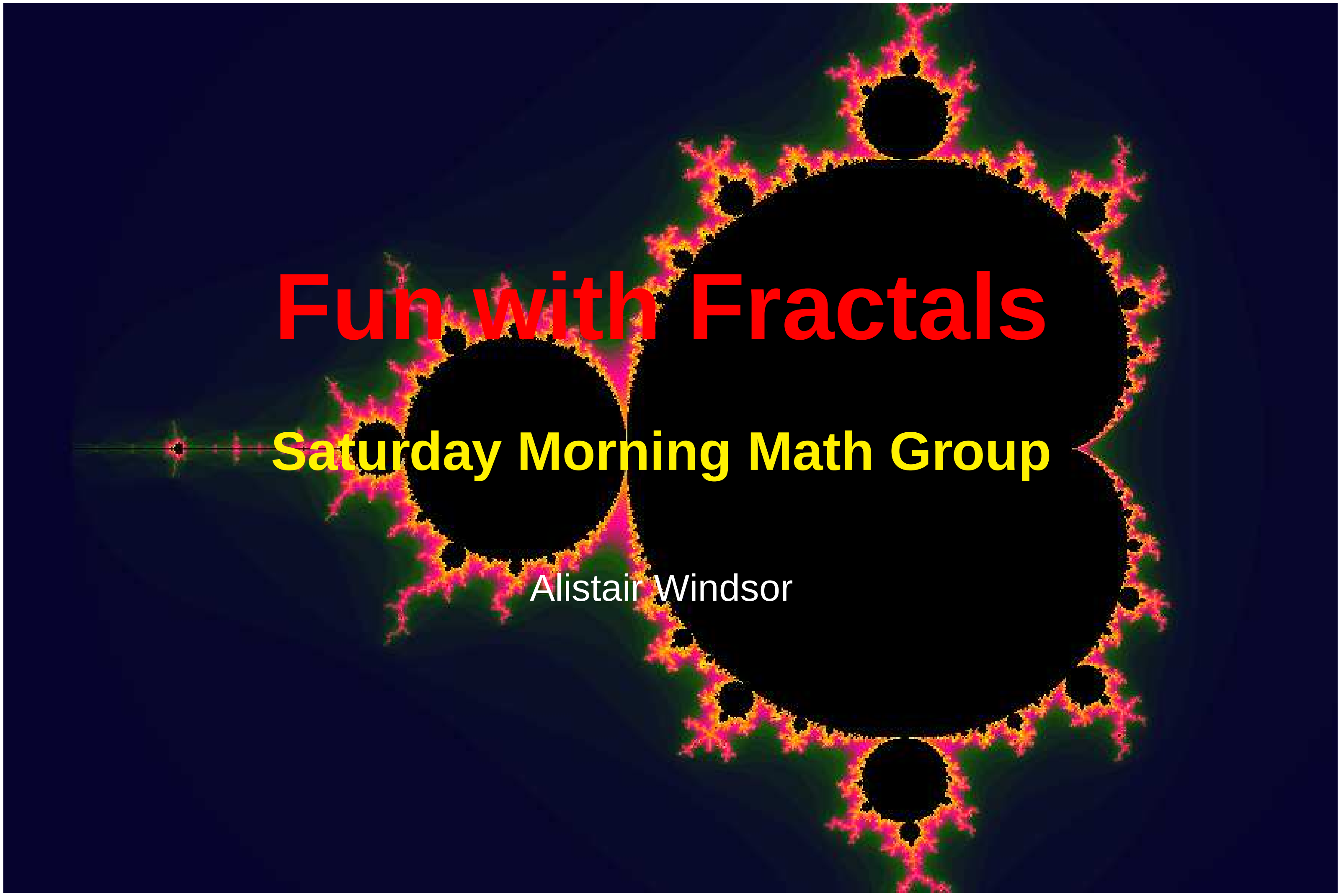

Fun with Fractals

Saturday Morning Math Group

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Fractals

Fractals are amazingly complicated patterns often produced by very simple processes.

