UT Saturday Morning Math Group

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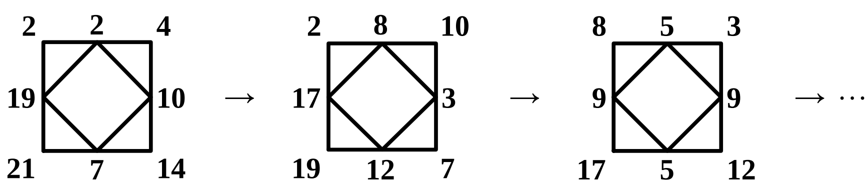
Saturday, February 27, 2010, 10 AM - 12 PM, RLM 4.102 (On the UT Campus)

Polygon Differencing Games

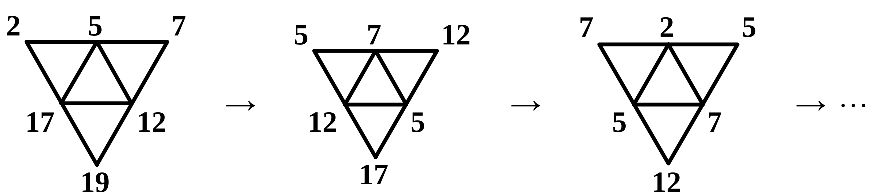
Abstract: We will investigate the following games which you can try in advance:

1. Draw a square and write a (small) non-negative integer at each vertex. (See the example diagram below.)
   - **a.** Write the (absolute value of the) differences at the midpoints of each side.
   - **b.** Connect the midpoints to produce a new square with a non-negative integer at each vertex. To get the square to line up with the original square, rotate it counterclockwise by $45°$ and enlarge slightly.
   - **c.** Repeat (a) and (b) until the process "terminates". This is a Square Differencing Game.
   - **d.** Repeat the Square Differencing Game with different initial numbers.
   - **e.** What do you conjecture? Will this happen for all initial numbers?



3 more steps

2. Repeat the above process but starting with an equilateral triangle. This is a Triangle Differencing Game. What do you conjecture?



5 more steps

3. Repeat the above process but starting with other regular polygons. If the polygon has N sides, this is called an N-gon Differencing Game. What do you conjecture? For which N's will the game end like the square or like the triangle?

Some Notation:

A configuration of an N-gon can be described as an N-tuple (an ordered list of N numbers) $\vec{a} = (a_1, a_2, \cdots, a_N)$ where $a_1, a_2, \ldots, a_N$ are the numbers on the vertices starting at a fixed vertex (say top left) and going clockwise around the N-gon. The differencing operation on an N-gon takes the configuration $\vec{a} = (a_1, a_2, \cdots, a_N)$ to the configuration

$$D\vec{a} = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_N - a_1|).$$

We write D^p for the operation of applying D, the differencing operator, p times. We also write $G(\vec{a})$ for the game starting from $\vec{a}$. Thus $G(\vec{a})$ is the sequence of configurations

$$\vec{a}, \quad D\vec{a}, \quad D^2\vec{a}, \quad D^3\vec{a}, \quad D^4\vec{a}, \quad \cdots$$