Homework 4

due Thursday, September 26, 11:00

Problem 1. Does the definition of congruence classes and $\mathbb{Z}_m$ also make sense for a negative integer m? What would $\mathbb{Z}_m$ be then? What about $\mathbb{Z}_0$?

Problem 2. To understand the multiplication operation on $\mathbb{Z}_m$ better, it can be helpful to compile a multiplication table, that is to list the product $x \cdot y$ for all pairs $x, y \in \mathbb{Z}_m$. For this to be of any use, we should make sure to fix a single representative for every congruence class, for example by using the least non–negative representatives. For example, the multiplication table for $\mathbb{Z}_3$ would look like this:

	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{1}$

Create a multiplication table for $\mathbb{Z}_4$, $\mathbb{Z}_5$ and $\mathbb{Z}_6$. Do you see interesting patterns?

Problem 3. Show that there are no integer solutions to $x^2 + y^2 = 1000003$.

Problem 4. Prove Theorem 4.3: Let $m \in \mathbb{N}$. Then

$$\mathbb{Z}_m = \{0 \bmod m, 1 \bmod m, \dots, (m-1) \bmod m\},$$

and all of these elements are different. In particular $|\mathbb{Z}_m| = m$.

Problem 5. Compute the following expressions. Give the answer as a least non–negative representative. You should (and can!) do this without a calculator, and write down intermediate steps.

a) $2^6 \bmod 11$

b) $2^{12} \bmod 11$

c) $5^{1030} \bmod 3$

d) $78^3 \bmod 3$

e) $(1! + 2! + \dots + 10!) \bmod 5$

f) $(1! + 2! + \dots + 100!) \bmod 15$