$G \curvearrowright X$ proper if $\left(x_n \to x \ \& \ g_n x_n \to x'\right) \Rightarrow g_{n_k} \to g$

$x \in X$, $g \in G_x$, $g : X \to X \Rightarrow G_x \curvearrowright T_x X$

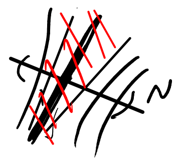
$dg_x : T_x X \to T_x X$ linear

$$T_x X = T_x(G \cdot x) \oplus \underset{\left(T_x(G\cdot x)\right)^{\perp}}{\overset{\shortparallel}{N}} \qquad G_x\text{- invariant}$$

$$G \times_{G_x} N = (G \times N)/\sim \qquad (g, n) \sim (gh^{-1}, hn) \ \forall h \in G_x$$

Theorem (slice theorem):

If $U \subset N$ is a small enough nbh of the origin (G_x-invariant), there exists a G-equivariant smooth map

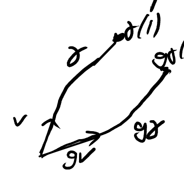


$$G \times_{G_x} U \longrightarrow X$$

which is a diffeomorphism onto a nbh of the orbit $G \cdot x$, and whose restriction to $G \times_{G_x} \{0\} = G/G_x$ is the map $G/G_x \to G \cdot x$, $gG_x \mapsto gx$.

Lemma

Assume G is compact and $x \in X$ is a fixed point. Then there exists a G-equivariant diffeomorphism from a nbh of 0 in $T_x X$ to a nbh. of x in X.



Proof:

Take a G-invariant Riemannian metric on X and let $\exp_x : U \subset T_x X \to X$ be the (Riemannian) exponential map. It is a diffeomorphism if U is small enough and G-equivariant by construction. $\square$

Proof of the slice theorem:

Because $G \curvearrowright X$ properly, G_x is compact. By the Lemma we have a G_x-equivariant diffeo E from a nbh of 0 in $T_x X$ to a nbh of x in X, such that $E(0) = x$ and $d_0 E = id_{T_x X}$.

Fix a G_x-invariant Riemannian metric on X and let $U_n \subset N$ be a ball of radius $1/n$, for all n. For n large enough, we can define

$$\varphi : G \times_{G_x} \overset{\subset N \subset T_xX}{U_n} \longrightarrow X, \quad [g,v] \longmapsto g\,E(v) \qquad h \in G_x$$

$$g\cdot[h,v] = [gh,v] \qquad \qquad \overset{\shortparallel}{[gh^{-1}, hv]} \longmapsto gh^{-1}E(gv) = gE(v)$$

φ is well-defined and G-equivariant. It is a local diffeomorphism at $[1,0]$ because

$$\dim X = \dim G\cdot x + \dim N = \dim G - \dim G_x + \dim N = \dim(G \times_{G_x} N).$$

and $d_{[1,0]}\varphi$ is surjective onto $\qquad ev_x(g) = gx : G \to X$

$$T_xX = T_x(G\cdot x) \oplus N = d\,ev_x(\mathfrak{g}) \oplus d_0E(N) = \operatorname{im}\varphi.$$

It remains to show that φ is injective if U_n is small enough.

Assume the contrary. Then there are $u_n, v_n \in U_n$ and $g_n, h_n \in G$ s.t. $[g_n, u_n] \neq [h_n, v_n]$ but

$$h_n^{-1}g_n \underbrace{E(u_n)}_{\to x} = \varphi(h_n^{-1}g_n, u_n) = \varphi(1, v_n) = \underbrace{E(v_n)}_{\to x} \implies g_\infty x = x$$

By properness and passing to a subsequence, $h_n^{-1}g_n \to g_\infty \in G$. Actually $g_\infty \in G_x$. So

$$\begin{array}{ccc} [h_n^{-1}g_n, u_n] & \longrightarrow & [g_\infty, 0] \\ \neq & & \shortparallel \\ [1, v_n] & \longrightarrow & [1, 0] \end{array}$$

Since φ is a diffeo close to $[1,0]$, $[h_n^{-1}g_n, u_n]$ and $[1, v_n]$ have to eventually be equal. This is a contradiction. $\square$

Corollary $\qquad G_x = \{1\} \qquad G \times \overset{\overset{N}{\shortparallel}}{N}/G \longrightarrow X/G$

If $G \curvearrowright X$ properly and freely, then X/G has a unique smooth structure s.t. $\pi : X \to X/G$ is a smooth submersion.

