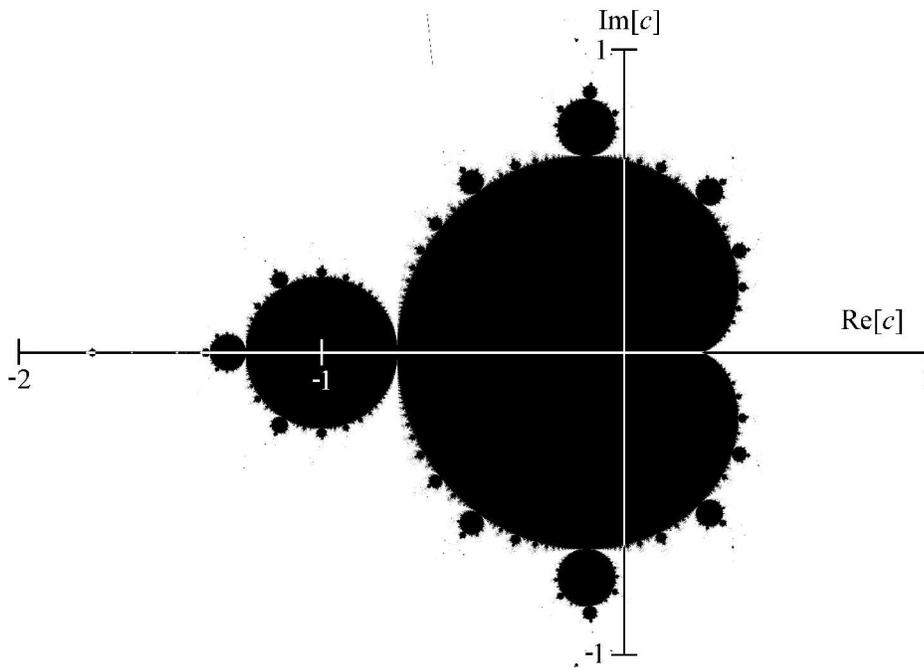


SMMG February 25th, 2006
featuring Dr. Alistair Windsor
"Fun with Fractals"

1. If $f(z)$ is a function, various behaviors can arise when f is iterated. Gaston Julia studied the iteration of polynomials and rational functions in the early twentieth century.

The set of all points $c = x+iy$ in the complex plane where the function $f_c(z) = z^2+c$ does not diverge under iteration, is called the **Mandelbrot set**, after Benoit Mandelbrot.



- a) Use this function $f_c(z) = z^2+c$ with initial value $z = 0$ and $c = -2$ and decide about convergence/divergence.

- b) Now use $c = i$ (where i is the complex number whose square is $i^2 = -1$) and calculate the first couple of iterations. What do you observe?

2. Some practice with infinite sums!

Suppose you want to calculate the sum of all the powers with positive integers of a number r with absolute value less than 1, that is $S = 1 + r + r^2 + r^3 + r^4 + r^5 + r^6 + r^7 + \dots$ with $|r| < 1$.

How can you do that in finite time?

a) Write down the product rS .

b) Subtract $S - rS$.

c) Solve for S . You found the formula for what mathematicians call a *geometric sum*.

3. What is the value of the sum

$$S = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^7 + \dots ?$$

4. The **Sierpinski Carpet**

Closely related to the Sierpinski Triangle is the Sierpinski Carpet, which is the corresponding square object. Instead of removing the central third of a triangle, the central square piece is removed from a square sliced into thirds horizontally and vertically. Here is the procedure:

Step 1: Start with a square of side length 1.

Step 2: Take out the center square of side length $1/3$.

