Filters and Social Choice

Definition 1. *A **total ordering** on the set of candidates is a true/false statement about two candidates, denoted $\geq$, satisfying the following properties:*

1. ***Reflexive Property** – for any candidate A, $A \geq A$.*
2. ***Anti-symmetric Property** – for any two distinct candidates A and B, if $A \geq B$, then $B \not\geq A$.*
3. ***Totality Property** – for any two candidates A and B, either $A \geq B$ or $B \geq A$.*
4. ***Transitive Property** – for any three candidates A, B, and C, if $A \geq B$ and $B \geq C$, then $A \geq C$.*

For each of the following true/false statements about two candidates, determine whether or not it is a total ordering. If it is not a total ordering, which of the four properties does it satisfy, and which does it fail to satisfy?

1. $A \geq_v B$ – Voter v prefers candidate A to candidate B.
2. $A \heartsuit B$ – Candidate A loves candidate B.
3. $A \clubsuit B$ – Candidate A is precisely as smelly as candidate B.
4. $A \bigtriangledown B$ – When each candidate participated in a two-player bowling match against each of the others, candidate A beat candidate B.
5. $A \vdash B$ – Candidate A's name first appears before (or at the same time as) candidate B's in my choose-your-own-adventure novel.
6. $A \bigstar B$ – Candidate A is a real American.

We define V to be the set of all voters and $V_{A \geq B}$ to be the set of voters who prefer candidate A to candidate B. In fancy mathematical symbols, another way to write this is:

$$V_{A \geq B} = \{v \in V | A \geq_v B\}.$$

1. Use the reflexive property to show that $V_{A \geq A} = V$.

2. Use the anti-symmetric property to show that, if a voter v is in $V_{A \geq B}$, then v is **not** in $V_{B \geq A}$.

3. Use the totality property to show that, if a voter v is **not** in $V_{B \geq A}$, then v is in $V_{A \geq B}$. Use this, together with the previous exercise, to conclude that $V_{A \geq B}$ is precisely the set of voters that are **not** in $V_{B \geq A}$.

4. Use the transitive property to show that if a voter v is in both $V_{A \geq B}$ and $V_{B \geq C}$, then v is in $V_{A \geq C}$.

   If W and X are two sets of voters, then their **intersection** is defined to be the set of voters that are in both W and X. The intersection of W and X is written $W \cap X$.

5. Show, by example, that it is possible for $V_{A \geq B} \cap V_{B \geq C} = V_{A \geq C}$. Using another example, show that it is possible for there to exist a voter in $V_{A \geq C}$ who is not in $V_{A \geq B} \cap V_{B \geq C}$.

Now, suppose that we have a total ordering $\geq$ on the set of candidates. If $A \geq B$, we will call the set $V_{A \geq B}$ a **large set**.

1. Use the reflexive property to show that V is a large set.

2. Use the anti-symmetric property to show that, if $V_{A \geq B}$ is a large set, then the set of voters that are **not** in $V_{A \geq B}$ is **not** a large set.

3. Use the totality property to show that, if $V_{A \geq B}$ is **not** a large set, then the set of voters that are **not** in $V_{A \geq B}$ is a large set.

4. Use the transitive property to show that there is a large set that contains $V_{A \geq B} \cap V_{B \geq C}$.

Now, suppose that we have a voting system that satisfies independence of irrelevant alternatives. By definition, this means that we have a total ordering, the **social preference**, that is determined entirely by the sets $V_{A \geq B}$. We will use the following rule to determine the social preference:

$A \geq B$ if and only if $V_{A \geq B}$ is a **large set**.

Without a definition of large sets, this rule is meaningless, so we need to say which of our sets are large sets. There are lots of ways in which we might define our large sets, but, by the exercises on the previous page, we know that there are certain properties that they must satisfy. We collect these properties into a single definition:

Definition 2. *An **ultrafilter** on the set of voters V is a collection of subsets of V, called **large sets**, satisfying the following properties:*

1. *V is a large set.*
2. *If W is a large set, then the set of voters that are not in W is not a large set.*
3. *If W is not a large set, then the set of voters that are not in W is a large set.*
4. *If W and X are two large sets, then $W \cap X$ is a large set.*
5. *If W is a large set, then any set containing W is a large set.*

1. Let $V = \{v_1\}$ be a set consisting of just one voter. Find all of the ultrafilters of V. (Hint: there's only one!)
2. Let $V = \{v_1, v_2\}$ be a set consisting of just two voters. Find all of the ultrafilters of V. (Hint: there are only two. What does this have to do with an earlier part of this presentation?)
3. Let $V = \{v_1, \ldots v_N\}$ be a finite set of voters, and suppose that we have an ultrafilter on V. Show that, for some i, the set $\{v_i\}$ is a large set.
4. Show that, if you have a finite set of voters, then there is only one voting system they can use that satisfies independence of irrelevant alternatives. What would be an appropriate term for this voting system?