Proof:

$$\begin{array}{ccccc} G \times U_x & \xrightarrow{\ \varphi_x\ } & X & \xleftarrow{\ \varphi_y\ } & G \times U_y \\ \uparrow\downarrow \ {\scriptstyle v \mapsto (1,v)} & & \downarrow \pi & & \downarrow \\ U_x & \underset{\text{charts}}{\xrightarrow{\ \tilde\varphi_x\ }} & X/G & \xleftarrow{\ \tilde\varphi_y\ } & U_y \end{array}$$

$$\tilde\varphi_y^{-1}(\tilde\varphi_x(v)) = pr_U(\varphi_y^{-1}(\varphi_x(1,v))) \qquad \text{smooth.}$$

Uniqueness:

$$\begin{tikzcd} & X \arrow[dl, shift left] \arrow[dr, "\text{surj. submersions}"] & \\ (X/G, \text{smooth str \#1}) \arrow[ur, shift left, "s"] \arrow[rr, "\mathrm{id}"] & & (X/G, \text{smooth str. \#2}) \end{tikzcd}$$

smooth local section s

$\Rightarrow$ id is a smooth map $\Rightarrow$ smooth str. are equal.

Corollary:

If $H \subset G$ is a closed subgroup then G/H is a manifold ($H \curvearrowright G$ proper), called homogeneous space. If G acts smoothly and transitively on a mfd. X, then X is G-equivariantly diffeomorphic to G/G_x, $x \in X$, via the map

$$G/G_x \longrightarrow X, \quad gG_x \longmapsto gx.$$

Example: $SO_0(n,1) \curvearrowright \mathbb{H}^n$ transitively, stabilizer of $e_{n+1} \in \mathbb{H}^n$ is $SO(n)$.

$\Rightarrow \mathbb{H}^n \cong SO_0(n,1)/SO(n)$.

Proposition

$G \curvearrowright G/H$ properly if and only if H is compact.

Proof:

Assume H cpt. and $[g_n] \in G/H$, $g_n' \in G$ s.t. $[g_n]$ and $[g_n' g_n]$ converge. That means $g_n h_n \to g$, $g_n' g_n h_n' \to g'$ for some $h_n, h_n' \in H$, $g, g' \in G$. Passing to a subsequence $h_n \to h$, $h_n' \to h'$. Then $g_n \to g h^{-1}$ and $g_n' = (g_n' g_n h_n') h_n'^{-1} g_n^{-1} \longrightarrow g' h'^{-1} h g^{-1}$, so the action is proper.

Converse: If action is proper, then $H = G_{[1]}$ is compact. $\square$

Remark: So $SO_0(n,1) \curvearrowright \mathbb{H}^n$ is proper.

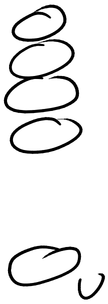
Related theorem: X Riemannian mfd. $\Rightarrow$ $\mathrm{Isom}(X) \curvearrowright X$ properly. (compact open topology)

(Helgason, "Differential Geometry, Lie Groups & Symm. Spaces", IV.2)

Corollary: If Γ is discrete (0-dim Lie group) and $\Gamma \curvearrowright X$ properly & freely, then X/Γ is a mfd., $\pi: X \to X/\Gamma$ is a regular covering map with deck transformation group Γ.

Proof: $x \in X$, let $V \subset X/\Gamma$ be the image of $\pi \circ \psi$ from the slice theorem

$\Gamma \times U$



U

$$\begin{tikzcd} \Gamma \times U \arrow[r, "\psi"] \arrow[d, "pr_U"] & \pi^{-1}(V) \subset X \arrow[d, "\pi"] \\ U \arrow[r] & V \end{tikzcd}$$

fibers $\cong \Gamma$

The cover is regular as Γ acts transitively of fibers. $\square$