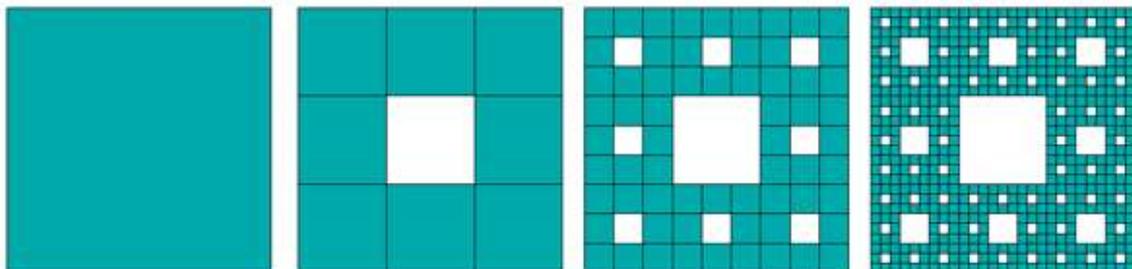
Step 3: Now, for each of the remaining squares with side lengths $1/3$, take out their middle squares.

If you repeat this process infinitely many times, the remaining object is called the Sierpinski Carpet. For now, do it at least twice and draw the corresponding pictures!

5. Use the table and the pictures below to help you organize your answers to a) – e).
- For each step as above, write down the number of holes. What is the number of holes at step n ?
 - Find the number of squares at each stage. What is the pattern of this sequence? Can you build a formula for the number of remaining squares at the n -th stage?
 - Find the sequence composed of the length of a side at each stage. What is the pattern? Can you build a formula for the length of one square at the n -th stage?
 - Find the sequence composed of area of one square at each stage. What is the pattern? Can you build a formula for the area of one shaded square at the n -th stage?
 - Find the sequence composed of total area colored in at each stage. What is the pattern? Can you build a formula for the total area at the n -th stage?

Iteration Step	1	2	3	4	...	n
Number of holes	0	1				
Number of remaining squares	1					
Length of the side of one square	1					
Area of one square	1					
Total remaining area	1					

- f) What do these sequences do as the number of stages gets larger and larger?



6. Calculate the fractional dimension of the Sierpinski Carpet.

7. Compare these results to those for the Sierpinski Triangle. Does the "three-ness" of a triangle and the "four-ness" of a square seem to play a role in these numbers?
8. Challenge! Consider the 3-dimensional object that is created by performing a similar process on a cube. It is often called the **Menger Sponge** and its construction can be visualized as follows:

Step 1: Begin with a cube.

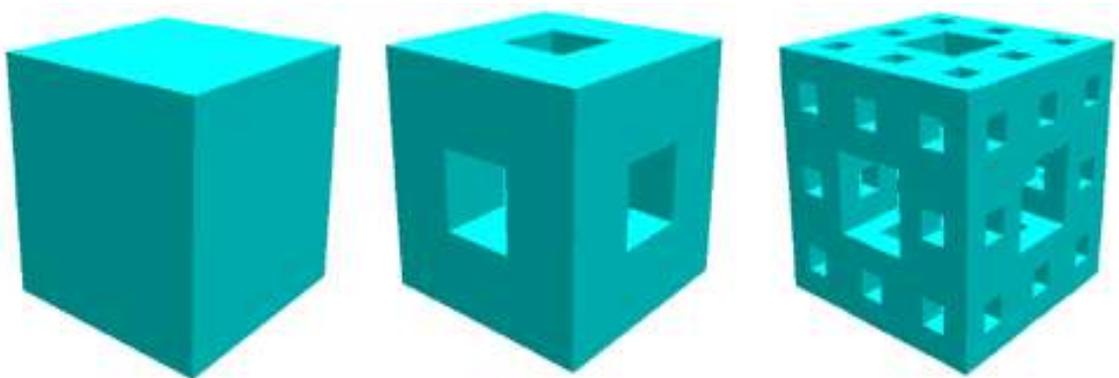
Step 2: Shrink the cube to a third of its original size and make 20 copies of it.

Step 3: Place the copies so they will form a new cube of the same size as the original one but lacking the center parts.

Repeat the process from step 2 for each of the remaining smaller cubes. After an infinite number of iterations, a Menger sponge will remain.

What interesting things can you say about it?

Alternatively, start with one cube and on each iteration split all existing cubes into 27 smaller equal size cubes. Remove the cube in the center of each side as well as the center of the original cube. Repeat this process for ever, the first three iterations are shown below.



