Austin Math Circle – September 2009

Striking Symmetries for Perplexing Polynomials

Terminology

A *polynomial* is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are numbers, and $a_n \neq 0$. The numbers $a_n, a_{n-1}, \ldots, a_1, a_0$ are called the *coefficients* of the polynomial. The number n is called the *degree* of the polynomial; it tells us the highest power of x that is represented in the polynomial. (*Note:* It is also possible to write a polynomial in more than one variable, but we'll be focusing most of our attention on polynomials in one variable.)

A number r is said to be a *root* or *zero* of this polynomial if substituting $x = r$ in the polynomial yields a value of zero; that is, if $a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$.

A polynomial is said to be *monic* if its lead coefficient a_n is 1. Given a polynomial, we can always find a monic polynomial with the same roots by dividing all of the coefficients by a_n.

Some Useful Tools

Factor Theorem: Suppose that $P(x)$ is a polynomial. Then r is a root of $P(x)$ if and only if $P(x) = (x - r)Q(x)$ for some polynomial $Q(x)$; in other words, if and only if $(x - r)$ is a *factor* of $P(x)$.

Example: Consider the polynomial $x^3 - 4x^2 - 15x + 18$. We know that 1 is a root of this polynomial because $1^3 - 4 \cdot 1^2 - 15 \cdot 1 + 18 = 0$. So we should be able to write this polynomial in the form $(x - 1)Q(x)$, where $Q(x)$ is a polynomial. If we divide $x^3 - 4x^2 - 15x + 18$ by $x - 1$, we get $x^2 - 3x - 18$ (with no remainder); thus we have $x^3 - 4x^2 - 15x + 18 = (x-1)(x^2 - 3x - 18)$. We can then observe that $x^2 - 3x - 18 = (x-6)(x+3)$, so we have $x^3 - 4x^2 - 15x + 18 = (x - 1)(x - 6)(x + 3)$. So the roots of this polynomial are 1, 6, and -3.

Fundamental Theorem of Algebra: Every polynomial with complex-number coefficients has a complex root.

Some notes on the Fundamental Theorem of Algebra:

1. The complex numbers include the real numbers. So when we say "complex-number coefficients," we include real numbers. Similarly, "complex root" could mean a real root.
2. We can use an easy inductive argument, together with the Factor Theorem, to prove the following "stronger" version of the Fundamental Theorem of Algebra: Every polynomial of degree n with complex coefficients factors into n linear polynomials. So every polynomial of degree n has n complex roots, though there may be repetitions among these roots. (For example, the polynomial $x^2 - 4x + 4$ "has two roots," but both of these roots are equal to 2. To see why this is, observe that $x^2 - 4x + 4$ factors into $(x - 2)(x - 2)$.)
3. By the Fundamental Theorem of Algebra, we know that a polynomial $P(x)$ of degree n has exactly n complex roots, including repetitions. So call these roots $r_1, r_2, \ldots, r_n$. Then by the Factor Theorem, $P(x)$ can be written in the form $P(x) = C(x - r_1)(x - r_2)\cdots(x - r_n)$ for some constant C. (Observe that we did this with the polynomial $x^3 - 4x^2 - 15x + 18$ in the example above.)
4. The Fundamental Theorem of Algebra isn't really a theorem about algebra; it's a theorem about the topology of the complex numbers. There are several really neat proofs of this theorem; one involves showing that a bounded differentiable function on the complex plane must be constant, while another looks at a family of paths and shows that they wrap around the origin a certain number of times.

Quadratic Formula: The roots of the polynomial $ax^2 + bx + c$ are given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Remark: There is also a **Cubic Formula**; it's quite a bit more complicated, and in order to evaluate it, you have to know how to find complex cube roots of a number. There is also a **Quartic Formula** for finding roots of fourth-degree polynomials; it's downright scary.

Suppose that S is a set of numbers, such as the set of integers, the set of rational numbers, or the set of real numbers. We say that a polynomial with coefficients in S is *irreducible over* S if we cannot factor it into two polynomials of smaller degree, both having coefficients in S. For example, the polynomial $x^2 - 2$ is not irreducible over $\mathbb{R}$, the set of real numbers, because we can factor it as $(x - \sqrt{2})(x + \sqrt{2})$. However, this polynomial is irreducible over $\mathbb{Z}$, the set of integers, because we cannot factor it into two linear polynomials with integer coefficients.

Here are some useful theorems about roots and irreducibility:

Conjugate Root Theorem: Let $P(x)$ be a polynomial with real coefficients. If the complex number $a + bi$ (where a and b are real) is a root of $P(x)$, then so is $a - bi$, the *complex conjugate* of this number.

