$SO_0(n,1) = \mathrm{Isom}^+(\mathbb{H}^n) \curvearrowright \mathbb{H}^n$ proper

$\Gamma \subset SO_0(n,1)$ discrete, in particular closed $\Rightarrow$ $\Gamma \curvearrowright \mathbb{H}^n$ proper

Γ torsion free $\qquad x \in \mathbb{H}^n$, $\mathrm{Stab}_\Gamma(x) \subset \Gamma$ compact $\Rightarrow$ ~~finite~~ trivial

$\Rightarrow$ free action

slice theorem
$\Rightarrow$ $\mathbb{H}^n/\Gamma$ is a manifold, $\mathbb{H}^n \longrightarrow \mathbb{H}^n/\Gamma$ is a regular cover with deck transformations $= \Gamma$

The covering induces a complete metric on $\mathbb{H}^n/\Gamma$ with curvature -1. Since Γ preserves orientation, $\mathbb{H}^n/\Gamma$ is oriented.

Conversely

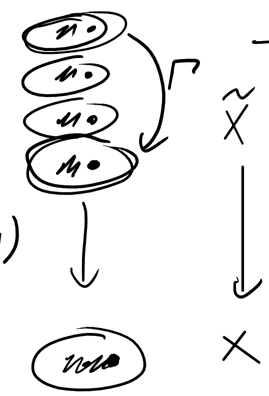
X oriented complete Riemannian mfd. with constant curvature -1, $\tilde{X} \longrightarrow X$ universal cover

$\Rightarrow$ hyperbolic metric on $\tilde{X}$ $\overset{\text{Killing-Hopf}}{\Longrightarrow}$ $\tilde{X}$ is isometric to $\mathbb{H}^n$.

⌜A complete Riem mfd. with constant curvature -1 which is simply connected, is isometric to $\mathbb{H}^n$ ⌟

$\Gamma =$ deck transformations of $\tilde{X} \longrightarrow X \cong \pi_1(X)$

acts by isometries on $\tilde{X} \cong \mathbb{H}^n \Rightarrow \rho: \Gamma \longrightarrow \mathrm{Isom}^+(\mathbb{H}^n) \cong SO_0(n,1)$



ρ is a group homomorphism, is injective

ρ has a discrete image:

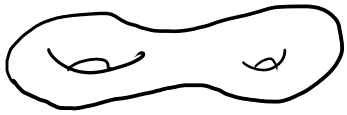
every orbit is just a fiber of the universal cover, hence discrete.

correspondence

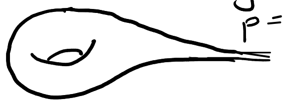
discrete torsion-free subgroups of $SO_0(n,1)$ / conjugation $\longleftrightarrow$ complete oriented hyperbolic manifolds / isometry

$\underline{n=2:}$ $\Gamma < SO_0(2,1)$ discrete torsion-free finitely generated $\Rightarrow$ $\mathbb{H}^2/\Gamma$ complete oriented hyperbolic surface of finite genus g and with finitely many punctures/ends p.

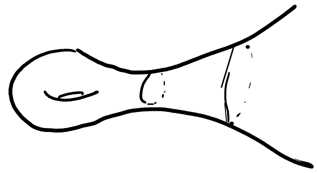
$\Gamma \cong \pi_1(\mathbb{H}^2/\Gamma)$



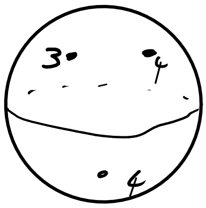
$g=2$
$p=0$



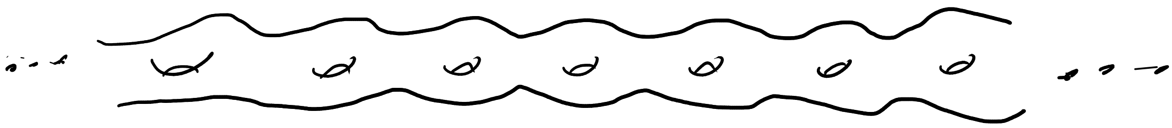
$g=1$
$p=1$



$g=1$
$p=1$

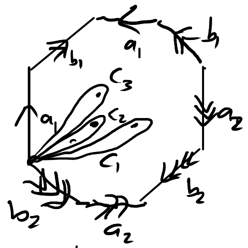


orbifold (for groups with torsion)



(not finitely generated)

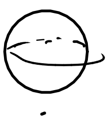
$$\Gamma \cong \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_p \mid [a_1,b_1]\cdots[a_g,b_g]c_1\cdots c_p = 1\rangle$$



If $p \geq 1$: $\quad c_p = \left([a_1,b_1]\cdots[a_g,b_g]c_1\cdots c_{p-1}\right)^{-1}$

Γ = free group in $\underline{2g+p-1}$ generators

$g=0, p=0$

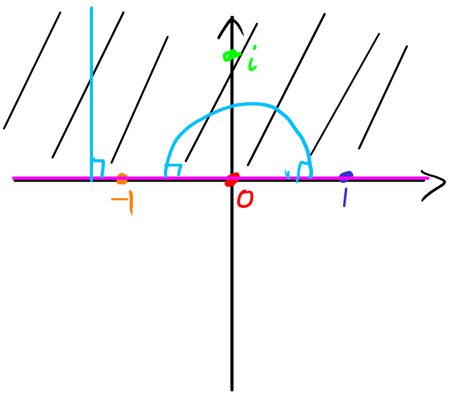


$g=1, p=0$

can not appear by Gauß-Bonnet

<u>Example</u>: free group in 2 generators $\quad (g=1, p=1 \text{ or } g=0, p=3)$

