Moving Through Space with Geometric Algebras

Part II: Reflections and Rotations

Eric Katerman
The University of Texas at Austin

The Austin Math Circle
April 19th, 2009

Introduction

Geometric Algebra: objects

- What are the elements in the geometric algebra?
- Scalars: α, β
- Vectors: $\mathbf{a}, \mathbf{b}$
- Bivectors $\mathbf{A}, \mathbf{B}$
- ...and sums of those! $a = \alpha + \mathbf{a} + \mathbf{A}$
- To draw a proper picture of this, we would need eight dimensions!

The vector product

- The geometric algebra product is a combination of the outer and inner products
- $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$
- The product of two vectors is a scalar plus a bivector!
- Many familiar properties:
  - $\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c}$
  - $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$
  - $(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{ac} + \mathbf{bc}$
  - $\alpha\mathbf{a} = \mathbf{a}\alpha$ for scalars α...
- BUT $\mathbf{ab} \neq \mathbf{ba}$

0 and 1

- There is a 0 element: $0 + a = a = a + 0$ for all a
- Subtraction: if $a = \alpha + \mathbf{a} + \mathbf{A}$, then $-a = -\alpha - \mathbf{a} - \mathbf{A}$, so $a - a = 0$
- There is a 1 element: $1(a) = a = (a)1$
- So we have addition, subtraction, multiplication, ...

Vector inversion: $\mathbf{a}^{-1}$

- Given $\mathbf{a}$, we want a vector $\mathbf{a}^{-1}$ such that $\mathbf{a}\mathbf{a}^{-1} = \mathbf{1} = \mathbf{a}^{-1}\mathbf{a}$
- Recall: $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^{\mathbf{2}} \cos\theta = |\mathbf{a}|^{\mathbf{2}}$
- So define

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{|\mathbf{a}|^{\mathbf{2}}}$$

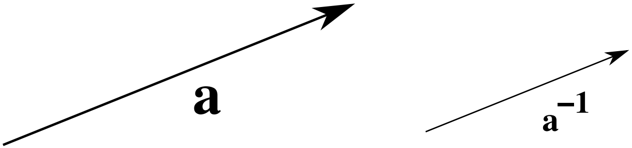


Figure: The length of $\mathbf{a}^{-1}$ is $\frac{1}{|a|^2}$

EXERCISE SET 3

- If $\mathbf{a}^{-1} = \mathbf{a}/|\mathbf{a}|^2$ is the inverse of $\mathbf{a}$, then what is the inverse of $2\mathbf{a}$ (in terms of $\mathbf{a}^{-1}$)?
- Check that our definition for $\mathbf{a}^{-1}$ is the right one by verifying $\mathbf{a}\mathbf{a}^{-1} = \mathbf{1}$.
- Challenge: Consider the silly equation $\mathbf{x} = \mathbf{x}(\mathbf{a}\mathbf{a}^{-1}) = (\mathbf{x}\mathbf{a})\mathbf{a}^{-1}$. Expand this using the product $\mathbf{x}\mathbf{a}$, the distributive property, and our definition for $\mathbf{a}^{-1}$. What two terms do you get?? Hint: draw a picture.

A SILLY COMPUTATION?

Let's look at that last exercise:

$$\begin{aligned} \mathbf{x} &= \mathbf{x}(\mathbf{a}\mathbf{a}^{-1}) \\ &= (\mathbf{x}\mathbf{a})\mathbf{a}^{-1} \\ &= (\mathbf{x}\cdot\mathbf{a} + \mathbf{x}\wedge\mathbf{a})\mathbf{a}^{-1} \\ &= (\mathbf{x}\cdot\mathbf{a})\mathbf{a}^{-1} + (\mathbf{x}\wedge\mathbf{a})\mathbf{a}^{-1} \end{aligned}$$

But $(\mathbf{x}\cdot\mathbf{a})\mathbf{a}^{-1}$ is projection!

$$(\mathbf{x}\cdot\mathbf{a})\mathbf{a}^{-1} = \frac{\mathbf{x}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$$

VECTOR REJECTION!

