ADAM

1. Play the following "hat game" with Adam. Each member of your team will receive a hat with a colored dot on it (either red or black). Place the hat on your head so that everyone can see the color EXCEPT FOR YOU. Each member of your team now gets one try to guess the color of their own hat. Other than the fact that you can all hear the guesses made by your teammates, you are not allowed to communicate with each other in any way. You win if all but one of member of your team guesses correctly. (Note: you are allowed to come up with a strategy before you receive your hats.)

   Materials required: Hats with colored dots on them

   Solution: The standard solution is to have one member of your team count the number of red hats she sees, and say "red" if she sees an odd number, and "black" if she sees an even number. From this, everyone else can deduce the color of their own hat. If, for example, this team member tells you that she sees an even number of red hats, but you see an odd number, then your hat must be red. This is an example of the mathematical concept of **parity**, or the evenness/oddness of a situation. There are many variations of this standard hat problem. For example, there are versions where you can only see some of the other people's hats, or where there are more colors. There is even a version involving an infinite number of people!

2. Play the following dice game with Adam. There are four dice. The first player picks one of the dice to play with for the rest of the game, and then the second player picks one of the remaining dice, also to play with for the rest of the game. The two players take turns rolling their die, and the person who rolls the higher number wins that turn. The player who wins the majority of turns after 7 rolls by each player wins. You decide whether to go first or second.

   Materials required: Four dice, labeled as follows: (four 4's, two 0's), (six 3's), (four 2's, two 6's), (three 1's, three 5's)

   Solution: The dice are ordered as follows: (four 4's, two 0's) is better than (six 3's), which is better than (four 2's, two 6's), which is better than (three 1's, three 5's), which in turn is better than (four 4's, two 0's). Notice that, if you go first, the other player can always pick a die that is better than yours! This is an example of a mathematical property known as **non-transitivity**. We typically think that, if A is better than B, and B is better than C, then A is better than C as well. This is known as the transitive property. In this game, however, the transitivity property does not hold! Another game in which transitivity does not hold is rock-paper-scissors, in which paper beats rock, rock beats scissors, and scissors beats rock. There are even examples of non-transitivity in genetics – there is a type of lizard that can be found in three different colors, such that the gene that controls for each color dominates exactly one of the other two and is dominated by the other.

3. How many ping pong balls would it take to fill the UT Tower?

   Solution: Using the official dimensions of the UT Tower (according to the school) and of a ping pong ball (according to the world's governing body of the sport) and assuming a spherical packing, I found that it would take 585 million ping pong balls to fill the UT Tower. I would accept any number of this order of magnitude – between 100 million and 1 billion.

   Estimation can be a powerful tool in math, science, and life. There are many situations in which we encounter numbers that are very large, but such numbers are notoriously difficult for people to understand. When politicians talk about spending a million dollars or a billion dollars, the majority of people do not have an intuitive notion of the difference between those two numbers. We often encounter outrageous claims involving large numbers that only seem reasonable because we are not used to working with numbers so colossal. Questions like this stretch our minds, familiarizing us with the world of the gargantuan.

4. Write down a 10-digit number such that the first digit is the number of 0's that appear in the number, the second digit is the number of 1's that appear in the number, and so on.

   Solution: The only solution is 6210001000.

CHRISTY

1. You need your whole team to play this game with Christy. Christy will say "FLIP", and each of you must point to someone else on your team. Then Christy will say "FLOP", and each of you must point to someone else on your team. Remember the two people you pointed to – your FLIP person and your FLOP person. Christy will give frisbees to two members of your team, and will designate two other members of your team as people to whom those frisbees must be delivered. The method of delivery is this: at any time when there are two people with frisbees in their hands (NOT while one of the frisbees is in the air), anyone on your team may say either "FLIP" or "FLOP". If someone says "FLIP", anyone with a frisbee must throw that frisbee to their "FLIP" person. Similarly, if someone says "FLOP", anyone with a frisbee must throw that frisbee to their "FLOP" person. Your objective is to deliver the frisbees to the two designated people.

   Materials required: Some frisbees

   Solution: A **permutation** of objects is a rearrangement of those objects into a different order. The FLIPS and FLOPS in this problem can be thought of as permutations of the members of your team. It is an amazing fact from abstract algebra that two permutations are enough to build every other permutation, so it is possible to solve this problem no matter how many people are on your team, and no matter who your volunteer points to! Truthfully, however, you don't need to be able to build every permutation. You just need to be able to switch any 2 team members with any other 2 team members. This property is known as being **2-transitive**.

