Homework 9

due Thursday, November 7, 11:00

Problem 1. Show the following facts about Euler's ϕ–function:

a) $\phi(n)$ is even for every $n \geq 3$,

b) $\phi(n^k) = n^{k-1}\phi(n)$ for all $n, k \in \mathbb{N}$,

c) $\phi(n) \geq \sqrt{n}$ for all $n \in \mathbb{N} \setminus \{2, 6\}$,

d) If $m \mid n$, then $\phi(m) \mid \phi(n)$.

Problem 2. Let $m \in \mathbb{N}$. The goal of this problem is to find all integers which are congruent modulo m to their own square. In other words, we want to find all solutions of $x^2 - x = \overline{0}$ in $\mathbb{Z}_m$.

a) Show that, if m is prime, then the only solutions are $\overline{0}$ and $\overline{1}$.

b) Show that, if m is a prime power, then the only solutions are still $\overline{0}$ and $\overline{1}$.

c) For general m, let $m = p_1^{i_1} \cdots p_k^{i_k}$ be the prime–power decomposition of m with $p_1 < \cdots < p_k$ prime and $i_1, \dots, i_k \in \mathbb{N}$. Show there are 2^k different solutions of $x^2 - x = \overline{0}$ in $\mathbb{Z}_m$, and that these are given by

$$\sum_{j=1}^{k} \delta_j \left(\frac{m}{p_j^{i_j}}\right)^{p_j^{i_j} - p_j^{i_j - 1}} \mod m$$

for all choices of $\delta_1, \dots, \delta_k \in \{0, 1\}$.

Hint: For the last part, use Euler's theorem and the Chinese remainder theorem.

Problem 3. Let $m \in \mathbb{N}$, $x \in \mathbb{Z}_m$ (not necessarily coprime to m) and $n = \frac{m}{(x,m)}$. Show that in $\mathbb{Z}_m$

$$x^{\phi(n)} = \overline{(x,m)}^{\phi(n)}$$