Zeckendorf representation* of N .

For example, the Zeckendorf representation of 100 is $100 = 89 + 8 + 3$.

There are other ways of representing 100 as the sum of Fibonacci numbers – for example $100 = 89 + 8 + 2 + 1$ or $100 = 55 + 34 + 8 + 3$ – but these are not Zeckendorf representations because 1 and 2 are consecutive Fibonacci numbers, as are 34 and 55.

Similarly, the (only) Zeckendorf representation of 14 is $13 + 1$.

Although $14 = 5 + 3 + 2 + 2 + 2$, this is not a Zeckendorf sum because 2 is used three times while the definition prohibits even two occurrences of the same number. Second, 3 and 5 are consecutive Fibonacci numbers, while the definition prohibits the use of two such numbers.

Try it yourself!

Find the Zeckendorf representation of the following numbers:

a) $52 =$

b) $1111 =$

c) $127 =$

(Hint: For any given positive integer N , a representation that satisfies the conditions of Zeckendorf's theorem can be found by using the following algorithm: one chooses the largest Fibonacci number not greater than N , say F_{n_1} . This Fibonacci number is the first term in your sum. Now subtract it from N . Unless the difference is zero, one then finds the largest Fibonacci number that does not exceed this new number $N - F_{n_1}$, say F_{n_2} . This Fibonacci number is the second number in your sum. Continue this process. If N is reduced to zero after k steps, one obtains a Zeckendorf representation of the form $N = F_{n_1} + F_{n_2} + \dots + F_{n_k}$, where $n_1, \dots, n_k$ is a decreasing sequence of positive integers.)

Challenge Questions:

- Why can this representation not include two consecutive Fibonacci numbers?
- Why is this representation unique, i.e. why is it the only representation that satisfies these conditions?

5. Fun and Games with Fibonacci

Play *Fibonacci Nim* with your friends, parents, teachers, neighbors, ... Explain the rules, but do not reveal the secret of winning as you learned it today!

Start with any (non-Fibonacci) number of sticks. Play carefully and beat your friend. Play again with another number of sticks to start. Record the number of sticks removed at each stage of the game.

Finally, reveal the secret strategy – and record your friend's reaction ☺

Here is a little warm-up exercise for you before you face your Nim opponent.

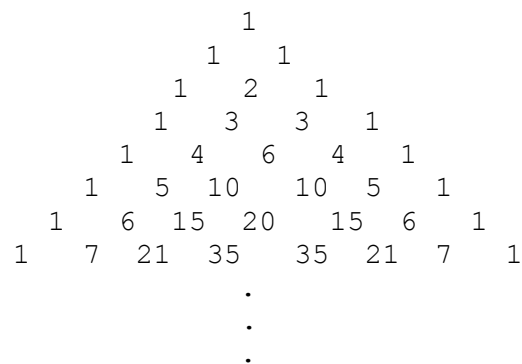
- a) Suppose you are to begin a game of Fibonacci Nim, starting with 100 sticks. What is your first move?
- b) Suppose you are playing a round of Fibonacci Nim and the game starts with 50 sticks. You start by removing three sticks; your friend then takes five; you then take eight; your friend then takes ten. How many sticks should you take next to win?
- c) Suppose you are playing a round of Fibonacci Nim. You start with 15 sticks. You first remove two sticks; your friend then takes one; you take two; your friend takes one. What should your next move be? Can you make it without breaking the rules of the game? Did you make a mistake at some point? If so, where?

6. More Fun Fibonacci Facts

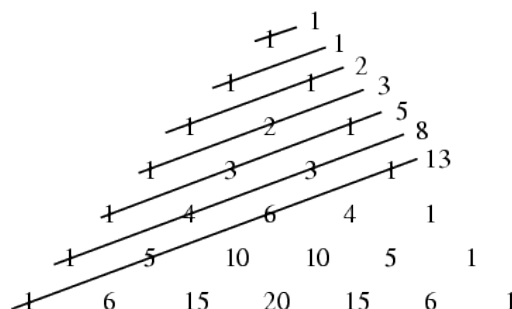
- Take any three adjacent numbers in the sequence, square the middle number, multiply the first and third numbers. The difference between these two results is always 1.
- Take any four adjacent numbers in the sequence. Multiply the outside ones. Multiply the inside ones. The first product will be either one more or one less than the second.
- The sum of any ten adjacent numbers equals 11 times the seventh one of the ten. (Mesoamericans thought the numbers 7 and 11 were special.)
- The smallest integer whose Zeckendorf representation is the sum of k Fibonacci numbers is $F_2 + \dots + F_{2k} = F_{2k+1} - 1$.
- There is a simple formula for $(F_{n+1})^2 + (F_n)^2$, that is a formula for the sum of the squares of two consecutive Fibonacci numbers. Experiment with numerous examples in search of a pattern and find the formula!
- The Fibonacci numbers appear in Pascal's Triangle!