As with most fractals we can only ever create approximations to the ideal fractal form. The Menger sponge in its final form (unlike the approximations we can create) has zero volume but an infinite surface area!

9. The **Koch Snowflake**

You will create the Koch snowflake and calculate its perimeter and area step by step:

Step One: Start with a large equilateral triangle.

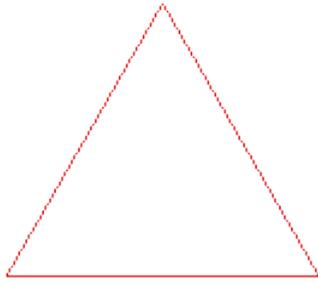
Step Two: Make a Star.

That is, divide one side of the triangle into three equal parts and remove the middle section. Replace it with two lines the same length as the section you removed to create a new equilateral triangle.

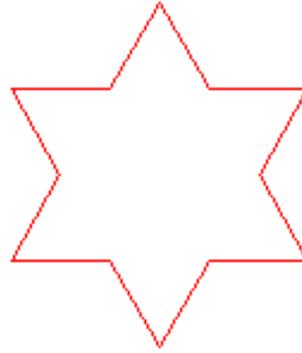
Do this to all three sides of the triangle.

Do it again and again. When you do it infinitely many times, you have a fractal.

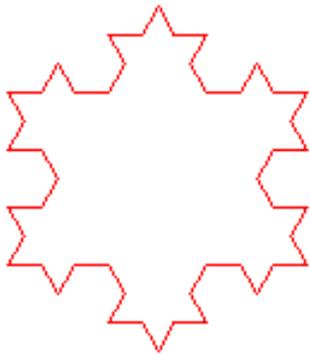
Try it yourself before taking a look at the next page where you see a few of the steps.



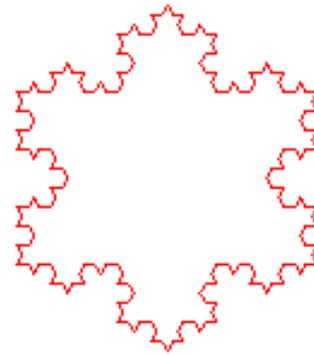
Step One



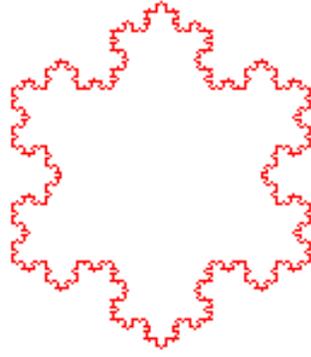
Step Two



Step Three



Step Four

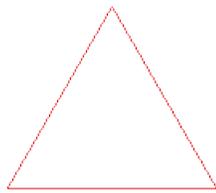


Step Five

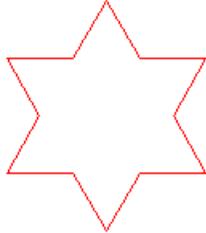
10. The perimeter of the Koch snowflake.

A really interesting characteristic of the Koch Snowflake is its perimeter. Ordinarily, when you increase the perimeter of a geometric figure, you also increase its area. If you have a square with a huge perimeter, it also has a huge area. But wait till you see what happens here!

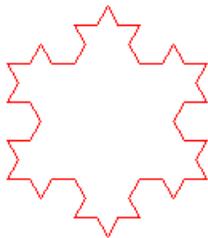
- a) If the perimeter of the equilateral triangle that you start with is 9 units, what is the perimeter of the other figures?