We will look at two different types of fractals

- Escape Time Fractals like the Mandelbrot set and its associated Julia sets.
- Iterated Function Systems Generating self similar fractals using iterative procedure.

The Mandelbrot Set

First plotted by the Polish born and French educated mathematician Benoit Mandelbrot at IBM in early 1980. This complicated set is determined entirely iterating by the map

$$f_c(z) = z^2 + c$$

where c is a complex number.

Benoit Mandelbrot (1924–)

Complex Numbers

Recall that we multiply complex numbers

$$z_1 = u + i\,v \qquad z_2 = x + i\,y$$

by expanding the brackets and using the relation $i^2 = -1$

$$\begin{aligned} z_1 \times z_2 &= (u + i\,v)(x + i\,y) \\ &= ux + i\,uy + i\,vx + i^2\,vy \\ &= (ux - vy) + i\,(uy + vx) \end{aligned}$$

The “size” of a complex number is given by

$$|x + i\,y| = \sqrt{x^2 + y^2}.$$

Defining the Mandelbrot Set

The Mandelbrot set is the set of constants c such that the sequence of numbers

$$
\begin{aligned}
f_c(0) &= c \\
f_c^2(0) &= f_c(f_c(0)) = c^2 + c \\
f_c^3(0) &= f_c(f_c(f_c(0))) = (c^2 + c)^2 + c \\
f_c^4(0) &= ((c^2 + c)^2 + c)^2 + c \\
&\vdots
\end{aligned}
$$

remains bounded (does not escape to infinity). In all my pictures the Mandelbrot set will be colored black.

Computing the Mandelbrot Set

If after computing some terms we get a number big enough then we can stop. If after m steps we have $|f_c^m(0)| > 2$ then the sequence of numbers $f_c^n(0)$ escapes to infinity and c is not in the Mandelbrot set.

Example: Is 1 in the Mandelbrot set ?

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Example: Is 1 in the Mandelbrot set ?

We compute the first few terms of the sequence

$$f_1(0) = 0^2 + 1 = 1$$
$$f_1^2(0) = f_1(1) = 1^2 + 1 = 2$$
$$f_1^3(0) = f_1(2) = 2^2 + 1 = 5$$

Since $|f_1^3(0)| = 5 > 2$ we see that this sequence “escapes” and thus 1 is **not** in the Mandelbrot set.

The m when $|f_c^m(0)| > 2$ for the first time is called the escape time. For $c = 1$ we found the escape time was 3. The escape time is what we use to color the points outside of the Mandelbrot set.

Computing the Mandelbrot Set

Exercise: Is -2 in the Mandelbrot set ?

Computing the Mandelbrot Set

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Answer: We compute

$$f_{-2}(0) = 0^2 - 2 = -2$$

$$f^2_{-2}(0) = f_{-2}(-2) = (-2)^2 - 2 = 2$$

$$f^3_{-2}(0) = f_{-2}(2) = 2^2 - 2 = 2$$

$$f^4_{-2}(0) = f_{-2}(2) = 2^2 - 2 = 2$$

Clearly we are just going to get 2 at every step from now on and thus -2 is in the Mandelbrot set.

Computing the Mandelbrot Set

Exercise: Is i in the Mandelbrot set ?

Computing the Mandelbrot Set

Exercise: Is i in the Mandelbrot set ?

Answer: We compute

$$
\begin{aligned}
f_i(0) &= 0^2 + i = i \\
f_i^2(0) &= f_i(i) = i^2 + i = -1 + i \\
f_i^3(0) &= f_i(-1+i) = (-1+i)^2 + i \\
&= (-1)^2 - 2i + i^2 + i \\
&= 1 - 2i - 1 + i = -i \\
f_i^4(0) &= f_i(-i) = (-i)^2 + 1 = -1 + i \\
f_i^5(0) &= f_i(-1+i) = -i
\end{aligned}
$$

Clearly we are stuck in a cycle and from now on we just get alternating $-i$ and $-1+i$ terms. Thus the sequence never "escapes" and i is in the Mandelbrot set.

