$SO_0(n,1) \cong \mathrm{Isom}^+(\mathbb{H}^n)$

$\iota: PSL(2,\mathbb{R}) \longrightarrow \cancel{SO_0(2,1)}$ $\quad SO_0(J') = Q^{-1} SO_0(2,1)\, Q$

$$\begin{array}{ccc} A & \longmapsto & A \otimes A \\ \circlearrowright & & \\ \mathbb{R}^2 & & \mathbb{R}^3 \cong Sym^2 \mathbb{R}^2 \subset \mathbb{R}^2 \otimes \mathbb{R}^2 \end{array}$$

$\omega(x,y) = x^T \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} y \qquad B' = \omega \otimes \omega$

$B'(u_1 \otimes v_1, u_2 \otimes v_2) = \omega(u_1,u_2)\,\omega(v_1,v_2)$

basis: $\left(e_1 \otimes e_1, \ \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1), \ e_2 \otimes e_2\right)$

$$B'(x,y) = x^T \underbrace{\begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \end{pmatrix}}_{J'} y \qquad \forall\, x,y \in \mathbb{R}^3$$

$\iota(A)$ preserves B'

$$Q = \begin{pmatrix} 1/\sqrt{2} & & -1/\sqrt{2} \\ & 1 & \\ 1/\sqrt{2} & & 1/\sqrt{2} \end{pmatrix} \qquad Q J' Q^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & & 1/\sqrt{2} \\ & 1 & \\ -1/\sqrt{2} & & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} = J$$

Both $PSL(2,\mathbb{R})$ and $SO_0(2,1)$ are 3-dimensional

ι injective immersion $\Rightarrow$ $\iota(PSL(2,\mathbb{R})) \subset \underline{SO_0(2,1)}$ open subgroup

open subgroups are closed! $\Rightarrow$ $\iota(PSL(2,\mathbb{R})) = SO_0(2,1)$

$H \subset G = \bigcup_{g \in G/H} \overbrace{gH}^{\text{open}} \qquad G \setminus H = \bigcup_{\substack{g \in G/H \\ g \neq id}} gH \quad$ open

1.2 Proper actions

- A smooth (left) action of a Lie group G on a manifold X is a smooth map $\cdot : G \times X \longrightarrow X$ s.t. $(gh)\cdot x = g\cdot(h\cdot x)$ $\forall g,h \in G$, $x \in X$.

- A smooth action of G on X is <u>proper</u> if the map

$$G \times X \longrightarrow X \times X$$
$$(g, x) \longmapsto (gx, x)$$

is proper, i.e. the preimage of every compact set is compact.

- Equivalently, the action is proper if, for all sequences g_n in G and x_n in X s.t. x_n and $g_n x_n$ converge, a subsequence of g_n converges.

$$\left(K = \{(g_1 x_1, x_1), (g_2 x_2, x_2), \dots, \lim_{n\to\infty}(g_n x_n, x_n)\} \right)$$

<u>Examples</u>

1) If G is compact, the action is proper.
2) If G is not compact but X is compact, the action is <u>not</u> proper.
3) G acting on itself by left- or right-translation is proper.

$$G \times G \longrightarrow G \qquad (g,h) \mapsto gh \quad \text{or} \quad (g,h) \mapsto hg^{-1}$$

4) If the action $G \circlearrowright X$ is proper, its restriction to a closed subgroup $H \subset G$ is proper, and its restriction to a G-invariant open subset $U \subset X$ is proper.

<u>Note</u>: A closed subgroup of a Lie group is an embedded submanifold. (closed subgroup theorem).

$$\underset{\text{orbit of } x \in X}{G\cdot x = \{gx \mid g \in G\}} \qquad \underset{\text{stabilizer of } x \in X}{G_x = \{g \in G \mid gx = x\}}$$

$G_x \subset G$ is a closed subgroup. If the action is proper, then G_x is compact.

$G_x = \{1\}$ $\forall x \in X \iff$ the action is <u>free</u>.