perimeter = 9 units



perimeter = _____ units

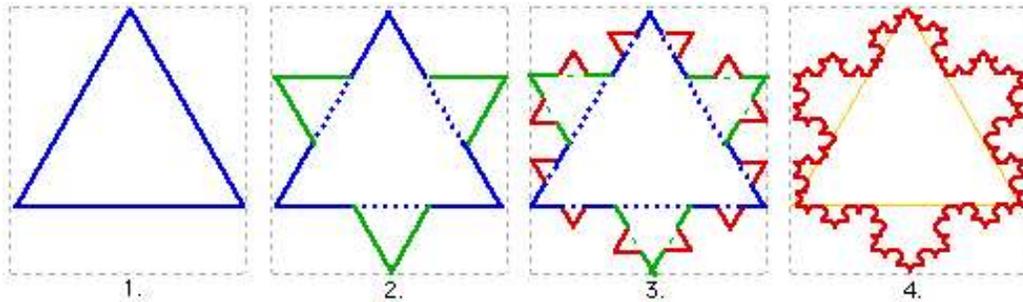


perimeter = _____ units

Hint: Think of the original triangle with sides of three parts, and the next figure with sides of four parts.

- b) Is there a pattern here?
The perimeter of each figure is ____ times the perimeter of the figure before.
- c) If the original triangle has a perimeter of 9 units, how many iterations would it take to obtain a perimeter of 100 units? (Or as close to 100 as you can get.)

Let's investigate this a little further. For your convenience, here are again the pictures of the first four iterations:



d) Suppose now you started with a triangle of side length 1. How does the perimeter of the Koch Snowflake change in each step?

Iteration Step	1	2	3	4	...	n
Number of sides	3					
Length of one side	1	1/3				
Number of triangular bumps added	0	3				
Total perimeter added	0	1				
Total perimeter of the object	3					

e) Now think of doing this many, many times. Is the perimeter going to be finite?

11. We have seen that the perimeter of the Koch Snowflake gets huge. But does the area?

a) Suppose you start with an equilateral triangle of area 1. Fill in the table to see how the area of our object changes in each step:

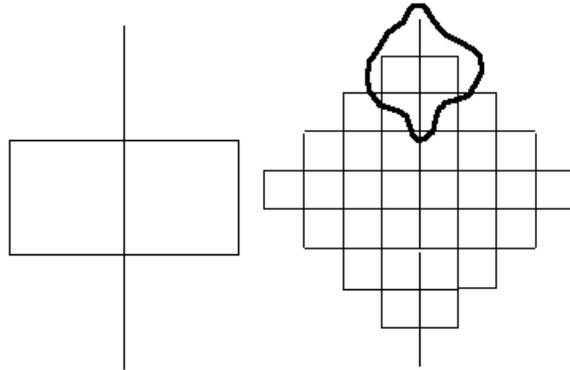
Iteration Step	1	2	3	4	...	n
Area of each new triangle	1	1/9				
Number of new triangles	0	3				
Total area added in this step	0					
Total area of the whole object	1					

b) Can you make a guess of what happens to the area after infinitely many steps? Is it going to be finite?

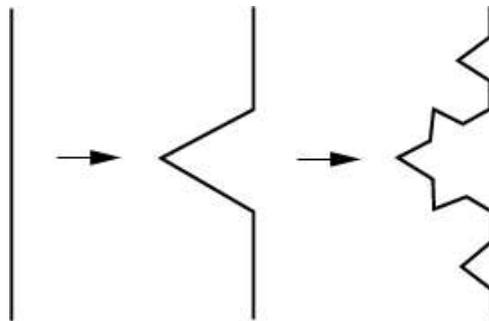
An infinite perimeter encloses a finite area... **Now that's amazing!!**

12. What is self-similarity?

It is the property of an object, not one of the steps used to build it. Think of it this way – when we look at stage 2 of the Hilbert Curve, we can find a copy of stage 1:



This copy is a smaller scale version of the original. We can find this in the Koch Snowflake, too. Circle the copy of stage 1 of one of its sides in the second picture and then a copy of one side of stage 2 in the third!



Can you find self-similarity in the other fractals?





13. What makes a fractal a fractal?

a) List some of the properties that all the objects we looked at today have in common:

b) When you go home, explain to your family and friends what fractals are!

Here is a pretty picture of the Mandelbrot set and a few examples of corresponding Julia sets.

