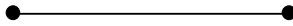# Compass and Straightedge Constructions
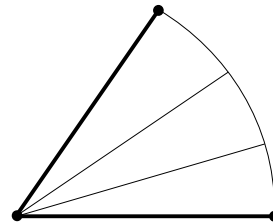## Worksheet/Outline
Travis Mandel
September 23, 2012

Problem 0. Using only a compass and straightedge, construct an equilateral triangle with a given line segment as a side.
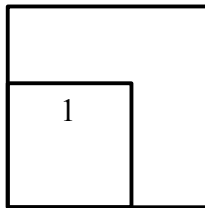
Classical Questions:

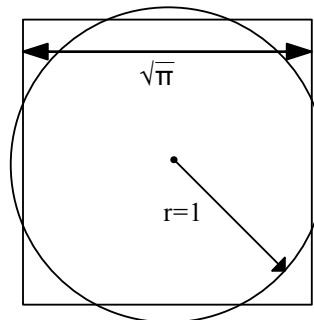**Trisecting an Angle:** Can we trisect any given angle θ?

Doubling a Cube: Given the base of a cube of volume V, is it possible to construct the base of a cube of volume 2V?
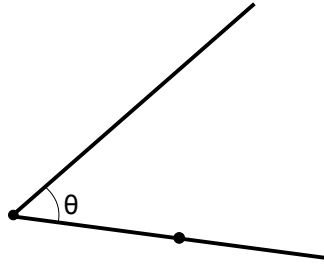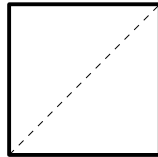
1

$\sqrt[3]{2}$

Squaring the Circle: Given a circle of area A, is it possible to construct a square of area A?

$\sqrt{\pi}$

r=1

Problem 1. Given an angle θ, show how to bisect the angle with only a compass and straightedge.

Problem 2. Given a square of area A, construct a square of area 2A.

What is a constructible point?
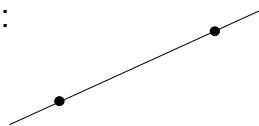
(i) Start with (0,0) and (1,0). Call these points **constructible**.

$$(0,0) \qquad (1,0)$$
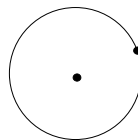
(ii) Given two constructible points, we may draw
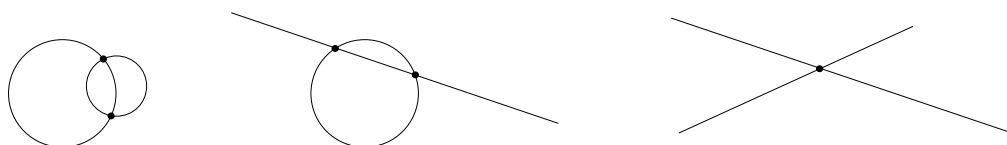   (a) A line through them:

   (b) A circle with center at one point and passing through the other.

Such lines and circles are **constructible**.

(iii) The points of intersection of constructible lines and circles are also **constructible**.

Problem 3. Show that (a,0) is constructible for every integer a.

───

(0,0)   (1,0)

Problem 4. Given a line ℓ and a point p, show that you can construct a line through p and perpendicular to ℓ.

    Case 1: p is not in ℓ.

p •

─────────────────────────────── ℓ

    Case 2: p is in ℓ. (Hint: Use Problem 1).

──────────•─────────── ℓ
       p
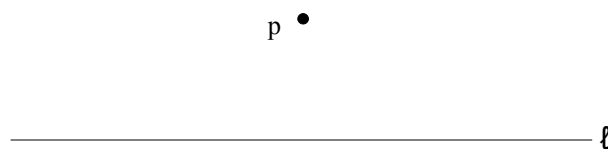
Problem 5. Given a line ℓ and a point p not in ℓ, prove that you can draw a line through p and parallel to ℓ. You do **not** have to do the whole construction. Just refer to Problem 4 to draw perpendiculars.

p •

———————————————— ℓ

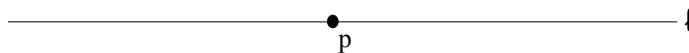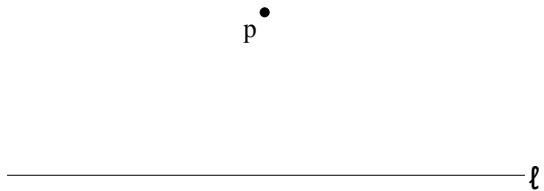**Problem 6.** Given two points q and r of distance d apart and a third point p, show that you can construct a circle of radius d centered at p. **Use the previous problem.**