Poincaré disk model:

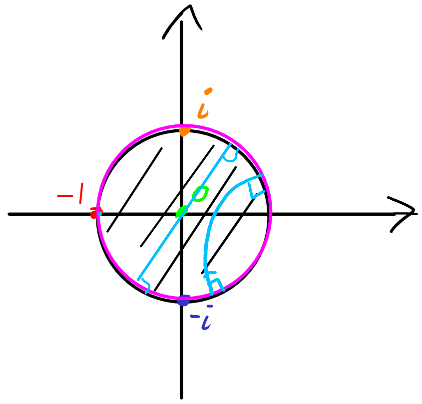


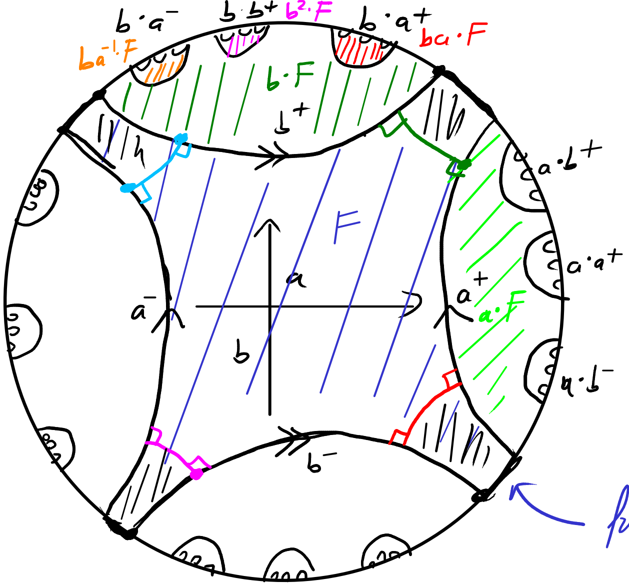
$$z \longmapsto \frac{z-i}{z+i}$$

Cayley transform

$0 \longmapsto -1$

$1 \longmapsto \frac{1-i}{1+i} = -i$





$a, b \in Isom^+(\mathbb{H}^2)$

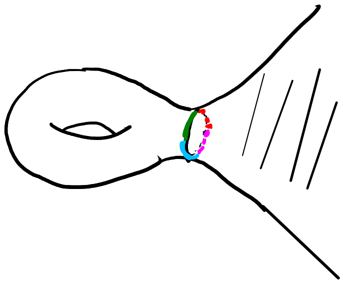
$a \cdot a^- = a^+$

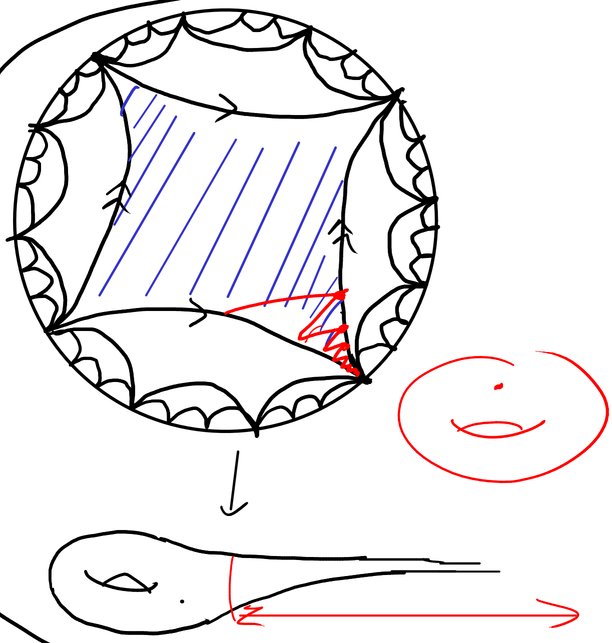
$b \cdot b^- = b^+$

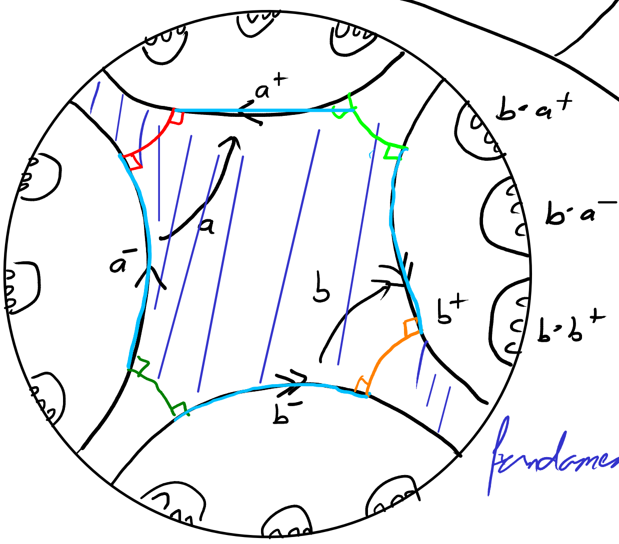
$\Gamma = \langle a, b \rangle \subset Isom^+(\mathbb{H}^2)$

fundamental domain

$\mathbb{H}^2/\Gamma$







$\Gamma = \langle a, b \rangle \subset Isom^+(\mathbb{H}^2)$ free group

fundamental domain

$\mathbb{H}^2/\Gamma$