Moving Through Space with Geometric Algebras

Part I: Adding and Multiplying Vectors

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INTRODUCTION

EUCLIDEAN MOTION

- The problem: modeling the motion of an object in space
  - designing a 3D video game
  - building a robot with arms and legs
- Basic Euclidean motions: translation and rotations
- Today: a mathematical model of rotation

Linear Algebra vs. Geometric Algebra

- Traditional linear algebra: vectors are objects and matrices are operations
- Rotation using linear algebra:

$$\begin{bmatrix} x^2 + (y^2 + z^2)c & xy(1-c) - zs & xz(1-c) + ys \\ xy(1-c) + zs & y^2 + (x^2 + z^2)c & yz(1-c) - xs \\ xz(1-c) - ys & yz(1-c) + xs & z^2 + (x^2 + y^2)c \end{bmatrix}$$

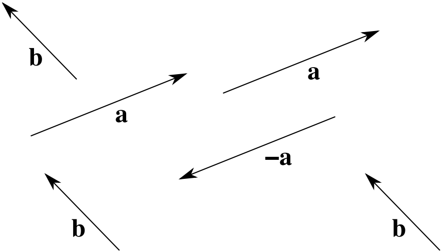
- Geometric algebra: scalars, vectors and bivectors are objects and operations!
- Rotation using geometric algebra: $\mathbf{x} \mapsto \mathbf{abxba}$

Scalars

- Scalars are numbers: $1, \frac{5}{2}, \pi$
- We will denote them by Greek letters: α, β, γ
- Adding and multiplying:
  - $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ (associative law)
  - $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ (distributive law)
  - $\alpha\beta = \beta\alpha$ (commutative law)

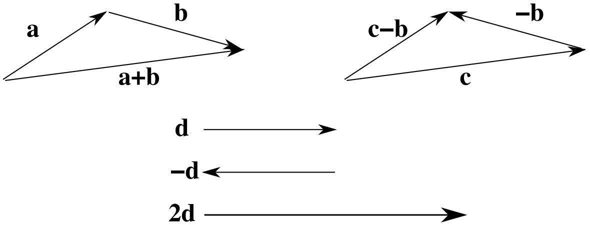
VECTORS

- Vectors are arrows: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$
- $|\mathbf{a}| = \text{Length}(\mathbf{a})$, "line", and orientation
- $\mathbf{a} = \mathbf{b}$ if $|\mathbf{a}| = |\mathbf{b}|$, $\mathbf{a} \parallel \mathbf{b}$, and same orientation



Adding and scaling vectors

Add vectors tip to tail: $\mathbf{a} + \mathbf{b}$
Scale $\mathbf{a}$ by α to get $\alpha\mathbf{a}$:
Subtract $\mathbf{b}$ from $\mathbf{a}$ by adding $-\mathbf{b}$ to $\mathbf{a}$:



Bivectors

Generalize "vector" to two dimensions:

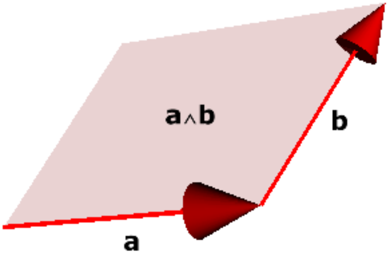
- Bivectors are planar segments (parallelograms or flat disks): $\mathbf{A}, \mathbf{B}, \mathbf{C}$
- $|\mathbf{A}| = \text{Area}(\mathbf{A})$, "plane", and orientation
- $\mathbf{A} = \mathbf{B}$ if $|\mathbf{A}| = |\mathbf{B}|$, $\mathbf{A} \parallel \mathbf{B}$, and same orientation
- (examples on board)

EXERCISE SET 1

- Given $\mathbf{a}$ and $\mathbf{b}$, compare $\mathbf{a}+\mathbf{b}$, $(\mathbf{2a})+\mathbf{b}$, $\mathbf{a}+(\mathbf{2b})$, $\mathbf{2}(\mathbf{a}+\mathbf{b})$.
- Draw some planar segments that represent the same bivector as $\mathbf{A}$. What is $-\mathbf{A}$?
- Consider the bivectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Can you add $\mathbf{A}+\mathbf{B}$? What about $\mathbf{A}+\mathbf{C}$?
- Challenge: How do you add arbitrary bivectors in the plane? What about in 3-space?

Outer product

- First way to multiply vectors: the outer product
- $\mathbf{a} \wedge \mathbf{b}$ = the parallelogram spanned by **a** and **b**!

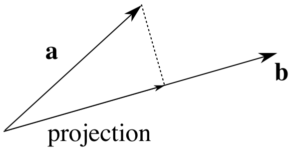


- If $\mathbf{a} \parallel \mathbf{b}$, then $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$!

INNER PRODUCT

- Second way to multiply vectors: the inner product
- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$
- Projection:

$$\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right) \mathbf{b} = (|\mathbf{a}| \cos \theta) \frac{\mathbf{b}}{|\mathbf{b}|}$$



EXERCISE SET 2

- Given $\mathbf{a}, \mathbf{b}$, draw and compare $(\mathbf{2a}) \wedge \mathbf{b}$, $\mathbf{a} \wedge (\mathbf{2b})$, and $\mathbf{2}(\mathbf{a} \wedge \mathbf{b})$. Are these all the same bivector?
- Explain this fact about the outer product:

$$\text{if } \mathbf{a} \| \mathbf{b}, \text{ then } \mathbf{a} \wedge \mathbf{b} = \mathbf{0}$$

- Challenge: given $\mathbf{a}, \mathbf{b}, \mathbf{c}$, draw and compare $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c})$ and $(\mathbf{a} \wedge \mathbf{b}) + (\mathbf{a} \wedge \mathbf{c})$. Explain the significance of this algebraically. Does the inner product behave the same way?

Review and Preview

- Review: scalars $\mathbf{a} \cdot \mathbf{b}$, vectors $\mathbf{a} + \mathbf{b}$, bivectors $\mathbf{a} \wedge \mathbf{b}$.
- After the break: use the geometric algebra product

$$\mathbf{ab} = ???$$

to understand rotation

$$\mathbf{x} \mapsto \mathbf{abxba}$$