r •

d

q

p

ℓ
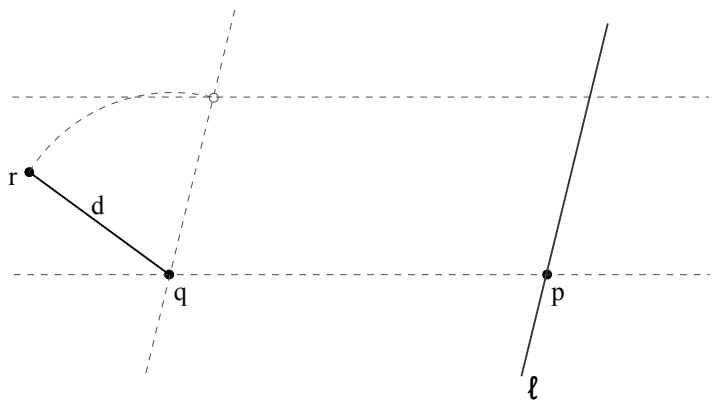
Definition:
We call a real number d **constructible** if |d| is the distance between 2 constructible points.

Example: Problem 6 says that if d is a constructible number, then we can construct a circle of radius |d| centered at any point.

Problem 7. Show that a point (x,y) is constructible if and only if |x| and |y| are constructible numbers. Use Problem 6.
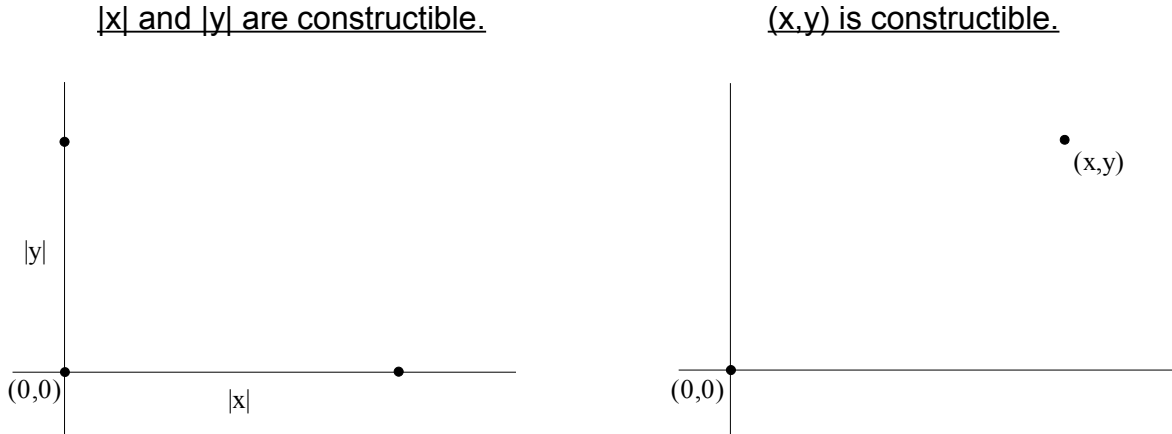
<u>|x| and |y| are constructible.</u>   <u>(x,y) is constructible.</u>


Note: I drew the above pictures assuming x and y were positive. Similar pictures in other quadrants work for the other cases. Alternatively, you can note that x is a constructible number if and only if -x is a constructible number.

<u>Proposition:</u> If x and y are constructible numbers, then so are x+y, x-y, xy, and x/y (if $y \neq 0$).

(A subset of the real numbers which satisfies these properties is called a **subfield** of the real numbers).

Problem 8. Prove this. Addition and subtraction come from Problem 6. For multiplication and division, show that the triangles at the top of the next page are constructible, and use the properties of similar triangles to express h in terms of x and y.
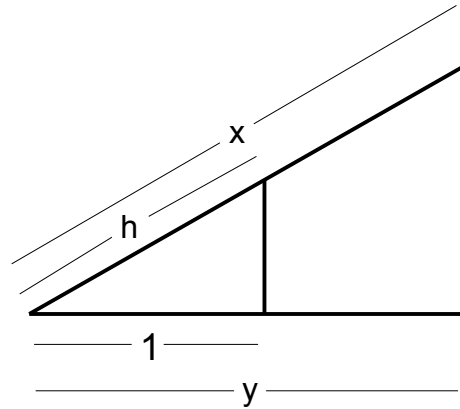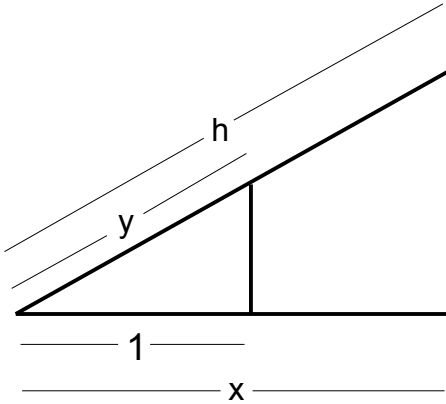
Problem 9. Prove that if x is a positive constructible number, then so is √x. HINT: In the figure below, use similar triangles to express h in terms of x.
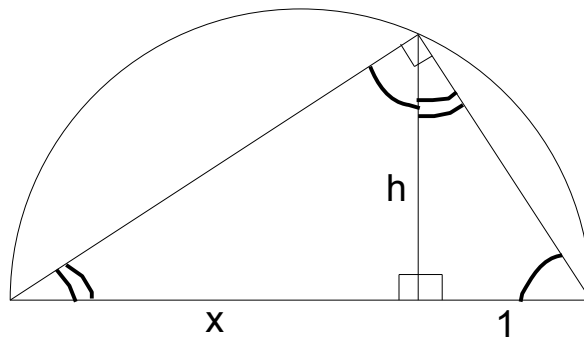
This proves the following theorem!