2. Send one member of your team to play the following game with Christy. Christy has a collection of coins, some of which are heads up and some of which are tails up. Your goal is to separate the coins into two collections, turning over whichever coins you wish, so that both collections contain the same number of heads up coins. Unfortunately, you will be blindfolded. You may ask Christy how many of the coins are lying heads up, but you may only ask this once.

   Materials required: Blindfolds, some coins

   Solution: Note to volunteers: it might be fun to make this person keep the blindfold on for the remainder of the Math Adventure – it's your call. Here's how to do this problem: ask your volunteer how many of the coins are heads up. Let's call that number N. Now, take N coins, turn them over, and put them in a collection by themselves. Notice that the number of coins that you turned over that were heads up to start with does not matter. If this number is H, then there are $N - H$ heads up coins still left in the original collection, and, after you turn over every coin in the new collection, there will be $N - H$ heads up coins in the new collection.

3. Play the following game with Christy. This game is played on a strip of squares, which are numbered from 0 to 20. Initially there are five coins: on squares 3, 4, 9, 11, and 17. This is a two-player game. At her turn, a player may move any coin to another square that has a lower number from the square it is currently at, but a coin may not jump over another coin and two coins cannot occupy the same square. For instance, initially, the coin in 9 could be moved to 8, 7, 6, or 5 only. The two players (Christy and you) take turns moving. The first player who cannot move loses. You choose whether you want to go first or second.

   Materials required: A couple of sheets of paper, divided into squares and numbered 0 through 20, and several coins or other kinds of tokens

   Solution: Amazingly, this game is Nim – the same game as you first played with Justin. To see this, color the coins black and white in alternation, so that you have 3 black coins and 2 white coins. Then, each black coin is equivalent to a Nim pile with as many beans as it has available moves. A move of a white coin is reversible, by moving the next black coin (with the next higher number) the same number of squares. Notice that the initial position is equivalent to (3, 4, 5) in Nim, so it is a first player win. A winning move is, for instance, to move 3 to 1.

4. Use each of the digits 1 through 9 once and only once to form a nine-digit number such that the first (leftmost) 8 digits form a number divisible by 8, the first 7 digits form a number divisible by 7, and so on.

   Solution: 381 654 729 is the only solution. You can solve this problem by using some of the tests for divisibility. For example, because a multiple of 5 always ends in a 0 or a 5, the middle digit has to be 5. Every digit in an even place must be even. The first, middle, and the last sets of three digits must all add up to a multiple of 9. The 2 and the 6 must be the 4th and 8th digits, not necessarily in that order. And there are many more.

JUSTIN

1. Play the game of Nim with Justin. Nim is played with three piles of stones, one with 3 stones, one with 4 stones, and one with 5 stones. It is a two-player game. At her turn, a player may take any number of stones, as long as they are all from one pile. The two players (Justin and you) take turns moving. The first player who cannot move loses. You choose whether you want to go first or second.

   Materials required: Some stones or other kinds of tokens

   Solution: Nim is a classic example of a **combinatorial game**, or a game in which, on each turn, a player has a finite number of possible moves, and there is no randomness involved. Other examples of combinatorial games include chess, checkers, connect four, and tic-tac-toe. Monopoly, Scrabble, and most card games are not combinatorial games because they have elements or randomness. In her Saturday Morning Math Group presentation this semester, Dr. Elaine Rich of the UT computer science department discussed strategies for winning at Nim. You can check out a video of her presentation on the SMMG website!

   There are many ways to win at Nim, but the standard solution is the following: write the number of stones in each pile as a binary number. So, in the example above, you would write 3 as 11, 4 as 100, and 5 as 101. Now, count the number of 1's in the ones place, the number of 1's in the twos place, the number of 1's in the fours place, and so on. You want to take away stones so that there are an even number of 1's in each place.

   To see why this is so, notice that the winning position (when all the piles have 0 stones) has an even number of 1's in each place. Next, notice that, if there are an even number of 1's in each place, then it is impossible to move in such a way so that there continue to be an even number of 1's in each place, because such a move would require you to take stones from more than one pile. If, on the other hand, there is an odd number of 1's in some place, then you can always remove stones to obtain an even number of 1's in each place.

   If you follow this strategy, you will see that (3,4,5) is a winning position for the first player. A good first move, for example, would be to remove two stones from the first pile.

2. Go play the following game with Justin. This is a two player game with piles of coins. On her turn, a player does one of the following:

   - either she can split one pile into two or more piles with the same amount of coins,

- or she can merge two piles with *different* numbers of coins together.

