1.1 Hyperbolic geometry

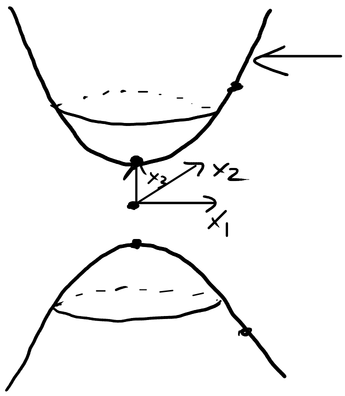
Different models of hyperbolic space $\mathbb{H}^n$, $n \geq 2$
- hyperboloid model
- upper half-space (mostly in $n=2$)
- Klein model

Let B be a $\overset{\text{symm.}}{\text{bilinear}}$ form of signature $(n,1)$ on $\mathbb{R}^{n+1}$

$$B(x,y) = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1} = x^T \underbrace{\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}}_{J} y$$

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} \mid B(x,x) = -1, \ \underline{x_{n+1} > 0} \}$$

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & -1 \end{pmatrix}$$



metric: $x \in \mathbb{H}^n$, $T_x\mathbb{H}^n \subset T_x\mathbb{R}^{n+1}$

$$g_{hyp,x} = B|_{T_x\mathbb{H}^n}$$

$$\underset{\shortparallel}{(Ax)^TJAy} \qquad \underset{\shortparallel}{x^TJy}$$

$$O(n,1) = \{ A \in GL(n+1,\mathbb{R}) \mid B(Ax,Ay) = B(x,y) \ \forall x,y \in \mathbb{R}^{n+1} \}$$
$$= \{ A \in GL(n+1,\mathbb{R}) \mid A^TJA = J \}$$

$O(n,1)$ has 4 connected components, distinguished by
- determinant $+1$ or -1
- "time-orientation": $B(Ae_{n+1}, e_{n+1})$ positive or negative ?

or any x with $B(x,x) < 0$

$$SO_0(n,1) = \{ A \in O(n,1) \mid \det A = 1, \ B(Ae_{n+1}, e_{n+1}) < 0 \}$$

$SO_0(n,1)$ acts on $\mathbb{H}^n$ and preseves the Riemannian metric.

Proposition

$SO_0(n,1)$ is the group of orientation-preserving isometries of $\mathbb{H}^n$.

Proof:

$(v_1, \dots, v_n) \subset T_x \mathbb{H}^n$ pos. oriented $\Longleftrightarrow$ $(x, v_1, \dots, v_n) \subset \mathbb{R}^{n+1}$ pos. oriented

$\boxed{\det A = 1}$

$\Longrightarrow$ $(Ax, Av_1, \dots, Av_n)$ pos. oriented $\Rightarrow$ $(Av_1, \dots, Av_n) \subset T_{Ax}\mathbb{H}^n$ pos. oriented

$$SO_0(n,1) \longrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$$

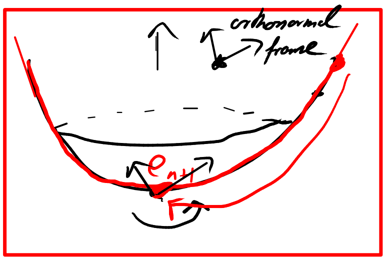
– is injective since $\mathbb{H}^n$ is not in a proper subspace of $\mathbb{R}^{n+1}$

For $U \in SO(n)$, $\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in SO_0(n,1)$ "rotation"

for $t \in \mathbb{R}$, $\begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \in SO_0(n,1)$ "translation"

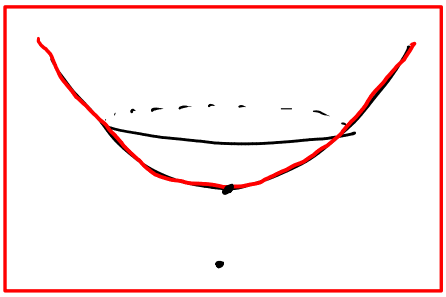
– $\underline{SO_0(n,1) \text{ acts transitively on } \mathbb{H}^n}$:

for $x \in \mathbb{H}^n$, use a rotation to bring x into the plane $\langle e_1, e_{n+1} \rangle$, then use a translation to move it to e_{n+1}.



$\langle e_1, e_{n+1} \rangle$

- The rotations act transitively on $\overbrace{\text{orthonormal}}^{\text{pos. oriented}}$ bases of $T_{e_{n+1}}\mathbb{H}^n$, so $SO_0(n,1)$ acts transitively on the pos. oriented orthonormal frame bundle of $\mathbb{H}^n$.
- If a isometry f of a $\overset{\text{conn.}}{}$ Riem. mfd. fixes x and $df_x = id$, then $f = id$. $\Rightarrow$ every element of $\mathrm{Isom}^+(\mathbb{H}^n)$ is realized by $SO_0(n,1)$. $\square$



The (unparametrized) geodesics of $\mathbb{H}^n$ are the intersections with planes through 0.

$$\gamma(t) = \begin{pmatrix} \sinh t \\ 0 \\ \vdots \\ 0 \\ \cosh t \end{pmatrix} \in \mathbb{H}^n$$

is unit-speed geodesic: $B(\dot\gamma(t), \dot\gamma(t)) = \cosh^2 t - \sinh^2 t = 1$.