Pascal's Triangle was originally developed by the ancient Chinese, but Blaise Pascal (1623-1662), a famous French Mathematician and Philosopher, was the first person to discover the importance of all of the patterns it contained.

To build the triangle, start with "1" at the top, then continue placing number below it in a triangular pattern in such a way that each number is just the sum of the two numbers above it.



The "shallow diagonals" of Pascal's triangle sum to Fibonacci numbers.



- Fibonacci Music (from <http://techcenter.davidson.k12.nc.us/Group2/music.htm>)

The basic structures of music and certain instruments display the use of Fibonacci numbers and the Golden Ratio. For instance, there are 13 notes that separate each octave of 8 notes in a scale. The foundation of a scale is based around the 3rd and the 5th tones while the 1st note of the scale, also called the root, is certainly special.



The keys of a piano also portray the Fibonacci numbers. Within the scale consisting of 13 keys, 8 of them are white, 5 are black, which are split into groups of 3 and 2. Look familiar? Well, it should, it's Fibonacci!

To see how the Golden Ratio is used to construct a violin check out this link: <http://www.violin.odessa.ua/method.html>

7. Miles versus Kilometers

An amusing trivial "application" of the Zeckendorf representation is a method of converting miles into kilometers and vice versa without having to perform a multiplication. It relies on the coincidence that the number of kilometres in a mile (approximately 1.609) is close to the golden ratio $(1+\sqrt{5})/2 \approx 1.618$ (the limit as n goes to infinity of the ratio F_{n+1}/F_n).

Thus, to roughly convert miles into kilometers one writes down the (integer) number of miles in Zeckendorf form and replaces each of the Fibonacci numbers by its successor. This will give the Zeckendorf form of the corresponding approximate number of kilometers.

For example, $50 = 34 + 13 + 3$ miles is approximately $55 + 21 + 5 = 81$ kilometers and 50 kilometers is approximately $21 + 8 + 2 = 31$ miles.

How many kilometers are there in 90 miles?

8. Take Home Challenge – Beyond Fibonacci

Suppose we create a new sequence of natural numbers starting with 0 and 1. Only this time, instead of adding the two previous terms to get the next one, let's generate the next term by adding 2 *times* the previous term to the term before it.

In other words,

$$W_0 = 0,$$

$$W_1 = 1 \text{ and then}$$

$$W_{n+1} = 2 W_n + W_{n-1}.$$

Such a sequence is called a *generalized Fibonacci sequence*.

- a) Write out the first 15 terms of this generalized Fibonacci sequence.

W_0	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	W_{11}	W_{12}	W_{13}	W_{14}

- b) Look at the ratio W_{n+1}/W_n for the first couple of terms. Can you detect a pattern? Do you think that this ratio converges to a decimal number when n gets large?

$$W_2/W_1 =$$

$$W_7/W_6 =$$

$$W_3/W_2 =$$

$$W_8/W_7 =$$

$$W_4/W_3 =$$

$$W_9/W_8 =$$

$$W_5/W_4 =$$

$$W_{10}/W_9 =$$

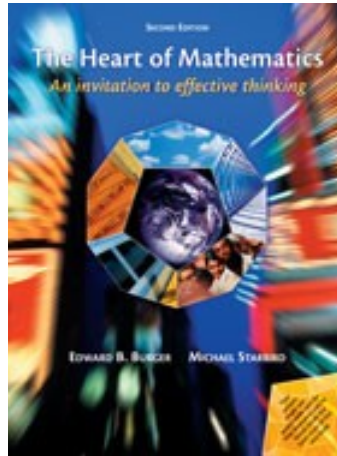
$$W_6/W_5 =$$

- c) Adapt the methods that were used to figure out that the quotient of consecutive Fibonacci numbers approaches $\phi = \frac{1 + \sqrt{5}}{2}$ to discover the exact number that W_{n+1}/W_n approaches as n gets large.

- d) Does the Zeckendorf Theorem generalize? That is, can any positive integer be expressed uniquely as a sum of distinct non-consecutive generalized Fibonacci numbers? Try it with the numbers 88 and 125!

For these and many more fun math facts, refer to the book
“The Heart of Mathematics: An Invitation to Effective Thinking”
by Edward B. Burger and Michael Starbird!

It was published by Key College Press and Springer Verlag in December 1999, a second edition in 2005; ISBN 978-1-931914-41-3 (text only) or 978-1-931914-51-2 (includes text and manipulative kit).



www.heartofmath.com

Hope to see you all again in the spring!