Theorem: Any point whose coordinates can be written using repeated sums, differences, products, quotients, and square roots of rational numbers is constructible.

Problem 10. Is the golden ratio $\varphi = (1+\sqrt{5})/2$ constructible?
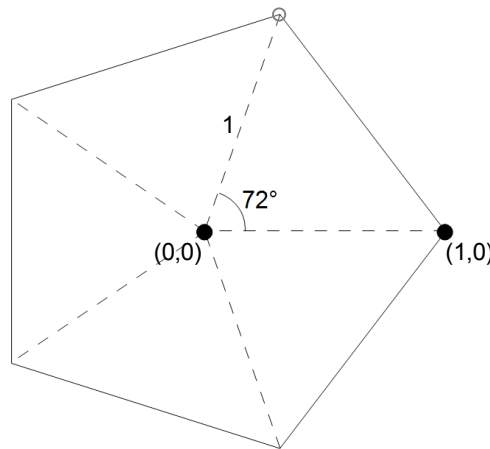
Problem 11. Is a regular pentagon constructible? Use

$$\cos(72°) = \frac{\sqrt{5}-1}{4} \qquad \text{and} \qquad \sin(72°) = \frac{\sqrt{2(\sqrt{5}+5)}}{4}\ .$$

The converse to the above theorem also holds:

Theorem: Constructible numbers are exactly the ones which can be expressed using only repeated sums, differences, products, quotients, and square roots of rational numbers.

Sketch of Proof:
The idea is to consider the equations for constructible lines and circles. The coefficients for these equations are constructible numbers, and intersection points can be found using the quadratic formula (or directly), which expresses solutions in terms of sums, differences, products, quotients, and square roots of the coefficients.

For example, if we want to know when

$$y - y_0 = m(x - x_0) \qquad \text{intersects} \qquad (x - x_1)^2 + (y - y_1)^2 = r^2,$$

we simply plug

$$m(x - x_0) + y_0$$

in for y in the second equation and then solve for x. □

Using some pretty advanced ideas from abstract algebra (taught in a college or graduate level course), the above theorem implies the following:

Theorem: If c is a constructible number, then there is some nonzero polynomial f(x) with rational coefficients such that f(c)=0. Furthermore, the lowest possible degree for f(x) is 2^k for some integer k≥0.

Examples: π is not a constructible number because it is transcendental, meaning it is not a root of any polynomial with rational coefficients.

2 is constructible. Its "minimal" polynomial is x-2.

$\sqrt{2}$ is constructible. What is its minimal polynomial?

However, $\sqrt[3]{2}$ is not constructible, because its minimal polynomial is x^3-2.

Problem 12. Prove that doubling a cube is impossible by showing that it is impossible to construct a side of a cube with volume 2.

Problem 13. Why is squaring a circle impossible? That is, given a circle of radius 1, why can't you construct a square with the same area as the circle?

Problem 14 (Hard). Prove that trisecting a 60° angle is impossible. HINT: Plug θ=20° into the identity:

$$\cos(3\theta)=4\cos^3\theta-3\cos\theta$$

Use cos(60°)=1/2. Let z=2cos(20°). So if a 20° angle is constructible, so is z.

Then use the (advanced) fact that solutions to the equation $8z^3 - 6z - 1 = 0$ cannot be expressed with sums, differences, products, quotients, and square roots of rational numbers (because their minimal polynomial has degree 3).

Problem 15. In Problem 14, we showed that a 20° angle is not constructible. Why does this imply that a regular 18-gon is not constructible?