Proposition:

If the action $G \circlearrowright X$ is proper, every orbit is a closed subset of X and X/G is Hausdorff.

with the quotient topology

Proof:

Let $g_n x \to x'$ be a converging sequence in an orbit. By properness, a subsequence of g_n converges, $g_{n_k} \to g$. So $g_{n_k} x \to gx = x'$, i.e. the limit is still in the orbit. So $G \cdot x$ is closed.

(Assume X/G is not Hausdorff.) Say the orbits of $u, v \in X$ cannot be separated. Let U_n, V_n be open balls around u, v with radius $1/n$ (in some metric). Then there exist $u_n \in U_n$, $v_n \in V_n$ and $g_n \in G$ s.t. $v_n = g_n u_n$.

$$\underset{u}{\downarrow} \qquad \underset{v}{\downarrow}$$

By properness, a subsequence g_{n_k} converges to $g \in G$. So $v = gu$, i.e. u and v represent the same orbit. $\square$

Theorem (Existence of the Haar measure)

Let G be a compact Lie group. There exists a probability measure on G which is invariant by left- and right-translations.

$$l_g, r_g : G \longrightarrow G \qquad l_g(h) = gh, \quad r_g(h) = hg.$$

Proof:

Let ω be a volume form on G which invariant under left translations. We get it by choosing $\omega_1 \in \underbrace{\Lambda^{\dim g} g^*}$ arbitrarily non-zero and defining $\omega_g = l_{g^{-1}}^* \omega_1$.

Let $g \in G$ and define $\omega' = r_g^* \omega$. Since $\Lambda^{\dim g} g^*$ is 1-dim., $\underbrace{\omega_1' = \lambda \omega_1}$ for some $\lambda \in \mathbb{R}$. For every $h \in G$

$$\omega_h' = r_g^* \omega_{hg} = r_g^* l_{(hg)^{-1}}^* \omega_1 = r_g^* l_{h^{-1}}^* l_{g^{-1}}^* \omega_1 = l_{h^{-1}}^* r_g^* \underbrace{l_{g^{-1}}^* \omega_1}$$

$$= l_{h^{-1}}^* \underbrace{r_g^* \omega_g} = l_{h^{-1}}^* \omega_1' = \lambda l_{h^{-1}}^* \omega_1 = \lambda \omega_h.$$

Because G is compact, we can integrate ω and ω'. They have the same total volume, so $\lambda = 1$. Hence ω is right-invariant. $\square$

Corollary

1) Let G be compact and acting linearly on a vector space V. Then there is a G-invariant inner product $\langle \cdot , \cdot \rangle$ on V.

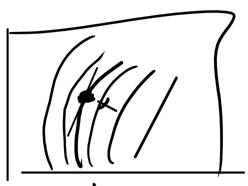
2) Let G be compact and act smoothly on a manifold X. Then there is a G-invariant Riemannian metric g.

Proof

Take any inner product $\langle \cdot , \cdot \rangle'$ / any Riemannian metric g' and average over G using the Haar measure:

$$\langle u, v \rangle := \int \langle hu, hv \rangle' \, dh \qquad g(u,v) := \int g'(h_* u, h_* v) \, dh . \qquad \square$$

Assume $G \circlearrowright X$ properly. If $g \in G_x$, then $dg : T_x X \to T_x X$ is a linear map. This defines a linear action of G_x on $T_x X$, called the isotropy action.



X

Choose a G_x-invariant inner product on $T_x X$ and let N_x be the orthogonal complement of $T_x(G \cdot x)$. Then we have a G_x-invariant splitting:

$$T_x X = T_x(G \cdot x) \oplus N_x$$

Define: $\quad G \times_{G_x} N_x = (G \times N_x)/\sim \quad$ where $(gh^{-1}, hv) \sim (g, v) \ \forall h \in G_x$

Slice theorem: locally $\underset{\underset{G}{\circlearrowright}}{X}$ looks like $\underset{\underset{G}{\circlearrowright}}{G \times_{G_x} N_x}$!