Descartes' Rule of Signs: Let $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients. The number of positive (real) roots of this polynomial is less than or equal to the number of sign changes in the sequence $a_n, a_{n-1}, \ldots, a_1, a_0$. (We count one "sign change" in this sequence each time we see a negative coefficient after seeing a positive coefficient, or vice versa. Zeroes do not, in themselves, count as "sign changes.") The number of negative roots is less than or equal to the number of sign changes in the sequence $(-1)^n a_n, (-1)^{n-1} a_{n-1}, \ldots, -a_3, a_2, -a_1, a_0$.

Gauss's Lemma: Let $P(x)$ be a polynomial with integer coefficients. If this polynomial is irreducible over $\mathbb{Z}$, then it is irreducible over $\mathbb{Q}$, the set of rational numbers. (Put differently, if you can't factor this polynomial into smaller polynomials with integer coefficients, then you won't be able to factor it into polynomials with rational coefficients either.)

Rational Root Theorem: Let $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. Then if $\pm\frac{p}{q}$, a rational number in lowest terms, is a root of this polynomial, then p must be a divisor of a_0, and q must be a divisor of a_n.

Eisenstein's Criterion: Let $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. Suppose there is a prime number p such that p divides every coefficient of the polynomial except a_n, p does *not* divide a_n, and p^2 does not divide a_0. Then the polynomial is irreducible over $\mathbb{Z}$.

Symmetric Polynomials

Let $x_1, x_2, \ldots, x_n$ be variables, and let $d \leq n$. The *elementary symmetric polynomial of degree d in n variables*, which we'll call $S_d(x_1, \ldots, x_n)$, is the sum of all the different d-fold products of distinct variables in the set $\{x_1, x_2, \ldots, x_n\}$. For example:

$$\begin{aligned}
S_1(x_1, x_2) &= x_1 + x_2 \\
S_2(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3 \\
S_3(x_1, x_2, x_3) &= x_1x_2x_3 \\
S_2(x_1, x_2, x_3, x_4) &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4
\end{aligned}$$

Note that these are examples of polynomials in more than one variable. We will find that these illuminate a useful connection between the coefficients of a polynomial $P(x)$ and the roots of the polynomial.

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Challenge Problems

1. Give two different polynomials $P(x)$ and $Q(x)$ such that both $P(x)$ and $Q(x)$ have 3, 5, and -1 as roots.

2. Give a monic polynomial $P(x)$, with degree as small as possible, that has $1+i$, $1-i$, $-1+i$, and $-1-i$ as roots.

3. Find a polynomial $ax^4 + bx^3 + cx^2 + bx + a$, where a, b, and c are integers (with $a \neq 0$), having $2+i$ as a root.

4. The polynomial $x^3 - 3x^2 - 10x + 24$ has three integer roots. Find them.

5. Let r and s be the roots of the polynomial $3x^2 - 7x - 11$. Compute the following:

1. $r + s$
2. rs
3. $r^2 + s^2$
4. $|r - s|$
5. $\frac{1}{r} + \frac{1}{s}$

6. Two real numbers have a sum of 13 and a product of 32. What are the two numbers?

7. The edges of a rectangular box have lengths equal to the three roots of the polynomial $x^3 - 10x^2 + 21x - 5$. Find the following:

1. The volume of the box.
2. The surface area of the box.
3. The length of an inner diagonal of the box.

8. Let r_1, r_2, r_3, r_4, r_5 be the five complex roots of the polynomial $x^5 - 5x^4 - 7x^3 + 2x^2 + 10x - 16$. Find the value of the expression $(1+r_1)(1+r_2)(1+r_3)(1+r_4)(1+r_5)$. (This is a slightly-adjusted version of a problem from a past American Regions Math League contest.)

9. Suppose that a triangle has sides whose lengths are equal to the three roots of the polynomial $x^3 - 16x^2 + 84x - 145$. What is the area of this triangle? (This is a slightly-adjusted version of a problem from a past American Regions Math League contest.)

10. Let r, s, and t be the complex roots of the polynomial $x^3 + 3x^2 + 15x - 7$. Find a cubic polynomial with integer coefficients whose roots are $r+s$, $s+t$, and $t+r$.

11. Let r_1, r_2, r_3, r_4 be the complex roots of the polynomial $2x^4 - 5x^3 + 8x^2 + 10x - 1$. Find a fourth-degree polynomial whose roots are $\frac{1}{r_1}$, $\frac{1}{r_2}$, $\frac{1}{r_3}$, and $\frac{1}{r_4}$.

12. Let $P(x)$ be a polynomial of degree 2008 such that $P(n) = \frac{1}{n}$ for $n = 1, 2, 3, \ldots, 2009$. Find the value of $P(2010)$. (This is a slightly-adjusted version of a problem from a past American Regions Math League contest.)