Approximating the Mandelbrot Set

We got lucky in our exercises.

For most values of c the numbers we get just jump around and never cycle. In this case we cannot know whether the sequence remains bounded.

We approximate the set by picking $n_{\max}$ a big number and guessing that if we haven't escaped by $f_c^{n_{\max}}(0)$ then we will never escape.

The black set in our pictures is not exactly the Mandelbrot set but rather the set of points that has not "escaped" by $n_{\max}$.

Julia Sets

For each value of c there is an associated filled Julia set. The filled Julia sets are computed in nearly the same way as the Mandelbrot set. The filled Julia set contains the values of z for which

$$f_c(z), f_c^2(z), f_c^3(z), \ldots$$

remain bounded.

Gaston Julia (1893–1978)

Julia Sets

Exercise: What is the Julia set for $c = 0$?

Exercise: Can we see a relationship between the Mandelbrot set and the Julia sets for different c ?

Julia Sets

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Answer: The sequence we get is

$$z^2, z^4, z^8, z^{16}, z^{32}, z^{64} \ldots$$

which remains bounded only if $|z| \leq 1$. The Julia set is a circle.

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Exercise: Can we see a relationship between the Mandelbrot set and the Julia sets for different c ?

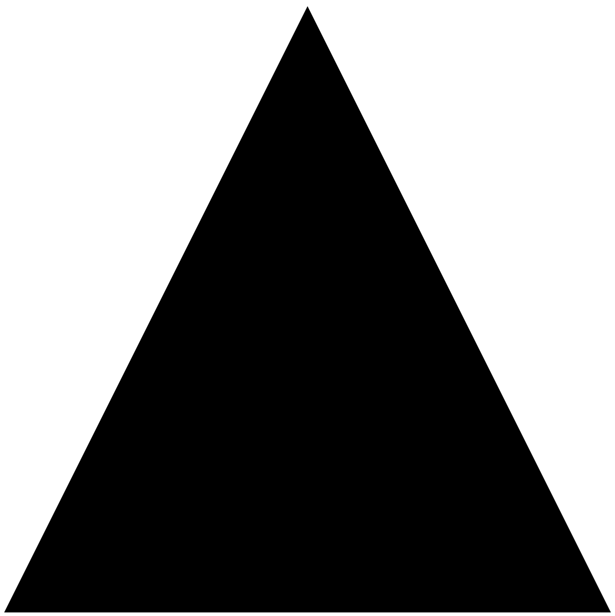
Answer: The Mandelbrot set is precisely the values of c for which the associated Julia set is connected. For c outside of the Mandelbrot set the filled Julia set is “dust”.

Sierpinski Triangle

Another way of constructing fractals is by a step by step (mathematicians say “iterative”) process.

Lets build a simple fractal, the Sierpinski Triangle, in this way

Step 1: Start with an equilateral triangle. We have 1 triangle, no holes, and area 1.



Sierpinski Triangle

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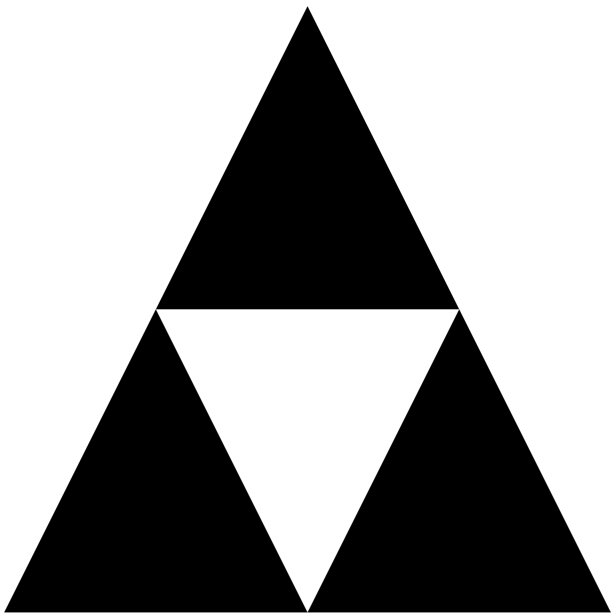
Lets build a simple fractal, the Sierpinski Triangle, in this way

Step 2: Shrink the first triangle by a factor $\frac{1}{2}$. Build a new "triangle" using 3 copies of this smaller triangle. We have 3 triangles, 1 hole, and area $\frac{3}{4}$.

