Fall 2019 – Math 328K – Florian Stecker

Homework 10

due Thursday, November 14, 11:00

Problem 1.

a) Show that $\tau(n) \leq n$ for all $n \in \mathbb{N}$ and find all $n \in \mathbb{N}$ with $\tau(n) = n$.

b) Let $n \in \mathbb{N}$, $n \geq 2$. Write $\tau^k(n)$ for the k–fold application of τ, that is $\tau^0(n) = n$ and $\tau^k(n) = \tau(\tau^{k-1}(n))$ for all $k \in \mathbb{N}$. Show that there exists $s \in \mathbb{N}$ so that $\tau^k(n) = 2$ for all $k \geq s$.

Problem 2.

a) Show that $f(1) = 1$ for every multiplicative function f which is not constant 0.

b) Let $\mathcal{P} = \{p^k \mid p \text{ prime}, k \in \mathbb{N}\}$ be the set of all prime powers and let $f\colon \mathcal{P} \to \mathbb{Z}$ be any function. Show that there is a unique multiplicative function $g\colon \mathbb{N} \to \mathbb{Z}$ extending f (i.e. $g|_{\mathcal{P}} = f$).

Problem 3. Given two arithmetic functions $f, g\colon \mathbb{N} \to \mathbb{R}$, their *Dirichlet product* is the arithmetic function $f \star g\colon \mathbb{N} \to \mathbb{R}$ defined by

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d) \qquad \forall n \in \mathbb{N}.$$

a) Show that $f \star g = g \star f$ and $(f \star g) \star h = f \star (g \star h)$ for all arithmetic functions f, g, h.

b) Define the arithmetic function ι by $\iota(1) = 1$ and $\iota(n) = 0$ for all $n > 1$. Show that $f \star \iota = \iota \star f = f$ for all arithmetic functions f.

c) Show that if f and g are multiplicative functions, then $f \star g$ is multiplicative.

d) The arithmetic function g is said to be the *inverse* of the arithmetic function f if $f \star g = \iota$. Show that f has an inverse if and only if $f(1) \neq 0$, and that the inverse is unique if it exists.

e) Let ν be the constant 1 function, i.e. $\nu(n) = 1$ for all $n \in \mathbb{N}$ and let μ be the inverse of ν. Let F be the summatory function of an arithmetic function f. Show that $f = F \star \mu$.

f) Show that the function μ defined in e) is multiplicative and $\mu(p) = -1$ and $\mu(p^k) = 0$ for every prime p and integer $k \geq 2$.

g) Let F be the summatory function of an arithmetic function f. Deduce from the above that, if F is multiplicative, then f is multiplicative.