The first player who cannot move loses. You will start with exactly two piles. The starting position is $(6, 2)$. You may choose to go first or second.

Materials Required: A bunch of coins, or other tokens

Solution: I do not actually know a winning strategy for this game. Do you?

3. Let's play tetris! The 7 tiles below are known as tetrominoes. Find a tiling of a 4×4 square using 4 distinct tiles. Then assign each of your team members to be one of the tiles, and demonstrate this tiling to Justin.

   Solution: There are lots of fun puzzles involving **tetrominoes** and their cousins, the **pentominoes**. Search the web to find some more!

4. Perform the following magic trick for Justin. Tell him to write down a very large number. Then, tell him to scramble the digits of this number anyway he chooses. Then tell him to subtract the second number from the first one, to obtain yet another number. Finally, ask him to cross out one of the digits of this third number (he's not allowed to cross out a zero). Now, use your magical powers to determine the digit that he crossed out!

   Solution: This is based on an old trick known as "casting out nines". It relies on the fact that, whatever the third number is, the sum of its digits will be a multiple of 9. Thus, if you cross out one of the numbers, you will be able to tell which number was crossed out by subtracting the sum of the remaining digits from a multiple of 9.

   You might wonder why the sum of the digits of the third number will always be a multiple of 9. This has to do with **modular arithmetic**. We say that two numbers a and b are equivalent mod 9 if $a - b$ is a multiple of 9. In particular, the number 10 is equivalent to 1 mod 9. What this means is, if you have a number whose digits are $a_N a_{N-1} \dots a_1 a_0$, you can write

$$a_N a_{N-1} \dots a_1 a_0 = a_N 10^N + a_{N-1} 10^{N-1} + \dots + a_1 10 + a_0$$

$$= a_N + a_{N-1} + \ldots + a_1 + a_0 (mod 9).$$

Since the first two numbers have the same digits, by the above they are equivalent mod 9, and so their difference, the third number, is a multiple of 9. This means, again by the above, that the sum of its digits is a multiple of 9.

SHANNON

1. Shannon is going to hold out her arms at the same height and a few inches apart. Your goal is to hang a picture on her arms with a piece of string attached to the picture's two upper corners. Moreover, you must hang the picture so that if **either** arm is removed, the picture will fall to the ground.

   Materials required: a picture with a piece of string attached to its two upper corners (hopefully one that is not too heavy!)

   Solution: The standard solution is to loop the string clockwise around the first arm, then clockwise around the second arm, then counterclockwise around the first arm, and then counterclockwise around the second arm. In fact, there is a sense in which all solutions to this problem look like "multiples" of this one. Mathematically, we say that a loop is **homotopically trivial** if it can be squished to a point. What this problem is really asking for is to build a loop that is not homotopically trivial, but becomes homotopically trivial when either of the arms is removed. One of the standard problems in **algebraic topology** is to determine whether loops in a space are homotopically trivial or not.

   PS: If you want to be really fancy about this problem, what we're looking for is an element of the kernel of

   $$\pi_1(\mathbb{R}^2 - \{p, q\}) = \mathbb{Z} * \mathbb{Z} \to \pi_1(\mathbb{R}^2 - \{p\}) \oplus \pi_1(\mathbb{R}^2 - \{q\}) = \mathbb{Z} \oplus \mathbb{Z}.$$

2. The vertices of a cube can be grouped into two disjoint sets of four, each of which comprise the vertices of a regular tetrahedron. The intersection of these two tetrahedra is another polyhedron. Form the graph of this polyhedron with your bodies. (The graph of a polyhedron is what you get when you look only at its edges.) If in doubt, ask Shannon.

   Solution: This polyhedron is the regular octahedron. There are five regular polyhedra – the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. The construction above is one of many beautiful relationships between these five shapes. These relationships arise because the five regular polyhedra have an enormous amount of symmetry. Polyhedra and their symmetries are of great interest to **geometric group theorists** and other mathematicians.

3. Shannon has a square piece of paper for you. Without using any tools other than the sheet of paper, accurately construct a 60 degree angle.

Materials required: A whole bunch of square sheets of paper (they really need to be pretty close to square. It's also good if they're fairly large, like $8\frac{1}{2}$ by $8\frac{1}{2}$)

Solution: One possible solution is this: first, fold the piece of paper in half. Then, unfold it, and now fold along a diagonal meeting one corner so that the next corner over meets the crease. You will then see a triangle with one side length equal to one half the length of its hypotenuse. The constructed angle is therefore $\cos^{-1}(\frac{1}{2}) = 60$ degrees.