- So the rejection is:

$$\frac{\mathbf{x} \wedge \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$$

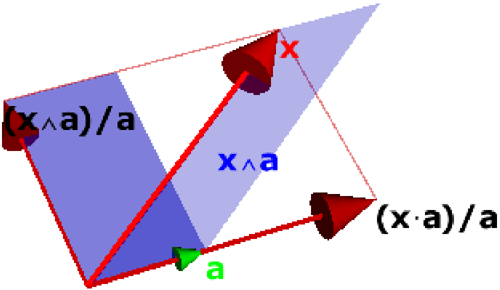
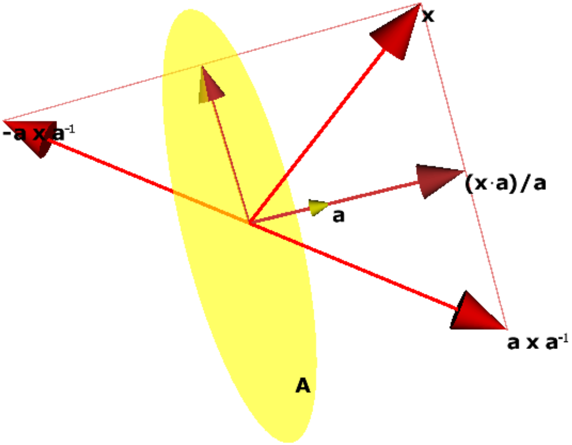


Figure: A vector is its projection plus its rejection.

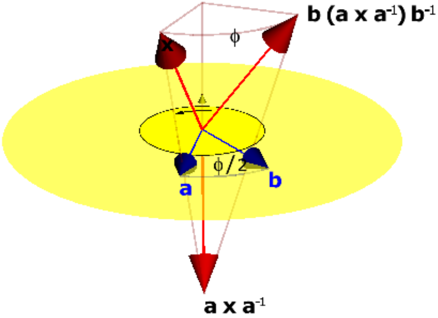
Projection minus rejection!

- Fact: $\mathbf{axa}^{-1} = (\mathbf{x} \cdot \mathbf{a})\mathbf{a}^{-1} - (\mathbf{x} \wedge \mathbf{a})\mathbf{a}^{-1}$
- What is projection *minus* rejection?



Two reflections is a rotation!

- Assume $\mathbf{a}, \mathbf{b}$ are unit vectors: $|\mathbf{a}| = \mathbf{1}, |\mathbf{b}| = \mathbf{1}$
- Then $\mathbf{a}^{-1} = \mathbf{a}$ and $\mathbf{b}^{-1} = \mathbf{b}$
- Reflect twice: $\mathbf{x} \mapsto \mathbf{bxb} \mapsto \mathbf{a}(\mathbf{bxb})\mathbf{a} = \mathbf{abxba}$



EXERCISE SET 4

- Use the fact that $\alpha a = a\alpha$ to show that $\mathbf{axa^{-1}} = (\mathbf{x} \cdot \mathbf{a})\mathbf{a^{-1}} - (\mathbf{x} \wedge \mathbf{a})\mathbf{a^{-1}}$
- Draw a picture representing the rotation $\mathbf{x} \mapsto \mathbf{abxba}$ in 3D.
- Challenge: we've been rotating using vectors $\mathbf{a}$ and $\mathbf{b}$. What would it mean to rotate using bivectors $\mathbf{A}$ and $\mathbf{B}$? Can you draw a picture to represent $\mathbf{x} \mapsto \mathbf{ABxBA}$?

Final remarks

- Geometric algebra is a powerful theoretical tool for Euclidean motion
- Generalizes to other motions: translations, rotations about arbitrary lines
- Although the ideas were discovered in the mid-1800s, scientists and engineers are only now starting to appreciate its full power...

Moving Through Space with Geometric Algebras

Main reference: **Geometric Algebra for Computer Science** by Dorst et al.

Eric Katerman
The University of Texas at Austin

The Austin Math Circle
April 19th, 2009