Fall 2019 – Math 328K (52890)

Midterm exam 1

Tuesday, October 1

- Write your name clearly readable on the top of **every page** you write!
- Do not use a red pen.
- No phones, calculators, books, notes, or other aids are permitted.
- Good luck!

Problem 1. Define prime numbers and composite numbers.

Solution 1. A natural number n is composite if it can be written as a product $n = ab$ with $a, b \in \mathbb{N}$ and $a, b > 1$. A natural number n is prime if it is not composite.

Problem 2. What is the last decimal digit of the number 4^{44}?

Solution 2. The question asks for $\overline{4}^{44}$ in $\mathbb{Z}_{10}$. We get this by the square and multiply algorithm:

$$\begin{aligned}
\overline{4}^2 &= \overline{6} \\
\overline{4}^4 &= (\overline{4}^2)^2 = \overline{6} \\
\overline{4}^5 &= \overline{4} \cdot \overline{4}^4 = \overline{4} \\
\overline{4}^{10} &= (\overline{4}^5)^2 = \overline{6} \\
\overline{4}^{11} &= \overline{4} \cdot \overline{4}^{10} = \overline{4} \\
\overline{4}^{22} &= (\overline{4}^{11})^2 = \overline{6} \\
\overline{4}^{44} &= (\overline{4}^{22})^2 = \overline{6}
\end{aligned}$$

so the last digit of 4^{44} is 6.

Problem 3. The *Fermat numbers* F_n are defined by

$$F_n = 2^{2^n} + 1$$

for all non–negative integers n.

a) Show that $F_{n+1} = F_0 F_1 \cdots F_n + 2$ for all non–negative integers n.

b) Show that F_n and F_m are coprime, for any integers $0 \le n < m$.

Solution 3.

a) We prove $F_0 F_1 \cdots F_n = F_{n+1} - 2$ by induction over n. In the basic case $n = 0$, the statement is $2^{2^0} + 1 = 2^{2^1} + 1 - 2$, which is true because both sides are 3. For the inductive step, assume we already know $F_0 F_1 \cdots F_n = F_{n+1} - 2$. Then

$$\begin{aligned} F_0 F_1 \cdots F_n F_{n+1} &= (F_{n+1} - 2) F_{n+1} = (2^{2^{n+1}} - 1)(2^{2^{n+1}} + 1) \\ &= (2^{2^{n+1}})^2 - 1 = 2^{2^{n+2}} - 1 = F_{n+2} - 2. \end{aligned}$$

This completes the inductive step and the proof.

b) By part a) we can write $F_m = F_0 F_1 \cdots F_n \cdots F_{m-1} + 2$, or equivalently

$$F_m - F_0 F_1 \cdots F_n \cdots F_{m-1} = 2.$$

Now if $d = (F_m, F_n)$, then d divides the left hand side of this equation, so $d \mid 2$. This means either $d = 1$ or $d = 2$. But the case $d = 2$ cannot occur, since the Fermat numbers are all odd.

Problem 4. Do the following equations have solutions x in $\mathbb{Z}_{96}$? How many? If there are solutions, list all of them.

a) $\overline{9}x = \overline{5}$

b) $\overline{5}x = \overline{9}$

Solution 4.

a) $(9, 96) = 3$, so this equation has a solution if and only if $3 \mid 5$, which is false. So there are no solutions.

b) $(5, 96) = 1$, so this equation has a solution if and only if $1 \mid 9$, which is true. So there is a unique solution. We can get it either by guessing or by using the extended Euclidean algorithm. In either case we obtain $x = \overline{21}$, since $\overline{5} \cdot \overline{21} = \overline{105} = \overline{9}$.

Problem 5. Let $a_1, \ldots, a_n, b \in \mathbb{N}$ be positive integers and let $A = a_1 a_2 \cdots a_n$. Show that if $(a_i, b) = 1$ for all $i \in \{1, \ldots, n\}$, then $(A, b) = 1$.

Solution 5. Assume for contradiction that $(A, b) \neq 1$. Then there exists a prime p such that $p \mid (A, b)$, so $p \mid A$ and $p \mid b$. We proved that if $p \mid a_1 \cdots a_n$ then $p \mid a_i$ for some i. p is therefore a common divisor of a_i and b, which contradicts $(a_i, b) = 1$.