4. Play the game of Dots and Boxes with Shannon. In this game, two players take turns connecting adjacent dots. When it is your turn, you may connect two dots that are next to each other, either horizontally or vertically. When you draw an edge that completes one or more of the 25 small squares, you put your initial in that box (or those boxes) and take one more turn immediately. You can take many turns in a row if you keep closing boxes. You are not required to take a box if you prefer some other move. When all 25 squares are taken, whoever took more of them wins.

   Materials required: Some Dots and Boxes boards. These can be printed out from the SMMG website.

   Solution: Dots and Boxes is a rich and fascinating game that has delighted mathematicians for years. Last Fall, Dr. Daniel Allcock, a professor of mathematics here at UT and a former Dots and Boxes tournament champion, led a SMMG presentation on this game. To learn all of his techniques and strategies, check out the video on our website!

TRACY

1. Form the Borromean rings with your bodies. The Borromean rings consist of three circles which are linked, but removing any one of the three rings results in two unlinked rings. If in doubt, ask Tracy.

   Solution: The Borromean rings are just one of many interesting links that are studied by **knot theorists**. These are mathematicians who think about strings and tangles all day long! (They don't usually build them out of people, though.)

2. Play the game of Sprouts with Tracy. The game begins with a sheet of paper with two dots on it. On your turn, you may draw a curve that starts at a dot and ends at a dot (they may be the same dot), and then draw a new point somewhere on the curve you just drew. The curves are not allowed to cross each other, and no point is allowed to have more than 3 curves coming out of it (if a curve starts and ends at the same point, that counts as two). Players alternate turns, and the first player who cannot move loses. You choose whether to go first or second.

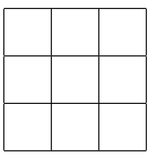
   Solution: Sprouts is a second-player win. It is possible for a game to last 5 moves, but the second player can always force the game to last only 4 moves. When you are finished playing Sprouts, the picture you are left with is an object called a **graph**. (These are not like the graphs of functions you see in school!) This graph has a special property – the fact that it can be drawn on a piece of paper without any of the edges crossing each other means that it is **planar**. Many beautiful properties of planar graphs have been discovered over the centuries. The most famous of these results is the 4-color theorem, which states that the vertices of any planar graph can be colored, using only 4 colors, in such a way so that no two vertices of the same color are connected by an edge.

3. Tracy has an unmarked ruler and a piece of string for you. The line segment below is exactly one inch long. Using only these tools, draw a line segment that is exactly $\sqrt{3}$ inches long.

Materials required: Unmarked rulers, string

Solution: One way to do this is the following: use the string to mark out all the points that are 1 inch away from one end of the given line segment. Similarly, mark out all the points that are 1 inch away from the other end of the line segment. You will find that these two circles intersect in two points. If you then use the unmarked ruler to draw the line segment between these two points, this line segment has length $\sqrt{3}$ inches. To see this, notice that your picture contains two equilateral triangles with side length 1 inch. The height of such a triangle is $\frac{\sqrt{3}}{2}$ inches.

Such drawings are known as **straightedge and compass constructions**, and have been studied as far back as the ancient Greeks. Although the ancient Greek mathematicians were very good at solving problems using straightedge and compass, there were certain problems that simply seemed intractable. The most famous of these was the problem of **squaring the circle**, or trying to construct a square whose area was equal to that of a given circle. Similarly, there was the problem of **trisecting the angle**, or trying to construct an angle that is $\frac{1}{3}$ the size of a given angle. A major insight was that these problems were equivalent to the problems of constructing line segments of a given length – namely, of length π and length $\sin(20)$. It was not until the advent of **Galois theory** in the 19th century that these constructions were finally proven to be impossible.

4. Play misere torus tic-tac-toe with Tracy. In misere torus tic-tac-toe, the object is to *avoid* making a row, column, or diagonal of three X's or O's on a 3×3 torus grid. The fact that the grid is a torus means that the right side of the grid is glued to the left side, and the top is glued to the bottom. This means, for example, that there is a "diagonal" of the grid that starts in the bottom right corner and goes up and to the right. You decide whether to go first or second. If you do not understand this game, ask Tracy. 

Solution: This is a first-player win. Place an X anywhere to start, and then every time they place an O, place an X along the line through your first X and the O they just placed. A **torus** is what a mathematician calls a doughnut. The picture above may not look much like a doughnut, but if you take a piece of paper, glue the top end to the bottom end, and then glue the right side to the left side, a doughnut is exactly what you get! The idea of cutting the torus open so that it looks flat, but all the while remembering how it is glued, is very important in the field of **topology**.