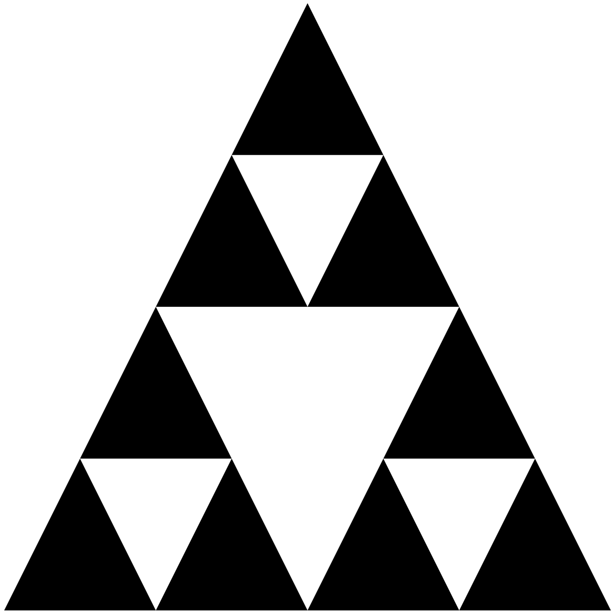


Sierpinski Triangle

Another way of constructing fractals is by a step by step (mathematicians say “iterative”) process.

Lets build a simple fractal, the Sierpinski Triangle, in this way

Step 3: Shrink the “triangle” from Step 2 by a factor $\frac{1}{2}$. Build a new “triangle” using 3 copies of this smaller “triangle”. We have 9 triangles, 4 holes, and area $\frac{9}{16}$.

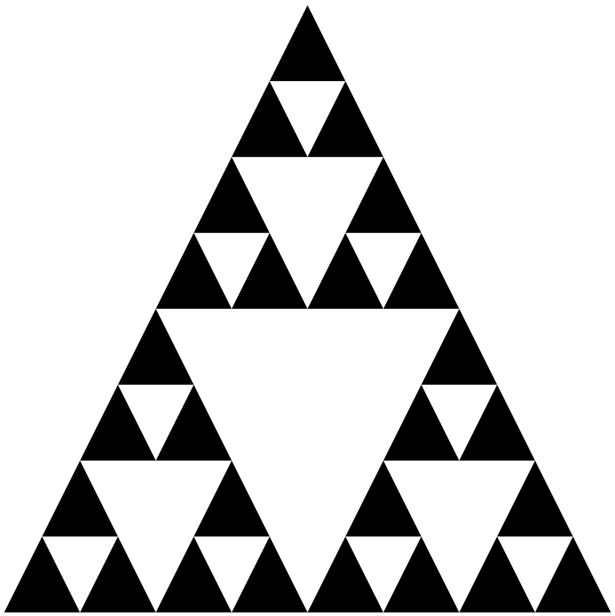


Sierpinski Triangle

Another way of constructing fractals is by a step by step (mathematicians say “iterative”) process.

Lets build a simple fractal, the Sierpinski Triangle, in this way

Step 4: Shrink the “triangle” from Step 3 by a factor $\frac{1}{2}$. Build a new “triangle” using 3 copies of this smaller “triangle”. We have 27 triangles, 13 holes, and area $\frac{27}{64}$.