$\Rightarrow \quad d_{\mathbb{H}^n}(\gamma(0), \gamma(t)) = |t| \qquad B(\gamma(0), \gamma(t)) = B\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sinh t \\ 0 \\ \vdots \\ 0 \\ \cosh t \end{pmatrix}\right) = -\cosh t$

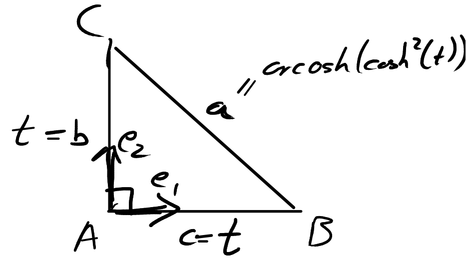
$$\Rightarrow \quad \boxed{d_{\mathbb{H}^n}(x,y) = \operatorname{arcosh}(-B(x,y))}$$

since $SO_0(n,1)$ preserves both $d_{\mathbb{H}^n}$ and B, and acts transitively on pairs of points with a fixed distance.

Also, since $\mathrm{Isom}^+(\mathbb{H}^n)$ acts transitively on pos. or. orthonormal frames, $\mathbb{H}^n$ has constant curvature.

Look at the triangle, $t > 0$

$$A = e_{n+1}, \quad B = \begin{pmatrix} \sinh t \\ 0 \\ \vdots \\ 0 \\ \cosh t \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ \sinh t \\ 0 \\ \vdots \\ 0 \\ \cosh t \end{pmatrix}$$



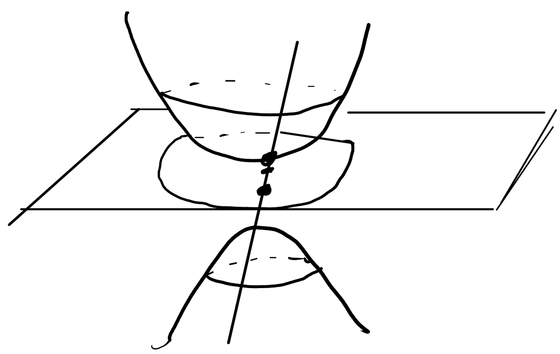
$$a = \operatorname{arcosh}\left(-B\left(\begin{pmatrix} \sinh t \\ 0 \\ \vdots \\ 0 \\ \cosh t \end{pmatrix}, \begin{pmatrix} 0 \\ \sinh t \\ 0 \\ \vdots \\ 0 \\ \cosh t \end{pmatrix}\right)\right) = \operatorname{arcosh}(\cosh^2 t) \neq \sqrt{2}\, t$$

"hyperbolic Pythagoras" in curvature $-k^2$

$$\boxed{\cosh(ka)\cosh(kb) = \cosh(kc)} \qquad \Rightarrow k = 1$$

<u>Klein model</u>

$$\{[x] \in \underbrace{\mathbb{R}P^n}_{= (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times} \mid B(x,x) < 0\}$$

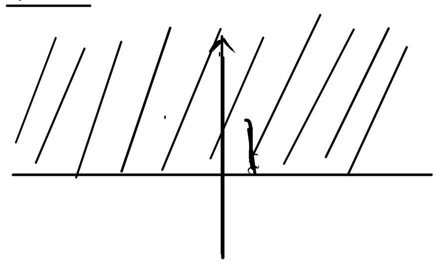


Upper half-space model

$$H^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$$

$$g_{H^n, x}(v,w) = \frac{1}{x_n^2}\, g_{\mathrm{Eucl}}(v,w)$$

<u>$n=2$</u>



If <u>$n=2$</u>: $\quad \mathrm{Isom}^+(H^n) \cong PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm 1\}$

$z \in H^n = \{x + iy \in \mathbb{C} \mid y > 0\}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$ Möbius transformation

$$\left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \cdot z = -\bar{z} \right)$$

$\Rightarrow$ there must be an isomorphism $\overset{\mathbb{R}^2}{\overset{\circlearrowleft}{PSL(2,\mathbb{R})}} \cong SO_0(2,1)$ [accidental isomorphism]

$$\underset{\mathbb{R}^2 \otimes \mathbb{R}^2}{\underset{\cap}{Sym^2 \mathbb{R}^2}} = \left\langle e_1 \otimes e_1 \,,\ \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) \,,\ e_2 \otimes e_2 \right\rangle \cong \mathbb{R}^3$$

$SL(2,\mathbb{R})$ acts on $\mathbb{R}^2$ and therefore on $Sym^2 \mathbb{R}^2 \cong \mathbb{R}^3$

$\Rightarrow$ gives a group hom. $\iota : PSL(2,\mathbb{R}) \longrightarrow GL(3,\mathbb{R})$

Let $\omega(x,y) = x^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y$. Every $A \in SL(2,\mathbb{R})$ satisfies $\omega(Ax, Ay) = \omega(x,y)$.

$$A^T \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} A = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$B' = \omega \otimes \omega$

$$B'(e_1 \otimes e_1 \,,\ e_1 \otimes e_1) = \omega(e_1, e_1)^2 = 0$$
$$B'(e_1 \otimes e_1 \,,\ e_2 \otimes e_2) = \omega(e_1, e_2)^2 = 1$$
$$B'\left(\frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) \,;\ \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1)\right) = -1$$

$B'(x,y) = x^T \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} y$ is preserved by $\iota(PSL(2,\mathbb{R}))$ and is a symmetric bilinear form of signature $(1,2)$.