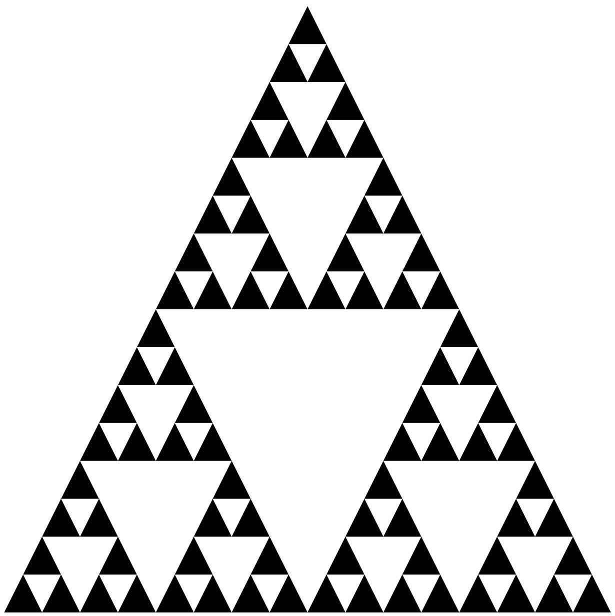


Sierpinski Triangle

Another way of constructing fractals is by a step by step (mathematicians say "iterative") process.

Lets build a simple fractal, the Sierpinski Triangle, in this way

Step 5: Shrink the "triangle" from Step 3 by a factor $\frac{1}{2}$. Build a new "triangle" using 3 copies of this smaller "triangle". We have 81 triangles, 30 holes, and area $\frac{81}{256}$.



Sierpinski Triangle

If we continue like this for ever we get the Sierpinski Triangle.

Exercise: What is the area of the Sierpinski Triangle ?

Sierpinski Triangle

If we continue like this for ever we get the Sierpinski Triangle.

Exercise: What is the area of the Sierpinski Triangle ?

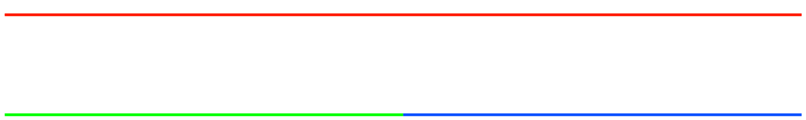
Answer: At each step the area goes down by a factor $\frac{3}{4}$. After n-steps the area will be $\left(\frac{3}{4}\right)^n$. Continuing like this we see that the final Sierpinski triangle must have area 0.

Fractal Dimension

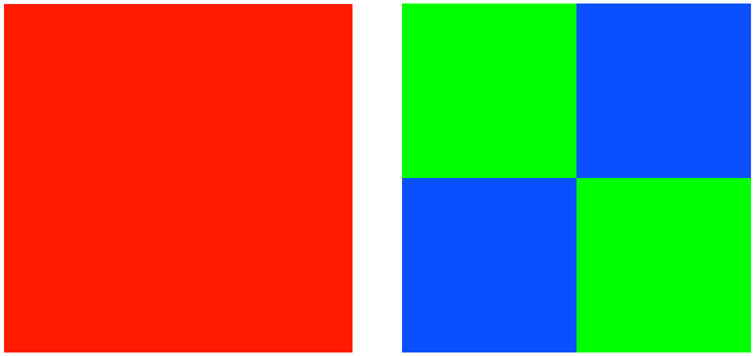
A solid square is 2 dimensional and a straight line is 1 dimensional.

What does this mean ?

If we try and cover a line by copies half the size then it takes 2 copies.



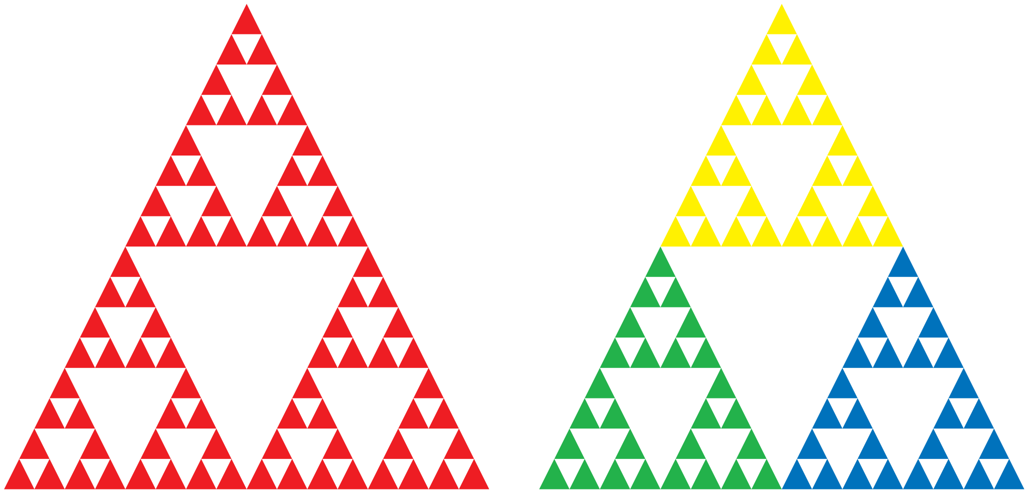
If we try and cover the square by copies half the size then it takes 4 copies.



Now $2^1 = 2$ so we say the line is dimension 1 and $2^2 = 4$ so say the solid square has dimension 2.

Fractal Dimension

If we try and cover the Sierpinski triangle by copies half the size then it takes 3 copies.



Analogously to the line and square we say that the dimension of the Sierpinski triangle is the number d such that $2^d = 3$.

Finding d is a little complicated. We have to use logarithms.

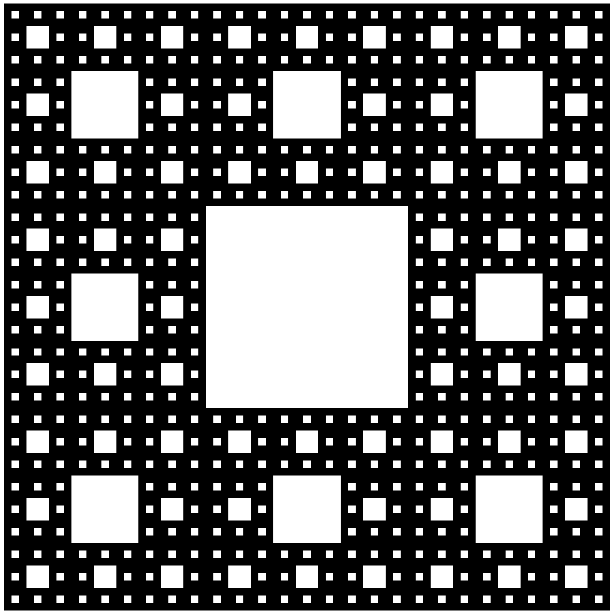
$$d \log 2 = \log 3$$

$$d = \frac{\log 3}{\log 2}$$

$$\approx 1.58$$

Fractal Dimension

There is nothing special about 2. You can instead cover using N copies one third the size and look for d such that $3^d = N$. This is useful for the Sierpinski carpet or for the Menger sponge (see the extension problems).



Fractals in Nature

Fractals can give very natural looking objects. Here is a plot of the Black Spleenwort fern generated in a similar way to the Sierpinski triangle. The big fern is made of 3 smaller ferns (the two lowest fronds plus the top of the fern) plus a straight line segment.

This plot is generated by a random algorithm - the same method can be used to make plots of the Sierpinski triangle.

L-Systems

L-Systems, named after Lindenmayer, are often used to describe plants. The actual objects are strings of characters. An L-system consist of an axiom and replacement rules.

For example: The Koch Snowflake is an L-system given by

Axiom: $F++F++F$

Replacement Rule: $F \longrightarrow F-F++F-F$

Using the replacement rule iteratively we generate a string. The first such string is $F-F++F-F++F-F++F-F++F-F++F-F$. Now we interpret F as meaning go forward by 1, $+$ as turn right by 60°, and $-$ as turn left by 60°.

Koch Snowflake